Weak Solutions of a Phase-Field Model for Phase Change of an Alloy with Thermal Properties

José Luiz Boldrini *
boldrini@ime.unicamp.br

Gabriela Planas †
gplanas@ime.unicamp.br

UNICAMP-IMECC
C.P. 6065
13083-970 Campinas, SP, Brazil

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Abstract

The phase-field method provides an alternative mathematical description for free-boundary problems corresponding to physical processes with phase transitions. It postulates the existence of a function, called the phase-field, whose value identifies the phase at a particular point in space and time, and it is particularly suitable for cases with complex growth structures occurring during phase transitions.

The mathematical model studied in this work describes the solidification process occurring in a binary alloy with temperature dependent properties. It is based on a highly nonlinear degenerate parabolic system of partial differential equations with three independent variables: phase-field, solute concentration and temperature.

Existence of weak solutions of such system is obtained via the introduction of a regularized problem, followed by the derivation of suitable estimates and the application of compactness arguments.

1 Introduction

We are interested in performing a mathematical analysis of a model for phase change processes occurring in binary alloys with thermal properties. Such a model, using a phase-field methodology, was proposed by Caginalp and Xie [3] and a detailed derivation of more comprehensive system was presented by Caginalp and Jones [2]. It is described as the following coupled system of nonlinear partial differential equations:

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \quad \text{in } \Omega\times(0,\infty), \tag{1}$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla\cdot K_1(\phi)\nabla\theta \quad \text{in } \Omega\times(0,\infty), \tag{2}$$

$$c_t = K_2\nabla\cdot c(1-c)\nabla\left(M\phi + \ln\frac{c}{1-c}\right) \quad \text{in } \Omega\times(0,\infty), \tag{3}$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,\infty), \tag{4}$$

$$\phi(0) = \phi_0, \qquad \theta(0) = \theta_0, \qquad c(0) = c_0 \qquad \text{in } \Omega. \tag{5}$$

Here Ω is an open bounded domain of $I\!R^N$, $N = 2, 3$, with smooth boundary $\partial\Omega$. The order parameter (phase-field) ϕ is the state variable characterizing the different phases; the function θ represents the temperature; the concentration $c \in (0,1)$ denotes the fraction of one of the two materials in the mixture. The parameter $\alpha > 0$ is the relaxation scaling; the parameter β is given by $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width, σ the surface tension and $[s]$ the entropy density difference between phases; $C_V > 0$ is the specific heat; the constant $l > 0$ the latent heat; θ_A , θ_B, are the respective melting temperatures of each of the two materials in the alloy; $K_2 > 0$ is the solute diffusivity; M is a constant related to the slopes of solidus and liquidus lines; $K_1 > 0$ denotes the thermal conductivity. Concerning this last physical parameter, throughout this paper we assume the conditions of Laurençot [11]:

(A) K_1 depends only on the order parameter ϕ and is a Lipschitz continuous function. Moreover, there exists $b > 0$ such that

$$0 \le K_1(r) \le b \qquad \text{for all } r \in I\!R.$$

We observe that one technical difficulty with the previous system is that, when K_1 vanishes the equation (2) degenerates and looses its parabolicity.

Note also that equation (3) can be rewritten as

$$\begin{aligned} c_t &= K_2 \left(\Delta c + M \nabla \cdot c(1-c) \nabla \phi \right) \\ &= K_2 \Delta c + K_2 M (1-2c) \nabla c \cdot \nabla \phi + K_2 M c (1-c) \Delta \phi \end{aligned} \tag{6}$$

and that for the pure materials, that is, when $c \equiv 0$ or $c \equiv 1$, the equations reduce to the usual phase field model for pure materials.

We should remark that in recent years the phase-field methodology has achieved considerable importance in modeling and numerically simulating a range of phase transitions and complex growth structures occurring during solidification. Phase-field models have been used to describe phase transitions of pure material due to thermal effects; they lead to nonlinear parabolic systems for the phase-field and the temperature. Such models have been studied, and we refer for instance to [1, 8, 11, 14], where existence and uniqueness of solutions are investigated for various types of nonlinearities. The phase-field governing equations have been derived in a thermodynamically consistent way by Penrose and Fife [15], which recovers the phase-field model employed by Caginalp [1] by linearization of the heat flux. Many papers has been devoted to the mathematical analysis of the Penrose-Fife model, for instance see [4, 5] and references therein. Several phase-field models have also been developed for binary alloys. The first work in this direction was due to Wheeler *et al* [19] and is concerned isothermal solidification. Warren and Boettinger [18] extended this model, while recently Rappaz and Scheid [16] investigated well-posedness of solutions under suitable assumptions for the non-linearities. Another similar model including temperatures changes was developed by Caginalp and Xie [3]. In a such models, the governing equations for the phase-field and the concentration are derived from a free energy functional, and an appropriate balance equation for the temperature is then adjoined to complete the model.

Concerning notations, in this paper we use standard ones, with the remark that we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. Also, for a given fixed $T > 0$, we denote $Q = \Omega \times (0, T)$.

The main result of this paper is the following.

Theorem 1 *Suppose that* $\phi_0 \in H^1(\Omega)$, $\theta_0 \in L^2(\Omega)$, $c_0 \in H^1(\Omega)$, $0 < c_0 < 1$ *a.e. in* $\bar{\Omega}$, *satisfy the compatibility conditions. Under the assumption* **(A)**, *there exist functions* (ϕ, θ, c, J) *satisfying, for any fixed* $T > 0$,

i) $\phi \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \quad \phi_t \in L^2(Q), \quad \phi(0) = \phi_0,$

ii) $\theta \in L^\infty(0,T;L^2(\Omega)), \quad \theta_t \in L^2(0,T;H^1(\Omega)'), \quad \theta(0) = \theta_0,$

iii) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \quad c_t \in L^2(0,T;H^1(\Omega)'),$
$c(0) = c_0, \quad 0 < c < 1 \quad$ *a.e. in* Q,

iv) $J \in L^2(Q, \mathbb{R}^n), \quad J = \nabla(K_1(\phi)\theta) - \theta \nabla K_1(\phi),$

and such that

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \qquad \text{a.e. in } Q, \quad (7)$$

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{a.e. on } \partial\Omega \times (0,T), \quad (8)$$

$$C_\nu \int_0^T \langle \theta_t, \eta \rangle dt + \frac{l}{2}\int_0^T \int_\Omega \phi_t \eta \, dxdt + \int_0^T \int_\Omega J \cdot \nabla\eta \, dxdt = 0, \quad (9)$$

$$\int_0^T \langle c_t, \eta \rangle dt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla\eta \, dxdt + K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\eta \, dxdt = 0, \quad (10)$$

for any $\eta \in L^2(0,T;H^1(\Omega))$.

If, in addition, $K_1 \geq b_0$ *for some* $b_0 > 0$, *then* $\theta \in L^2(0,T;H^1(\Omega))$ *for each* $T > 0$, *and* $J = K_1(\phi)\nabla\theta$.

We remark that because the possibility of degeneracy of the parabolic character of the model, we were able neither to prove uniqueness nor to improve the regularity of the constructed solution.

Finally, the outline of this paper is as follows. In Section 2, we study a family of regularized problems depending on a small accessory positive parameter; this study will be auxiliary to prove the main existence result, and it will be introduced to avoid the above mentioned possibility of degeneracy of parabolicity in equation (3). Section 3 is devoted to proof of Theorem 1; it will be done with the help of suitable estimates derived for the regularized problem together with compactness arguments.

2 A regularized problem

In this section we introduce a regularized problem related to (1)–(5); for it, we prove a result of existence of solutions by using Leray-Schauder fixed point theorem ([6] pp. 189).

Before doing so, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space

$$W_2^{2,1}(Q) = \left\{ w \in L^2(Q) \,/\, D_x w, D_x^2 w \in L^2(Q), w_t \in L^2(Q) \right\}$$

and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^{N+1})$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^{N+1})} \leq C \, \|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [13] pp.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^{N+1})$ be a family of symmetric positive mollifier functions converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_q^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^{N+1})$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

Now, we define K_1^δ for each $\delta \in (0,1)$ by

$$K_1^\delta(r) = K_1(r) + \delta$$

for all $r \in I\!R$. We infer from **(A)** that,

(B) K_1^δ is a Lipschitz continuous function and

$$0 < \delta \leq K_1^\delta(r) \leq b + 1 \qquad \text{for all } r \in I\!R.$$

Now, we are in position to define the following family of regularized problems. For $\delta \in (0,1)$, we consider the system

$$\alpha\epsilon^2\phi_t^\delta - \epsilon^2\Delta\phi^\delta \;=\; \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) + \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q, \quad (11)$$

$$C_V\theta^\delta_t + \frac{l}{2}\phi^\delta_t \;=\; \nabla\cdot K_1^\delta(\rho_\delta(\phi^\delta))\nabla\theta^\delta \qquad \text{in } Q, \tag{12}$$

$$c^\delta_t - K_2\Delta c^\delta \;=\; K_2 M\nabla\cdot c^\delta(1-c^\delta)\nabla\left(\rho_\delta(\phi^\delta)\right) \qquad \text{in } Q, \tag{13}$$

$$\frac{\partial\phi^\delta}{\partial n}=0, \qquad \frac{\partial\theta^\delta}{\partial n}=0, \qquad \frac{\partial c^\delta}{\partial n}=0 \qquad \text{on } \partial\Omega\times(0,T), \tag{14}$$

$$\phi^\delta(0)=\phi^\delta_0, \qquad \theta^\delta(0)=\theta^\delta_0, \qquad c^\delta(0)=c^\delta_0 \qquad \text{in } \Omega. \tag{15}$$

We then have the following existence result.

Proposition 1 *For each $\delta\in(0,1)$, let $(\phi^\delta_0,\theta^\delta_0,c^\delta_0)\in H^1(\Omega)\times H^1(\Omega)\times C^1(\bar\Omega)$ satisfying the compatibility conditions and $0<c^\delta_0<1$ in $\bar\Omega$. Under the assumption* **(B)***, there exist functions $(\phi^\delta,\theta^\delta,c^\delta)$ satisfying, for any fixed $T>0$,*

i) $\phi^\delta\in L^2(0,T;H^2(\Omega)), \quad \phi^\delta_t\in L^2(Q)$,

ii) $\theta^\delta\in L^2(0,T;H^2(\Omega)), \quad \theta^\delta_t\in L^2(Q)$,

iii) $c^\delta\in C^{2,1}(Q), \quad 0<c^\delta<1$,

iv) $(\phi^\delta,\theta^\delta,c^\delta)$ *satisfies (11)-(15) almost everywhere.*

Proof: To ease the notation, in this proof we will omit the superscript δ of the variables $\phi^\delta,\theta^\delta,c^\delta$.

First of all, we consider the following family of operators, indexed by the parameter $0\le\lambda\le1$,

$$\mathcal{T}_\lambda : B\to B,$$

where B is the Banach space

$$B=L^2(Q)\times L^2(Q)\times L^2(Q),$$

and defined as follows: given $(\hat\phi,\hat\theta,\hat c)\in B$, let $\mathcal{T}_\lambda(\hat\phi,\hat\theta,\hat c)=(\phi,\theta,c)$, where (ϕ,θ,c) is obtained by solving the problem

$$\alpha\epsilon^2\phi_t-\epsilon^2\Delta\phi \;-\; \frac{1}{2}(\phi-\phi^3)=\lambda\beta\left(\hat\theta+(\theta_B-\theta_A)\hat c-\theta_B\right) \text{ in } Q, \tag{16}$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla\cdot K_1^\delta(\rho_\delta(\phi))\nabla\theta \qquad \text{in } Q, \tag{17}$$

$$c_t - K_2\Delta c = K_2 M\nabla\cdot c(1-c)\nabla\left(\rho_\delta(\phi)\right) \qquad \text{in } Q, \tag{18}$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{19}$$

$$\phi(0) = \phi_0^\delta, \qquad \theta(0) = \theta_0^\delta, \qquad c(0) = c_0^\delta \qquad \text{in } \Omega. \tag{20}$$

Before we prove that $\mathcal{T}_\lambda$ is well defined, we observe that clearly (ϕ,θ,c) is a solution of (11)-(15) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem.

To verify that $\mathcal{T}_\lambda$ is well defined, observe that since $\hat{\theta}$, $\hat{c}\in L^2(Q)$, we infer from Theorem 2.1 of [8] that there is a unique solution ϕ of equation (16) with $\phi\in W_2^{2,1}(Q)$.

Now, since K_1^δ is a bounded Lipschitz continuous function and $\rho_\delta(\phi)\in C_0^\infty(I\!R^{N+1})$, we have that $K_1^\delta(\rho_\delta(\phi))\in W_r^{1,1}(Q)$ for $1\le r\le\infty$. Thus, since $\phi_t\in L^2(Q)$, we infer from L^p-theory of parabolic equations ([10] pp. 341) that there is a unique solution θ of equation (17) with $\theta\in W_2^{2,1}(Q)$.

We observe that equation (18) is a semilinear parabolic equation with smooth coefficients. Moreover, by looking at the right-hand side of this equation, written in form (6), it has the properties and growth conditions to apply semigroup results of Henry [7], pp.75. Thus, we conclude that there is a unique global classical solution c.

In addition, note that equation (18) does not admit constant solutions, except $c\equiv 0$ and $c\equiv 1$. Thus, by using Maximum Principles together with conditions $0<c_0^\delta<1$ and $\dfrac{\partial c^\delta}{\partial n}=0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\,(x,t)\in Q. \tag{21}$$

Therefore, for each $\lambda\in[0,1]$, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$, let $(\hat{\phi}_n,\hat{\theta}_n,\hat{c}_n)\in B$ strongly converging to $(\hat{\phi},\hat{\theta},\hat{c})\in B$; for each n, let (ϕ_n,θ_n,c_n) the corresponding solution of the problem:

$$\alpha\epsilon^2\phi_{nt} - \epsilon^2\Delta\phi_n - \frac{1}{2}(\phi_n-\phi_n^3) = \lambda\beta\left(\hat{\theta}_n + (\theta_B-\theta_A)\hat{c}_n - \theta_B\right) \text{ in } Q, \tag{22}$$

$$C_V\theta_{nt} + \frac{l}{2}\phi_{nt} = \nabla\cdot K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n \qquad \text{in } Q, \tag{23}$$

$$c_{nt} - K_2\Delta c_n = K_2 M\nabla\cdot c_n(1-c_n)\nabla\left(\rho_\delta(\phi_n)\right) \qquad \text{in } Q, \tag{24}$$

$$\frac{\partial\phi_n}{\partial n} = 0, \qquad \frac{\partial\theta_n}{\partial n} = 0, \qquad \frac{\partial c_n}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{25}$$

$$\phi_n(0) = \phi_0, \qquad \theta_n(0) = \theta_0, \qquad c_n(0) = c_0 \qquad \text{in } \Omega. \tag{26}$$

We show that the sequence (ϕ_n, θ_n, c_n) converges strongly to $(\phi, \theta, c) = \mathcal{T}_\lambda(\hat\phi, \hat\theta, \hat c)$ in B. For that purpose, we will obtain estimates, uniformly with respect to n, for (ϕ_n, θ_n, c_n). We denote by C_i any positive constant independent of n.

We multiply (22) successively by ϕ_n, ϕ_{nt} and $-\Delta\phi_n$, and integrate over $\Omega\times(0,t)$. After integration by parts and the use the Hölder's and Young's inequalities, we obtain the following three estimates:

$$\begin{aligned}
\frac{\alpha\epsilon^2}{2}\int_\Omega |\phi_n|^2 dx &+ \int_0^t\int_\Omega \left(\frac{\epsilon^2}{2}|\nabla\phi_n|^2 + \frac{1}{4}\phi_n^4\right) dxdt \\
&\le C_1 + C_2\int_0^t\int_\Omega \left(|\hat\theta_n|^2 + |\hat c_n|^2 + |\phi_n|^2\right) dxdt, \qquad (27)\\
\frac{\alpha\epsilon^2}{2}\int_0^t\int_\Omega |\phi_{nt}|^2 dxdt &+ \int_\Omega \left(\frac{\epsilon}{2}|\nabla\phi_n|^2 + \frac{\phi_n^4}{8} - \frac{\phi_n^2}{4}\right) dx \\
&\le C_1 + C_2\int_0^t\int_\Omega \left(|\hat\theta_n|^2 + |\hat c_n|^2\right) dxdt, \qquad (28)\\
\frac{\alpha\epsilon^2}{2}\int_\Omega |\nabla\phi_n|^2 dx &+ \frac{\epsilon^2}{2}\int_0^t\int_\Omega |\Delta\phi_n|^2 dxdt \\
&\le C_1 + C_2\int_0^t\int_\Omega \left(|\nabla\phi_n|^2 + |\hat\theta_n|^2 + |\hat c_n|^2\right) dxdt. \qquad (29)
\end{aligned}$$

By multiplying (28) by $\alpha\epsilon^2$ and adding the result to (27), we find

$$\begin{aligned}
\int_\Omega \Big(|\phi_n|^2 &+ |\nabla\phi_n|^2 + \phi_n^4\Big) dx \\
&\le C_1 + C_2\int_0^t\int_\Omega \left(|\hat\theta_n|^2 + |\hat c_n|^2 + |\phi_n|^2\right) dxdt. \qquad (30)
\end{aligned}$$

Since $\|\hat{\theta}_n\|_{L^2(Q)}$ and $\|\hat{c}_n\|_{L^2(Q)}$ are bounded independent of n, we infer from (30) and Gronwall's Lemma that

$$\|\phi_n\|_{L^\infty(0,T;H^1(\Omega))} \leq C_1. \tag{31}$$

Then, thanks to (27)-(29) we have

$$\|\phi_n\|_{L^2(0,T;H^2(\Omega))} + \|\phi_{nt}\|_{L^2(Q)} \leq C_1. \tag{32}$$

Now, by multiplying (23) by θ_n, one obtains in a similar way as above that

$$\int_\Omega |\theta_n|^2 dx + \int_0^t \int_\Omega |\nabla \theta_n|^2 dx dt \leq C_1 + C_2 \int_0^t \int_\Omega \left(|\phi_{nt}|^2 + |\theta_n|^2 \right) dx dt. \tag{33}$$

Thus, with the help of (32) and Gronwall's Lemma, we infer that

$$\|\theta_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1. \tag{34}$$

Hence, it follows from (33) that

$$\|\theta_n\|_{L^2(0,T;H^1(\Omega))} \leq C_2. \tag{35}$$

Now, we take scalar product in $L^2(\Omega)$ of (23) with $\eta \in H^1(\Omega)$. By integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_{nt}\|_{H^1(\Omega)'} \leq C_1 \left(\|\nabla \theta_n\|_{L^2(\Omega)} + \|\phi_{nt}\|_{L^2(\Omega)} \right).$$

Thus, we infer from (32) and (35) that

$$\|\theta_{nt}\|_{L^2(0,T;H^1(\Omega)')} \leq C_1. \tag{36}$$

Next, by multiplying (24) by c_n, with the help of (21), as above we conclude that

$$\int_\Omega |c_n|^2 dx + \int_0^t \int_\Omega |\nabla c_n|^2 dx dt \leq C_1 + C_2 \int_0^t \int_\Omega |\nabla \phi_n|^2 dx dt.$$

Hence, from (32) we have,

$$\|c_n\|_{L^2(0,T;H^1(\Omega))} + \|c_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1. \tag{37}$$

In order to get an estimate for c_{nt} in $L^2(0,T;H^1(\Omega)')$, we return to equation (24) and use similar techniques as above to obtain

$$\|c_{nt}\|_{L^2(0,T;H^1(\Omega)')} \le C_1. \tag{38}$$

We now infer from (31) and (32) that the sequence (ϕ_n) is bounded in

$$W_1 = \left\{ v \in L^2(0,T;H^2(\Omega)),\ v_t \in L^2(0,T;L^2(\Omega)) \right\}$$

and in

$$W_2 = \left\{ v \in L^\infty(0,T;H^1(\Omega)),\ v_t \in L^2(0,T;L^2(\Omega)). \right\},$$

We also infer from (34)-(37) that the sequences (θ_n) and (c_n) are bounded in

$$W_3 = \left\{ v \in L^2(0,T;H^1(\Omega)),\ v_t \in L^2(0,T;H^1(\Omega)') \right\}$$

and in

$$W_4 = \left\{ v \in L^\infty(0,T;L^2(\Omega)),\ v_t \in L^2(0,T;H^1(\Omega)') \right\}.$$

Since W_1 is compactly embedded in $L^2(0,T;H^1(\Omega))$, W_2 in $C([0,T];L^2(\Omega))$, W_3 in $L^2(0,T;L^2(\Omega))$ and W_4 in $C([0,T];H^1(\Omega)')$ ([17] Cor.4), it follows that there exist

$$\begin{array}{rcl} \phi & \in & L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \text{ with } \phi_t \in L^2(Q), \\ \theta & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \text{ with } \theta_t \in L^2(0,T;H^1(\Omega)'), \\ c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \text{ with } c_t \in L^2(0,T;H^1(\Omega)'), \end{array}$$

and a subsequence of (ϕ_n, θ_n, c_n) (which we still denote by (ϕ_n, θ_n, c_n)), such that, as $n \to +\infty$,

$$\begin{array}{rclll} \phi_n & \to & \phi & \text{in} & L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)) \text{ strongly}, \\ \phi_n & \rightharpoonup & \phi & \text{in} & L^2(0,T;H^2(\Omega)) \text{ weakly}, \\ \theta_n & \to & \theta & \text{in} & L^2(Q) \cap C([0,T];H^1(\Omega)') \text{ strongly}, \\ \theta_n & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega)) \text{ weakly}, \\ c_n & \to & c & \text{in} & L^2(Q) \cap C([0,T];H^1(\Omega)') \text{ strongly}, \\ c_n & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega)) \text{ weakly}, \end{array} \tag{39}$$

It now remains to pass to the limit as n tends to ∞ in (22)-(26). Since the embedding of $W_2^{2,1}(Q)$ into $L^9(Q)$ is compact ([12] pp.15), and (ϕ_n) is bounded in $W_2^{2,1}(Q)$, we infer that ϕ_n^3 converges to ϕ^3 in $L^2(Q)$. We then pass to the limit in (22) and get

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \qquad \text{a.e. in } Q.$$

Since K_1^δ is bounded Lipschitz continuous function and $\rho_\delta(\phi_n)$ converges to $\rho_\delta(\phi)$ in $L^2(Q)$, we have that $K_1^\delta(\rho_\delta(\phi_n))$ converges to $K_1^\delta(\rho_\delta(\phi))$ in $L^p(Q)$ for any $p \in [1,\infty)$. This fact and (39) yield the weak convergence of $K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n$ to $K_1^\delta(\rho_\delta(\phi))\nabla\theta$ in $L^{3/2}(Q)$. Now, by multiplying (23) by $\eta \in L^2(0,T;H^1(\Omega))$ and integrating over $\Omega \times (0,T)$, after integration by parts, we obtain

$$C_V \int_0^T \int_\Omega \theta_{nt}\eta \, dxdt + \frac{l}{2}\int_0^T \int_\Omega \phi_{nt}\eta \, dxdt + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n \cdot \nabla\eta \, dxdt = 0.$$

Then, we may pass to the limit and find that

$$C_V \int_0^T \langle \theta_t, \eta\rangle dt + \frac{l}{2}\int_0^T \int_\Omega \phi_t\eta \, dxdt + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi))\nabla\theta \cdot \nabla\eta \, dxdt = 0$$

holds for any $\eta \in L^2(0,T;H^1(\Omega))$. This implies that

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla \cdot K_1^\delta(\rho_\delta(\phi))\nabla\theta \qquad \text{in } \mathcal{D}'(Q). \tag{40}$$

Now, by using the L^p-theory of parabolic equations, it is easy to conclude that (40) holds almost everywhere in Q.

It remains to pass to the limit in (24). We infer from (39) that $\nabla\rho_\delta(\phi_n)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q)$ and since $\|c_n\|_{L^\infty(Q)}$ is bounded, it follows that $c_n(1-c_n)$ converges to $c(1-c)$ in $L^p(Q)$ for any $p \in [1,\infty)$. Similarly, we may pass to the limit in (24) to obtain

$$c_t - K_2\Delta c = K_2 M \nabla \cdot c(1-c)\nabla\left(\rho_\delta(\phi)\right) \qquad \text{in } Q.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0 \le \lambda \le 1$. At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1 \times W_3 \times W_3$, and the embedding of this space in B is compact. We conclude that $\mathcal{T}_\lambda$ is a compact operator for each $\lambda \in [0,1]$.

To prove that for $(\hat\phi, \hat\theta, \hat c)$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and (ϕ_i, θ_i, c_i) $(i = 1,2)$ be the corresponding solutions of (16)-(20). We observe that $\phi = \phi_1 - \phi_2$, $\theta = \theta_1 - \theta_2$ and $c = c_1 - c_2$ satisfy the following problem:

$$\begin{aligned} \alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi &= \frac{1}{2}\phi(1-(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)) \\ &+ (\lambda_1 - \lambda_2)\beta\left(\hat\theta + (\theta_B - \theta_A)\hat c - \theta_B\right) \qquad \text{in } Q, \end{aligned} \tag{41}$$

$$\begin{aligned} C_V\theta_t + \frac{l}{2}\phi_t &= \nabla\cdot K_1^\delta(\rho_\delta(\phi_1))\nabla\theta \\ &+ \nabla\cdot\left[K_1^\delta(\rho_\delta(\phi_1)) - K_1^\delta(\rho_\delta(\phi_2))\right]\nabla\theta_2 \qquad \text{in } Q, \end{aligned} \tag{42}$$

$$\begin{aligned} c_t - K_2\Delta c &= K_2 M\nabla\cdot c_1(1-c_1)\left(\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)\right) \\ &+ K_2 M\nabla\cdot c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2) \qquad \text{in } Q, \end{aligned} \tag{43}$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{44}$$

$$\phi(0) = 0, \qquad \theta(0) = 0, \qquad c(0) = 0 \qquad \text{in } \Omega. \tag{45}$$

We remark that $d := \phi_1^2 + \phi_1\phi_2 + \phi_2^2 \geq 0$. Now, multiply equation (41) by ϕ and integrate over Q; after integration by parts and the use of Hölder's and Young's inequalities we, obtain

$$\begin{aligned} \int_\Omega |\phi|^2 dx + \int_0^t\int_\Omega |\nabla\phi|^2 dxdt &\leq C_1\int_0^t\int_\Omega |\phi|^2 dxdt \\ &+ C_2|\lambda_1-\lambda_2|^2\int_0^t\int_\Omega\left(|\hat{\theta}|^2 + |\hat{c}|^2\right)dxdt. \end{aligned}$$

By applying Gronwall's Lemma, we arrive at

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{46}$$

Now, multiply (41) by ϕ_t and use Hölder's inequality to obtain

$$\begin{aligned} \alpha\epsilon^2\int_0^t\int_\Omega |\phi_t|^2 dxdt &+ \frac{\epsilon^2}{2}\int_\Omega |\nabla\phi|^2 dx \\ &\leq C_1\int_0^t\int_\Omega |\phi|^2 dxdt + \frac{\alpha\epsilon^2}{2}\int_0^t\int_\Omega |\phi_t|^2 dxdt \\ &+ C_2\left(\int_0^t\int_\Omega |\phi|^{10/3} dxdt\right)^{3/5}\left(\int_0^t\int_\Omega |d|^5 dxdt\right)^{2/5} \\ &+ C_2|\lambda_1-\lambda_2|^2\int_0^t\int_\Omega\left(|\hat{\theta}|^2 + |\hat{c}|^2\right)dxdt. \end{aligned}$$

Since $W_2^{2,1}(Q) \hookrightarrow L^{10}(Q)$, the following interpolation inequality holds

$$\|\phi\|^2_{L^{10/3}(Q)} \leq \eta\,\|\phi\|^2_{W_2^{2,1}(Q)} + \tilde{C}\,\|\phi\|^2_{L^2(Q)} \text{ for all } \eta > 0.$$

Moreover, since $\|d\|_{L^5(Q)} \leq C$, with C depending on $\|\phi_1\|_{L^{10}(Q)}$ and $\|\phi_2\|_{L^{10}(Q)}$, by rearranging the terms in the last inequality, we obtain

$$\begin{aligned}\int_0^t \int_\Omega |\phi_t|^2 dxdt \;&+\; \int_\Omega |\nabla\phi|^2 dx \\ &\leq\; C_1 \int_0^t \int_\Omega |\phi|^2 dxdt + C_2\,\eta\,\|\phi\|^2_{W_2^{2,1}(Q)} \\ &+\; C_3|\lambda_1-\lambda_2|^2 \int_0^t \int_\Omega \left(|\hat\theta|^2 + |\hat c|^2\right) dxdt.\end{aligned} \tag{47}$$

By multiplying (41) by $-\Delta\phi$, we infer in a similar way that

$$\begin{aligned}\int_\Omega |\nabla\phi|^2 dx \;&+\; \int_0^t \int_\Omega |\Delta\phi|^2 dxdt \\ &\leq\; C_1 \int_0^t \int_\Omega \left(|\phi|^2 + |\nabla\phi|^2\right) dxdt + C_2\,\eta\,\|\phi\|^2_{W_2^{2,1}(Q)} \\ &+\; C_3|\lambda_1-\lambda_2|^2 \int_0^t \int_\Omega \left(|\hat\theta|^2 + |\hat c|^2\right) dxdt.\end{aligned} \tag{48}$$

By taking $\eta > 0$ small enough and considering (46) we conclude from (47) and (48) that

$$\|\phi\|^2_{W_2^{2,1}(Q)} + \|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{49}$$

By multiplying (42) by θ, integrating over Ω and using Hölder's inequality and **(B)**, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\theta|^2 dx + \delta\int_\Omega |\nabla\theta|^2 dxdt \;&\leq\; C_1 \int_\Omega \left(|\phi_t|^2 + |\theta|^2 + |\phi|\,|\nabla\theta_2|\,|\nabla\theta|\right) dxdt \\ &\leq\; C_1 \int_\Omega \left(|\phi_t|^2 + |\theta|^2\right) dxdt + \frac{\delta}{2}\int_\Omega |\nabla\theta|^2 \\ &+\; C_3\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))}\|\theta_2\|^2_{H^2(\Omega)}.\end{aligned}$$

Therefore, integration with respect to t and the use of Gronwall's Lemma and (49) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{50}$$

Now, we multiply (43) by c and integrate over $\Omega\times(0,t)$. By integration by parts and we use Hölder's and Young's inequalities and (21), we obtain

$$\begin{aligned}\int_\Omega |c|^2 dx + \int_0^t \int_\Omega |\nabla c|^2 dxdt \;&\leq\; C_1 \int_0^t \int_\Omega \left(|\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)|^2 + |c|^2\right) dxdt \\ &\leq\; C_1 \int_0^t \int_\Omega \left(|\nabla\phi|^2 + |c|^2\right) dxdt.\end{aligned}$$

By applying Gronwall's Lemma and using (49), we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{51}$$

Therefore, it follows from (49)-(51) that $\mathcal{T}_\lambda$ is uniformly continuous with respect λ on bounded sets of B.

Now we have to estimate the set of all fixed points of $\mathcal{T}_\lambda$, let $(\phi, \theta, c) \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi \;-\; \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \text{ in } Q, \tag{52}$$

$$C_V\theta_t + \frac{l}{2}\phi_t \;=\; \nabla\cdot K_1^\delta(\rho_\delta(\phi))\nabla\theta \qquad \text{in } Q, \tag{53}$$

$$c_t - K_2\Delta c \;=\; K_2 M\nabla\cdot c(1-c)\nabla\left(\rho_\delta(\phi)\right) \qquad \text{in } Q, \tag{54}$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{55}$$

$$\phi(0) = \phi_0^\delta, \qquad \theta(0) = \theta_0^\delta, \qquad c(0) = c_0^\delta \qquad \text{in } \Omega. \tag{56}$$

For this, we multiply the first equation (52) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over Ω. After integration by parts, using Hölder's and Young's inequalities, we obtain, respectively

$$\begin{aligned}
\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_\Omega |\phi|^2dx \;&+\; \int_\Omega\left(\epsilon^2|\nabla\phi|^2 + \frac{1}{4}\phi^4\right)dx \\
&\leq\; C_1 + C_2\int_\Omega\left(|\theta|^2 + |c|^2 + |\phi|^2\right)dx, &(57)\\
\frac{\alpha\epsilon^2}{2}\int_\Omega |\phi_t|^2dx \;&+\; \frac{d}{dt}\int_\Omega\left(\frac{\epsilon^2}{2}|\nabla\phi|^2 + \frac{1}{8}\phi^4 - \frac{1}{4}|\phi|^2\right)dx \\
&\leq\; C_1 + C_2\int_\Omega\left(|\theta|^2 + |c|^2\right)dx, &(58)\\
\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_\Omega |\nabla\phi|^2dx \;&+\; \int_\Omega \frac{\epsilon^2}{2}|\Delta\phi|^2dx \\
&\leq\; C_1 + C_2\int_\Omega\left(|\theta|^2 + |c|^2 + |\nabla\phi|^2\right)dx. &(59)
\end{aligned}$$

By multiplying (53) by θ and (54) by c, arguments similar to the previous ones lead to the following estimates

$$\frac{d}{dt}\int_\Omega \frac{C_\nu}{2}|\theta|^2 dx + \delta \int_\Omega |\nabla\theta|^2 dx \leq \frac{\alpha^2\epsilon^4}{4}\int_\Omega |\phi_t|^2 dx + C_1 \int_\Omega |\theta|^2 dx, \quad (60)$$

$$\frac{d}{dt}\int_\Omega |c|^2 dx + K_2 \int_\Omega |\nabla c|^2 dx \leq C_1 \int_\Omega |\nabla\phi|^2 dx, \quad (61)$$

where (21) was used to obtain the last inequality.

Now, multiply (58) by $\alpha\epsilon^2$ and add the result to (57), (59)- (61), to obtain

$$\frac{d}{dt}\int_\Omega \left(\frac{\alpha\epsilon^2}{4}|\phi|^2 + \left(\frac{\alpha\epsilon^2}{2} + \frac{\alpha\epsilon^4}{2}\right)|\nabla\phi|^2 + \frac{\alpha\epsilon^2}{8}\phi^4 + \frac{C_\nu}{2}|\theta|^2 + |c|^2\right) dx$$
$$\int_\Omega \left(\epsilon^2|\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{\alpha^2\epsilon^4}{4}|\phi_t|^2 + \frac{\epsilon^2}{2}|\Delta\phi|^2 + \delta|\nabla\theta|^2 + K_2|\nabla c|^2\right) dx$$
$$\leq C_1 + C_2 \int_\Omega \left(|\theta|^2 + |c|^2 + |\phi|^2 + |\nabla\phi|^2\right) dx. \quad (62)$$

Hence, the integration of (62) with respect t and the use Gronwall's Lemma give us

$$||\phi||_{L^\infty(0,T;H^1(\Omega))} + ||\theta||_{L^\infty(0,T;L^2(\Omega))} + ||c||_{L^\infty(0,T;L^2(\Omega))} \leq C_1,$$

where C_1 is independent of λ.

Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0, 1]$.

Finally, for $\lambda = 0$, we can reason as in the proof that $\mathcal{T}_\lambda$ is well defined to conclude that problem (16)-(20) has a unique solution. Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $(\phi, \theta, c) \in B \cap W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times C^{2,1}(Q)$ of the operator $\mathcal{T}_1$, i.e., $(\phi, \theta, c) = \mathcal{T}_1(\phi, \theta, c)$. This corresponds to a solution of problem (11)-(15) and the proof of Proposition 1 thus complete. ∎

3 Proof of Theorem 1

To prove Theorem 1, we start by taking the initial condition in the previous regularized problem as follows. For a sequence $\delta \to 0+$, we choose $\phi_0^\delta = \phi_0$

and pick two corresponding sequences $\theta_0^\delta \in H^1(\Omega)$ and $c_0^\delta \in C^1(\bar{\Omega})$ satisfying $0 < c_0^\delta < 1$, $\theta_0^\delta \to \theta_0$ in $L^2(\Omega)$ and $c_0^\delta \to c_0$ in $H^1(\Omega)$.

From Proposition 1, we know that there exist a sequence $(\phi^\delta, \theta^\delta, c^\delta)$ of corresponding solutions of problem (11)-(15). For such solutions, we will derive bounds, uniform with respect to δ; then, we will use compactness arguments to pass to the limit and establish the desired result.

Lemma 1 *There exists a constant* C_1 *such that, for any* $\delta \in (0,1)$

$$\|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\phi_t^\delta\|_{L^2(Q)} \leq C_1 \tag{63}$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dxdt \leq C_1 \tag{64}$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C_1. \tag{65}$$

Proof: From inequality (62), it follows estimates (63), (65) and also

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1. \tag{66}$$

By multiplying (12) by θ^δ and integrating over Q, we obtain

$$\int_\Omega |\theta^\delta|^2 dx + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dxdt \leq C_1 \int_0^T \int_\Omega \left(|\phi_t^\delta|^2 + |\theta^\delta|^2\right) dxdt.$$

In view of (63) and (66), this gives estimate (64). ∎

Lemma 2 *There exists a constant* C_1 *such that, for any* $\delta \in (0,1)$

$$\|\theta_t^\delta\|_{L^2(0,T;H^1(\Omega)')} \leq C_1 \tag{67}$$

$$\|c_t^\delta\|_{L^2(0,T;H^1(\Omega)')} \leq C_1. \tag{68}$$

Proof: We take the scalar product in $L^2(\Omega)$ of (12) with $\eta \in H^1(\Omega)$. By using Hölder's inequality and **(B)**, we find

$$C_V\|\theta_t^\delta\|_{H^1(\Omega)'} \leq \left((b+1)\int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dx\right)^{1/2} + \frac{l}{2}\|\phi_t^\delta\|_{L^2(\Omega)}.$$

Then, (67) follows from (63)-(64). Estimate (68) can be similarly obtained by using (63) and (65). ∎

We now infer from Lemma 1 and Lemma 2 that there exist

$$\begin{array}{rcl} \phi & \in & L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \text{ with } \phi_t \in L^2(Q), \\ \theta & \in & L^\infty(0,T;L^2(\Omega)) \text{ with } \theta_t \in L^2(0,T;H^1(\Omega)'), \\ c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \text{ with } c_t \in L^2(0,T;H^1(\Omega)'), \\ J & \in & L^2(Q) \end{array}$$

and a subsequence that to ease the notation we still denote $(\phi^\delta, \theta^\delta, c^\delta)$, such that as $\delta \to 0+$, satisfies

$$\begin{array}{rcll} \phi^\delta & \to & \phi & \text{in } L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)) \text{ strongly}, \\ \phi^\delta_t & \rightharpoonup & \phi_t & \text{in } L^2(Q) \text{ weakly}, \\ \theta^\delta & \to & \theta & \text{in } C([0,T];H^1(\Omega)') \text{ strongly}, \\ \theta^\delta & \rightharpoonup & \theta & \text{in } L^2(Q) \text{ weakly}, \\ c^\delta & \to & c & \text{in } L^2(Q) \cap C([0,T];H^1(\Omega)') \text{ strongly}, \\ c^\delta & \rightharpoonup & c & \text{in } L^2(0,T;H^1(\Omega)) \text{ weakly}, \\ K_1^\delta(\rho_\delta(\phi^\delta))\nabla\theta^\delta & \rightharpoonup & J & \text{in } L^2(Q) \text{ weakly}. \end{array} \tag{69}$$

It now remains to identify J in terms of ϕ and θ and pass to the limit as δ approaches zero in (11)-(15).

It follows from (69) that we may pass to the limit in (11) and find that (7) holds almost everywhere.

Since K_1 is a Lipschitz continuous function and $\rho_\delta(\phi^\delta)$ converges to ϕ in $L^2(0,T;H^1(\Omega))$, we have (see, e.g.[9] Thm 16.7)

$$K_1(\rho_\delta(\phi^\delta)) \to K_1(\phi) \text{ in } L^2(0,T;H^1(\Omega)) \text{ strongly}. \tag{70}$$

From (69)-(70), we conclude that

$$\begin{array}{rcll} \theta^\delta \nabla K_1(\rho_\delta(\phi^\delta)) & \rightharpoonup & \theta\nabla K_1(\phi) & \text{in } L^1(Q) \text{ weakly}, \\ K_1(\rho_\delta(\phi^\delta))\theta^\delta & \rightharpoonup & K_1(\phi)\theta^\delta & \text{in } L^1(Q) \text{ weakly}. \end{array} \tag{71}$$

Also, since $K_1(\rho_\delta(\phi^\delta)) \in L^\infty(0,T;H^1(\Omega))$ and $\theta^\delta \in L^2(0,T;H^1(\Omega))$, we have

$$K_1(\rho_\delta(\phi^\delta))\theta^\delta \in L^2(0,T;W^{1,p}(\Omega)) \quad \text{for } p = \min\left\{2, \frac{N}{N-1}\right\}$$

and

$$\nabla\left(K_1(\rho_\delta(\phi^\delta))\theta^\delta\right) = K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta + \theta^\delta\nabla K_1(\rho_\delta(\phi^\delta)).$$

It then follows from (71) that

$$K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta \to \nabla\left(K_1(\phi)\theta\right) - \theta\nabla K_1(\phi) \text{ in } \mathcal{D}'(Q). \tag{72}$$

Since K_1 is nonnegative, the definition of K_1^δ and (64) yield that $\|\delta^{\frac{1}{2}}\nabla\theta^\delta\|_{L^2(Q)} \le C$, with C independent of δ. Thus,

$$\delta\nabla\theta^\delta \to 0 \text{ in } L^2(Q). \tag{73}$$

From (69), (72) and (73), we conclude that

$$J = \nabla\left(K_1(\phi)\theta\right) - \theta\nabla K_1(\phi).$$

Moreover, we may pass to the limit in a weak sense in (12) and obtain (9).

In order to pass to the limit in (13), we take scalar product in $L^2(\Omega)$ of it with $\eta \in L^2(0,T;H^1(\Omega))$, to obtain

$$\begin{aligned}\int_0^T\int_\Omega c_t^\delta \eta \, dxdt &+ K_2\int_0^T\int_\Omega \nabla c^\delta\cdot\nabla\eta\,dxdt \\ &+ K_2 M\int_0^T\int_\Omega c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta))\cdot\nabla\eta\,dxdt = 0.\end{aligned}$$

Then, from (69), we have that

$$\int_0^T\langle c_t,\eta\rangle dxdt + K_2\int_0^T\int_\Omega\nabla c\cdot\nabla\eta\,dxdt + K_2M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\eta\,dxdt = 0$$

holds for any $\eta \in L^2(0,T;H^1(\Omega))$.

Moreover, since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$, we have that $0 < c < 1$ a.e. in Q.

Finally, it follows from (69) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$.

The proof of Theorem 1 is then complete. ∎

Remark. From the L^p-theory of parabolic equations, it is easy to conclude that $c \in W^{2,1}_{3/2}(Q)$, and therefore the equation for c holds almost everywhere.

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