Existence and uniqueness of strong solutions of nonhomogeneous incompressible asymmetric fluids in unbounded domains

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Abstract

We consider and initial boundary value problem for a system of equations describing nonstationary flows of nonhomogeneous incompressible asymmetric fluids in unbounded domains. Under conditions similar to the ones for the ones for the usual Navier-Stokes equations, we prove the existence and uniqueness of strong solutions.

1 Introduction

Let Ω be a bounded or unbounded domain in $I\!R^3$, $T>0$ and $Q_T = \Omega\times[0,T]$. The equations that describe the motion of nonhomogeneous asymmetric fluids are given by

$$\begin{cases} \rho\dfrac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mu+\mu_r)\Delta\mathbf{u} + \nabla p = 2\mu_r \text{ rot } \mathbf{w} + \rho\mathbf{f}, \\ \text{div } \mathbf{u} = 0\,, \\ \rho\dfrac{\partial \mathbf{w}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w} \\ \qquad +4\mu_r\mathbf{w} = 2\mu_r \text{ rot } \mathbf{u} + \rho\mathbf{g}\,, \\ \dfrac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho = 0\,, \end{cases} \tag{1}$$

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together with the following boundary and initial conditions

$$\begin{cases} \mathbf{u} = 0 \quad \text{on} \quad \sum_T = \partial\Omega \times (0,T)\ , \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in} \quad \Omega\ , \\ \mathbf{w} = 0 \quad \text{on} \quad \sum_T = \partial\Omega \times (0,T)\ , \\ \mathbf{w}(x,0) = \mathbf{w}_0(x) \quad \text{in} \quad \Omega\ , \\ \rho(x,0) = \rho_0(x) \quad \text{in} \quad \Omega\ . \end{cases} \tag{2}$$

The functions $\mathbf{u} = (u_1, u_2, u_3), \mathbf{w} = (w_1, w_2, w_3), p$ and ρ denote the velocity vector, the angular velocity vector of rotation of particles, the pressure and the density of the fluid, respectively. The functions $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ denote external sources of linear and angular momentum, respectively. The positive constants μ, μ_r, c_0, c_a and c_d are viscosities. We consider $c_0 + c_d > c_a$.

For the derivation and discussion of equations (1.1)- (1.2) which represent conservation laws, see [**?**], [**?**].

Existence of solutions to the system (1.1)-(1.2) in a bounded domain are considered in Lukaszewicz [**?**] (see, also [**?**]), Boldrini and Rojas-Medar [**?**] and Conca, Gormaz, Ortega-Torres and Rojas-Medar [**?**], the two last works obtain also the uniqueness of solutions.

The local existence of weak solutions for (1.1)-(1.2) was established by Lukaszewicz [**?**] under certain assumptions by using linearization and almost fixed point theorem.

Using the spectral semi-Galerkin method, Boldrini and Rojas-Medar [**?**] proved the existence and uniqueness of strong solutions (local and global). Analogous results were obtained in [**?**], in this work an iterative procedure was used.

However, no study of existence and uniqueness has been considered for system (1.1)-(1.2) in unbounded domains. We observe that this model includes as a particular case of the nonhomogeneous Navier-Stokes equations, which has been studied early by some authors, for example, Kazhikov [**?**] (see, also [**?**], [**?**]), Kim [**?**], for weak solutions, Ladyzhenskaya and Solonnikov [**?**], Okamoto [**?**], Salvi [**?**], Boldrini and Rojas-Medar [**?**] for existence and uniqueness of strong solutions. The above authors work in bounded domains. For exterior domains see the works of Padula [**?**], [**?**] and in unbounded domains Fernández-Caras and Guillén [**?**] and Itoh and Tani [**?**] (see, also Lions [**?**]).

This paper is organized as follows: in Section 2 we state some preliminary results, we also state the result of existence and uniqueness of strong solution. In Section 3, we study the linear problems associated with (1.1)-(1.2). Finally, in Section 4 we prove our result.

2 Preliminaries

We use the classical notations and results of the Sobolev spaces. For $k = 0, 1, 2, ...$ and $1 \leq q \leq \infty$,

$$W_q^k(\Omega) = \{\mathbf{u} \in L_q(\Omega) \ / \ \sum_{|\alpha| \leq k} \|D_x^\alpha \mathbf{u}\| < \infty\}$$

$$W_q^{2,1}(Q_T) = \{\mathbf{u} \in L_q(Q_T) \ / \ \|\mathbf{u}\|_{W_q^{2,1}(Q_T)} = \|\mathbf{u}_t\|_{L_q(Q_T)} + \sum_{|\alpha| \leq 2} \|D_x^\alpha \mathbf{u}\|_{L_q(Q_T)} < \infty\},$$

where $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}$ and $|\alpha| =_i \alpha_i$.

It is know that the values of the function from $W_q^{2,1}(Q_T)$ on the hyperplane $t =$ const. belong for $\forall\ t \in [0, T]$ to the Slobodetskii-Besov space $W_q^{2-\frac{2}{q}}(\Omega)$ and depend continuously on t in the norm of $W_q^{2-\frac{2}{q}}(\Omega)$, defined by

$$\|\mathbf{u}\|_{W_q^{2-\frac{2}{q}}(\Omega)} = \left(\sum_{|\alpha| \leq 1} \|D_x^\alpha \mathbf{u}\|_{L_q(\Omega)}^q + \sum_{|\alpha| = 1} \int_\Omega \int_\Omega \frac{|\ D_x^\alpha \mathbf{u}(x) - D_y^\alpha \mathbf{u}(y)\ |^q}{|\ x - y\ |^{1+q}} dx dy \right)^{\frac{1}{q}}.$$

Moreover, we have the Solonnikov'inequality

$$\|\mathbf{u}(\cdot, t)\|_{W_q^{2-\frac{2}{q}}(\Omega)} \leq \|\mathbf{u}(\cdot, 0)\|_{W_q^{2-\frac{2}{q}}(\Omega)} + \hat{c} \|\mathbf{u}\|_{W_q^{2,1}(Q_T)},$$

where the constant $\hat{c}$ does not depend on t.

For more details of the Solobodetskii-Besov space see [**?**]

Let $q > 3$. assume that

$$\rho_0(x) \in C^0(\overline{\Omega}), \nabla \rho_0(x) \in W_q^1(\Omega),$$

$$0 < m \leq \rho_0(x) \leq M < \infty,$$

$$\mathbf{u}_0(x) \in W_q^{2-\frac{2}{q}}(\Omega), \mathbf{u}_0|_S = 0,\ \text{div}\ \mathbf{u}_0 = 0,$$

$$\mathbf{w}_0(x) \in W_q^{2-\frac{2}{q}}(\Omega), \mathbf{w}_0|_S = 0,$$

$$\mathbf{f}, \mathbf{g} \in L_q(Q_T).$$

Then there exists $T_1 \in (0,T]$ such that problem (1.1)-(1.2) has a unique solution $(\rho, \mathbf{u}, \mathbf{w}, p)$ which satisfies

$$\rho \in C^0(\overline{Q_{T_1}}), \nabla\rho \in C^0([0,T_1]; W_q^1(\Omega)),$$

$$0 < m \leq \rho(x,t) \leq M < \infty,$$
$$\mathbf{u}(x,t) \in W_q^{2,1}(Q_{T_1}),$$
$$\nabla p \in L_q(Q_{T_1})$$
$$\mathbf{w} \in W_q^{2,1}(Q_{T_1}).$$

In the rest of work we assume that $q > 3$.

3 Linear problems

In this section, we study some linear problems associated with (1.1)-(1.2). The first Lemma is proved in Itoh and Tani [**?**].

Let $\rho \in C^{\alpha,\beta}(\overline{Q_T}), \alpha, \beta \in (0,1)$ such that $0 < m \leq \rho_0(x) \leq M < \infty$. Then for any $F \in L_q(Q_T)$ and $\mathbf{u}_0(x) \in W_q^{2-\frac{2}{q}}(\Omega)$ with $\mathbf{u}_0|_{\sum_T} = 0$ and div $\mathbf{u}_0 = 0$, problem

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} - (\mu+\mu_r)\triangle\mathbf{u} + \nabla p &= F, \\ \text{div } \mathbf{u} &= 0, \\ \mathbf{u}|_{\sum_T} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0(x) \end{aligned}$$

has a unique solution $\mathbf{u} \in W_q^{2,1}(Q_T)$, satisfying

$$\|\mathbf{u}\|_{W_q^{2,1}(Q_T)} + \|\nabla p\|_{L_q(Q_T)} \leq K_1(\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})}, T)(\|\mathbf{u}_0\|_{W_q^{2-\frac{2}{q}}(\Omega)} + \|F\|_{L_q(Q_T)}),$$

where K_1 is an increasing function of $\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})}$ and T, depending on m and M.

The next Lemma is proved in [**?**].

Let $\rho \in C^{\alpha,\beta}(Q_T)$, $\alpha, \beta \in (0,1)$, such that $0 < m \leq \rho(x,t) \leq M$. Then for any function $G \in L_q(Q_T)$, $q > 3$ and $\mathbf{w}_0(x) \in W_q^{2-2/q}(\Omega)$ with $\mathbf{w}_0|_{\sum_T} = 0$, problem

$$\rho\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} - \gamma\nabla \mathrm{div}\,\mathbf{w} + 4\mu_r\mathbf{w} = G(x,t) \text{ em } \Omega,$$
$$\mathbf{w} \equiv 0 \text{ sobre } \Sigma_T\ ,$$
$$\mathbf{w}(x,0) = \mathbf{w}_0(x) \text{ em } \Omega,$$

has a unique solution $\mathbf{w} \in W_q^{2,1}(Q_T)$, satisfying

$$\begin{aligned} &\left(\|\mathbf{w}\|_{W_q^{2,1}(Q_T)}\right)^q + \sup_{\tau\leq t}\left(\|\mathbf{w}(x,\tau)\|_{W_q^{2-2/q}(\Omega)}\right)^q \\ \leq\ & C\frac{M^{q+1}(t)}{m^{q+1}(t)}\left(1 + m^{-1}(t)\sup_{\tau\leq t}[\rho(x,\tau)]_\Omega^{(\alpha)}\right)^{\frac{2q}{\alpha}}\left[\tilde{G}^q(t) + \|\mathbf{w}\|^q_{L_a(Q_t)}\right], \end{aligned} \tag{3}$$

where

$$\begin{aligned} M(t) &= \max(1, \max_{Q_t}\rho), \\ m(t) &= \min(1, \min_{Q_t}\rho), \\ \tilde{G}^q(t) &= \|G\|^q_{L_q(Q_t)} + \|\mathbf{w}_0(x)\|^q_{W_q^{2-2/q}(\Omega)}. \end{aligned}$$

We observe that the above inequality can write of the following form

$$\|\mathbf{w}\|_{W_q^{2,1}(Q_T)} \leq K_2(\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})}, T)(\|\mathbf{w}_0\|_{W_q^{2-\frac{2}{q}}(\Omega)} + \|G\|_{L_q(Q_T)}), \tag{4}$$

where K_2 is an increasing function of $\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})}$ and T, depending on m and M (it is consequence from (3.1) by standard arguments see [**?**]}

The next Lemma is proved in Ladyszenskaya and Solonnikov [**?**] (see also [**?**]).

If $\mathbf{u}$ satisfies div $\mathbf{u} = 0, \mathbf{u}|_{\sum_T} = 0$ and

$$\|\mathbf{u}\|_{L_\infty(Q_T)} + \int_0^T \|\nabla\mathbf{u}(t)\|_{L_\infty(\Omega)}dt < \infty$$

then for any $\rho_0 \in C^1(\overline{\Omega})$ such that $0 < m \leq \rho_0(x) \leq M < \infty$, problem

$$\begin{aligned} \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho &= 0, \\ \rho(0) &= \rho_0(x) \end{aligned}$$

has a unique solution $\rho \in C^{1,1}(\overline{Q_T})$, which satisfies

$$m \le \rho(x,t) \le M$$

$$\|\nabla\rho\|_{L_\infty(Q_T)} \le \sqrt{3}\|\nabla\rho_0\|_{L_\infty(\Omega)} \exp(\int_0^T \|\nabla\mathbf{u}(t)\|_{L_\infty(\Omega)}dt),$$

$$\|\rho_t\|_{L_\infty(Q_T)} \le \sqrt{3}\|\mathbf{u}\|_{L_\infty(Q_T)}\|\nabla\rho_0\|_{L_\infty(\Omega)} \exp(\int_0^T \|\nabla\mathbf{u}(t)\|_{L_\infty(\Omega)}dt).$$

Moreover, if $\nabla\rho_0 \in W_q^1(\Omega)$ and $\mathbf{u} \in L_1(0,T;W_q^2(\Omega))$, then

$$\frac{d}{dt}\|\nabla\rho(t)\|_{W_q^1(\Omega)} \le c\|\mathbf{u}(t)\|_{W_q^2(\Omega)}\|\nabla\rho(t)\|_{W_q^1(\Omega)}.$$

The proof of the next Lemma is easily.

Let $\mathbf{u}$ be the same as in Lemma 3.3. If $\rho \in C^{1,1}(\overline{Q}_T)$ satisfies

$$\begin{aligned} \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho &= h, \\ \rho(0) &= \rho_0(x) \end{aligned}$$

where $h \in L_1(0,T;L_\infty(\Omega))$, then we have

$$\|\rho(t)\|_{L_\infty(\Omega)} \le \int_0^t \|h(\tau)\|_{L_\infty(\Omega)}d\tau.$$

4 Auxiliary result

We construct approximate solution inductively

$$\mathbf{u}^{(0)} = \mathbf{0},\ \mathbf{w}^{(0)} = \mathbf{0}$$

and for $k = 1, 2, 3, ...$, $\{\rho^{(k)}\}$, $\{\mathbf{u}^{(k)}, p^{(k)}\}$ and $\{\mathbf{w}^{(k)}\}$ are respectively, the solutions of problems

$$\begin{aligned} \frac{\partial\rho^{(k)}}{\partial t} + (\mathbf{u}^{(k-1)}\cdot\nabla)\rho^{(k)} &= 0, \\ \rho^{(k)}(0) &= \rho_0(x) \end{aligned} \tag{5}$$

$$
\begin{aligned}
\rho^{(k)}\frac{\partial \mathbf{u}^{(k)}}{\partial t}-(\mu+\mu_r)\triangle \mathbf{u}^{(k)}+\nabla p^{(k)} &= \rho^{(k)}\mathbf{f}+2\mu_r \text{ rot } \mathbf{w}^{(k-1)}-\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)},\\
\text{div } \mathbf{u}^{(k)} &= 0, \qquad (6)\\
\mathbf{u}^{(k)}|_{\sum_T} &= 0,\\
\mathbf{u}^{(k)}(0) &= \mathbf{u}_0(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho^{(k)}\frac{\partial \mathbf{w}^{(k)}}{\partial t}-(c_a+c_d)\triangle \mathbf{w}^{(k)}-(c_0+c_d-c_a)\nabla \text{ div } \mathbf{w}^{(k)}+4\mu_r\mathbf{w}^{(k)}\\
=\;& \rho^{(k)}\mathbf{g}+2\mu_r \text{ rot } \mathbf{u}^{(k-1)}-\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)} \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{w}^{(k)}|_{\sum_T} &= 0,\\
\mathbf{w}^{(k)}(0) &= \mathbf{w}_0(x).
\end{aligned}
$$

Now, we prove the boundness of above sequence.

For sufficiently small $T_1\in(0,T]$, the sequence $\{\mathbf{u}^{(k)},\nabla p^{(k)},\mathbf{w}^{(k)}\}$ is bounded in $W_q^{2,1}(Q_{T_1})\times L_q(Q_{T_1})\times W_q^{2,1}(Q_{T_1})$.

Let

$$
\Phi^{(k)}(T)=\|\mathbf{u}^{(k)}\|_{W_q^{2,1}(Q_T)}+\|\mathbf{w}^{(k)}\|_{W_q^{2,1}(Q_T)}+\|\nabla p^{(k)}\|_{L_q(Q_T)}.
$$

Lemmas 3.1 and 3.2 imply

$$
\begin{aligned}
\Phi^{(k)}(T) \;\le\;& K_1(\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})},T)(\|\mathbf{u}_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}+\|\rho^{(k)}\mathbf{f}\|_{L_q(Q_T)}+\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|_{L_q(Q_T)}\\
&+\|2\mu_r \text{ rot } \mathbf{w}^{(k-1)}\|_{L_q(Q_T)})\\
&+K_2(\|\rho\|_{C^{\alpha,\beta}(\overline{Q_T})},T)(\|\mathbf{w}_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}+\|\rho^{(k)}\mathbf{g}\|_{L_q(Q_T)}+\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_q(Q_T)}\\
&+\|2\mu_r \text{ rot } \mathbf{u}^{(k-1)}\|_{L_q(Q_T)}).
\end{aligned}
$$

Now, we estimate the right-hand side of the above inequality.

To estimate the term $\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|_{L_q(Q_T)}$ we will obtain first the following inequality

$$
\|\mathbf{u}^{(k-1)}\|_{L_\infty(Q_T)}\le C_1(\|\mathbf{u}_0\|_{W_q^{2-2/q}}+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}+T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)). \qquad (8)
$$

Since

$$\|\mathbf{u}^{(k-1)}(t)\|_{L_\infty(\Omega)} \le \|\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0\|_{L_\infty(\Omega)} + \|\mathbf{u}_0\|_{L_\infty(\Omega)} \ ,$$

using the interpolation inequality (see [?])with $q = \infty$, $q' = r$, $a = 3/q$ and the fact that $\mathbf{u}^{(k-1)}(t)|_{\partial\Omega} = 0$, $\mathbf{u}_0(x)|_{\partial\Omega} = 0$, we have

$$\|\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0\|_{L_\infty(\Omega)} \le C_2 \|\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0\|^{3/q}_{W^1_q(\Omega)} \|\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0\|^{1-3/q}_{L_q(\Omega)} \ . \quad (9)$$

Therefore,

$$\begin{aligned} \|\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0\|^q_{L_q(\Omega)} &= \int_\Omega |\mathbf{u}^{(k-1)}(t) - \mathbf{u}_0|^q dx \\ &= \int_\Omega \left| \int_0^t \mathbf{u}_t^{(k-1)}(s) ds \right|^q dx \\ &\le \int_\Omega \left| \left(\int_0^t ds \right)^{1/q'} \left(\int_0^t |\mathbf{u}_t^{(k-1)}(s)|^q ds \right)^{1/q} \right|^q dx \\ &\le t^{q/q'} \|\mathbf{u}^{(k-1)}\|^q_{W^{2,1}_q(Q_T)}, \end{aligned} \quad (10)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

By using the Sobolev embedding (see [?] pp. 108]), for $mq > 3$, we have

$$\|\mathbf{u}_0\|_{L_\infty(\Omega)} \le C_2 \|\mathbf{u}_0\|_{W^1_q(\Omega)} \le C_3 \|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)} \ . \quad (11)$$

Using the Solonnikov inequality , we obtain

$$\begin{aligned} \sup_{0\le t\le T} \|\mathbf{u}^{(k-1)}\|_{W^1_q(\Omega)} &\le \sup_{0\le t\le T} \|\mathbf{u}^{(k-1)}(t)\|_{W^{2-2/q}_q(\Omega)} \\ &\le C_4 \|\mathbf{u}^{(k-1)}\|_{W^{2,1}_q(Q_T)} + \|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}. \end{aligned}$$

Subtituying the inequalities (4.6) and (4.7) in (4.5), we obtain the inequality (4.4).

Now, we estimate the following term. By using, we get

$$\begin{aligned} \int_0^T \|\nabla \mathbf{u}^{(k-1)}(t)\|_{L_\infty(\Omega)} &\le C_5 \int_0^T \|\mathbf{u}^{(k-1)}(t)\|_{W^2_q(\Omega)} dt \\ &\le C_5 \left(\int_0^T dt \right)^{1/q'} \left(\int_0^T \|\mathbf{u}^{(k-1)}(t)\|^q_{W^2_q(\Omega)} dt \right)^{1/q} \\ &\le C_5 T^{1/q} \Phi^{(k-1)}(T). \end{aligned}$$

Using the interpolation inequality, we have

$$\begin{aligned}\|\nabla \mathbf{u}^{(k-1)}(t)\|_{L_q(\Omega)} &\le \|\mathbf{u}^{(k-1)}\|_{W_q^1(\Omega)} \\ &\le \|\mathbf{u}^{(k-1)}(t)\|^a_{W_q^2(\Omega)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)}_{L_\infty(\Omega)}.\end{aligned}$$

Consequently

$$\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|^q_{L_q(Q_T)} \le M^q\|\mathbf{u}^{(k-1)}\|^q_{L_\infty(Q_T)}\int_0^T \|\nabla \mathbf{u}^{(k-1)}\|^q_{L_q(\Omega)}dt$$
$$\le M^q\|\mathbf{u}^{(k-1)}\|^q_{L_\infty(Q_T)}\int_0^T \|\mathbf{u}^{(k-1)}\|^{aq}_{W_q^2(\Omega)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)q}_{L_\infty(\Omega)}dt$$
$$\le M^q\|\mathbf{u}^{(k-1)}\|^q_{L_\infty(Q_T)}\int_0^T \|\mathbf{u}^{(k-1)}\|^{aq}_{W_q^2(\Omega)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)q}_{L_\infty(\Omega)}dt$$
$$\le CM^q\|\mathbf{u}^{(k-1)}\|^{(2-a)q}_{L_\infty(Q_T)}\int_0^T \|\mathbf{u}^{(k-1)}(t)\|^{aq}_{W_q^2(\Omega)}dt$$
$$\le CM^q\|\mathbf{u}^{(k-1)}\|^{(2-a)q}_{L_\infty(Q_T)}T^{(1-a)}\left(\int_0^T \|\mathbf{u}^{(k-1)}\|^{q}_{W_q^2(\Omega)}dt\right)^a$$
$$\le CM^q\|\mathbf{u}^{(k-1)}\|^{(2-a)q}_{L_\infty(Q_T)}T^{(1-a)}\|\mathbf{u}^{(k-1)}\|^{aq}_{W_q^{2,1}(Q_T)}.$$

finally, we obtain

$$\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|_{L_q(Q_T)} \le CM\|\mathbf{u}^{(k-1)}\|^{(2-a)}_{L_\infty(Q_T)}\cdot T^{\frac{1-a}{q}}(\phi^{(k-1)}(T))^a.$$

Using the inequality and Young inequality $ab \le \varepsilon a^{\frac{1}{\alpha}} + (\frac{\alpha}{\varepsilon})^{\frac{\alpha}{1-\alpha}}(1-\alpha)b^{\frac{1}{1-\alpha}}$, with $\alpha = a/2$, we obtain

$$\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|_{L_q(Q_T)} \le CMT^{\frac{(1-a)}{q}}(\phi^{(k-1)^2}(T))^{\frac{a}{2}}\Big(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)}+$$
$$\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)} + T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)\Big)^{2(\frac{2-a}{2})}$$
$$\le \quad \alpha\varepsilon\phi^{(k-1)}(T)^2 + (1-\alpha)\varepsilon^{-\frac{\alpha}{1-\alpha}}(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} + \|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}$$
$$+T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T))^2(CMT^{\frac{1-a}{q}})^{\frac{1}{1-\alpha}}\ .$$

Chosing $\varepsilon = (T^{\frac{1-a}{q}})^{\frac{1}{\alpha}}$ and making some computations, we have

$$\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{u}^{(k-1)}\|_{L_q(Q_T)} \le CM\left((\|\mathbf{u}_0\|^2_{W_q^{2-2/q}} + \|\mathbf{w}_0\|^2_{W_q^{2-2/q}}) + T^\delta\phi^{(k-1)}(T)^2\right).$$

To estimate the rot operator, we observe

$$\begin{aligned}\| \text{ rot } \mathbf{w}^{(k-1)}\|_{L_q(\Omega)} &\le C\|\nabla \mathbf{w}^{(k-1)}\|_{L_q(\Omega)} \\ &\le C\|\mathbf{w}^{(k-1)}\|^a_{W^2_q(\Omega)}\|\mathbf{w}^{(k-1)}\|^{(1-a)}_{L_q(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\int_0^T \|\mathbf{w}^{(k-1)}(t)\|^{aq}_{W^2_q(\Omega)}dt &\le T^{(1-a)}\left(\int_0^T \|\mathbf{w}^{(k-1)}(t)\|^q_{W^2_q(\Omega)}dt\right)^a \\ &\le T^{(1-a)}\|\mathbf{w}^{(k-1)}\|^{aq}_{W^{2,1}_q(Q_T)} \\ &\le T^{(1-a)}(\Phi^{(k-1)}(T))^{aq}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\| \text{ rot } \mathbf{w}^{(k-1)}\|^q_{L_q(Q_T)} &= \int_0^T\int_\Omega | \text{ rot } \mathbf{w}^{(k-1)}|^q dxdt \\ &= \int_0^T \| \text{ rot } \mathbf{w}^{(k-1)}\|^q_{L_q(\Omega)}dt \\ &\le C\int_0^T \|\nabla \mathbf{w}^{(k-1)}\|^q_{L_q(\Omega)}dt \\ &\le C\int_0^T \|\mathbf{w}^{(k-1)}(t)\|^{(1-a)}_{L^\infty(\Omega)}\|\mathbf{w}^{(k-1)}(t)\|^{aq}_{W^2_q(\Omega)}dt \\ &\le C\|\mathbf{w}^{(k-1)}(t)\|^{(1-a)q}_{L_q(Q_T)}T^{\frac{(1-a)}{q}}(\phi^{(k-1)}(T))^{aq}.\end{aligned}$$

Applying the Young inequality, we obtain

$$\begin{aligned}&\| \text{ rot } \mathbf{w}^{(k-1)}\|_{L_q(Q_T)} \le C\|\mathbf{w}^{(k-1)}(t)\|^{(1-a)}_{L_\infty(Q_T)}T^{\frac{(1-a)}{q}}(\phi^{(k-1)}(T))^a \\ \le\;& CT^{\frac{1-a}{q}}(\phi^{(k-1)}(T))^a\left(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} + \|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)} + T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)\right)^{(1-a)} \\ =\;& a\varepsilon\phi^{(k-1)}(T) + (1-a)\varepsilon^{-\frac{a}{1-a}}\left(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} + \|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}\right. \\ &\left.+T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)\right)\left(CT^{\frac{1-a}{2}}\right)^{\frac{1}{1-a}}\end{aligned}$$

Taking $\varepsilon = \left(CMT^{\frac{1-a}{aq}}\right)^{\frac{1}{a}}$, we have

$$\| \text{ rot } \mathbf{w}^{(n-1)}\|_{L_q(Q_T)} \le CM\left(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} + \|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)} + T^{\delta_1}\phi^{(k-1)}(T)\right)$$

The above inequality also is verified by rot $\mathbf{u}^{(k-1)}$.

The estimate for the term $\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_q(Q_T)}$ is quite similar to the done to obtain the estimate, since we can obtain

$$\begin{aligned}&\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_q(Q_T)}\\ \leq\ & M\|\mathbf{u}^{(k-1)}\|_{L_\infty(Q_T)}.T^{\frac{1-a}{q}}.\|\mathbf{w}^{(k-1)}\|^{(1-a)}_{L_\infty(Q_T)}\|\mathbf{w}^{(k-1)}\|^{a}_{W^{2,1}_q(Q_T)}.\end{aligned}$$

Since

$$\|\mathbf{w}^{(k)}\|_{L_\infty(Q_T)}\leq C\left(\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)}+T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)\right),$$

we get

$$\begin{aligned}\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_q(Q_T)}\ \leq\ & T^{\frac{1-a}{q}}MC((\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)})\\ &+T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T))^{2-a}(\phi^{(k-1)}(T))^{a}\end{aligned}$$

and consequently

$$\|\rho^{(k)}(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_q(Q_T)}\leq MC\left(\|\mathbf{u}_0\|^2_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|^2_{W^{2-2/q}_q(\Omega)}+T^{\delta_1}\phi^{(k-1)}(T)^2\right).$$

Using the above estimates in the inequality, we obtain

$$\begin{aligned}\phi^{(k)}(T)\ \leq\ & K_1(\|\rho\|_{C^{1,1}(Q_T)},T)\Big\{\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+M\|\mathbf{f}\|_{L_q(Q_T)}\\ &+MC\left(\|\mathbf{u}_0\|^2_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|^2_{W^{2-2/q}_q(\Omega)}+T\cdot\phi^{(k-1)}(T)^2\right)\\ &+2\mu_r MC\left(\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)}+T^{\delta_1}\phi^{(k-1)}(T)\right)\Big\}\\ &+K_2(\|\rho\|_{C^{1,1}},T)\Big\{\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)}+M\|\mathbf{g}\|_{L_q(Q_T)}\\ &+MC\left(\|\mathbf{u}_0\|^2_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|^2_{W^{2-2/q}_q(\Omega)}+T^{\delta}\phi^{(k-1)}(T)^2\right)\\ &+2\mu_r MC\left(\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)}+T^{\delta_1}\phi^{(k-1)}(T)\right)\Big\}\end{aligned}$$

Setting $C=\max\{(1+MC),2MC,M\}$, we have

$$\begin{aligned}\phi^{(k)}(T)\ \leq\ & CK(\|\rho\|_{C^{1,1}},T)\Big\{\left(\|\mathbf{u}_0\|_{W^{2-2/q}_q(\Omega)}+\|\mathbf{w}_0\|_{W^{2-2/q}_q(\Omega)}\right)\\ &+\left(\|\mathbf{u}_0\|^2_{W^{2-2/q}_q(\Omega)}\|\mathbf{w}_0\|^2_{W^{2-2/q}_q(\Omega)}\right)+\\ &\left(\|\mathbf{f}\|_{L_q(Q_T)}+\|\mathbf{g}\|_{L_q(Q_T)}\right)+T^{\delta_1}\phi^{(k-1)}(T)+T\phi^{(k-1)}(T)^2\Big\}\end{aligned}$$

Now, we observe that

$$\begin{aligned}\frac{|\rho^{(k)}(x,t)-\rho^{(k)}(y,s)|}{|x-y|+|t-s|} &= \frac{|\rho^{(k)}(x,t)-\rho^{(k)}(y,t)+\rho^{(k)}(y,t)-\rho^{(k)}(y,s)|}{|x-y|+|t-s|} \\ &\leq \frac{|\rho^{(k)}(x,t)-\rho^{(k)}(y,t)|}{|x-y|}+\frac{|\rho^{(k)}(y,t)-\rho^{(k)}(y,s)|}{|t-s|}.\end{aligned}$$

For t fixed and $s \in [0,1]$, we define the function

$$\varphi(s) = \rho^{(k)}(sy+(1-s)x,t)$$

Moreover

$$\begin{aligned}|\rho^{(k)}(y,t)-\rho^{(k)}(x,t)| &= |\varphi(1)-\varphi(0)| = \left|\int_0^1 \varphi'(s)ds\right| \\ &\leq \int_0^1 |\nabla\rho^{(k)}(sy+(1-s)x,t).(y-x)|ds \\ &\leq \|\nabla\rho^{(k)}\|_{L_\infty(Q_T)}|y-x|.\end{aligned}$$

Therefore, if $t, s \in [0,T]$ are arbitrary, we have

$$|\rho^{(k)}(y,t)-\rho^{(k)}(y,s)| = \left|\int_s^t \rho_t^{(k)}(y,\theta)d\theta\right| \leq \|\rho_t^{(k)}\|_{L_\infty(Q_T)}|t-s|.$$

Using this identities, we get

$$\|\rho^{(k)}\|_{C^{1,1}(Q_T)} \leq M + \|\nabla\rho^{(k)}\|_{L_\infty(Q_T)} + \|\rho_t^{(k)}\|_{L_\infty(Q_T)}.$$

¿From Lemma (3.3), we obtain

$$\begin{aligned}\|\rho^{(k)}\|_{C^{1,1}(Q_T)} &\leq M+\sqrt{3}(1+\|\mathbf{u}^{(k-1)}\|_{L_\infty(Q_T)}) \\ &\quad \times\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp\left(\int_0^T \|\nabla\mathbf{u}^{(k-1)}(t)\|_{L_\infty(\Omega)}dt\right) \\ &\leq M+\sqrt{3}(1+C_1(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}+T^{(1-1/p)(1-3/p)}\phi^{(k-1)}(T)) \\ &\quad \times\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp(C_5T^{(1-1/q)(1-3/q)}\phi^{(k-1)}(T)) \\ &\equiv K_3(\phi^{(k-1)}(T),T).\end{aligned}$$

Consequently, we get

$$\begin{aligned}\phi^{(k)}(T) &\leq K(K_3(\phi^{(k-1)}(T),T),T)\times\Big\{(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)} \\ &\quad +(\|\mathbf{u}_0\|^2_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|^2_{W_q^{2-2/q}(\Omega)}+\|\mathbf{f}\|_{L_q(Q_T)}+\|\mathbf{g}\|_{L_q(Q_T)} \\ &\quad +T^{\delta_1}\phi^{(k-1)}(T)+T\phi^{(k-1)}(T)^2\Big\}.\end{aligned}$$

If we consider A_1 such that

$$\begin{aligned} A_1 &\geq K\left(M+\sqrt{3}\{1+C_1(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)})+1\}\|\nabla\rho_0\|_{L_\infty(\Omega)},T\right) \\ &\quad\times\left(\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}+\|\mathbf{u}_0\|^2_{W_q^{2-2/q}(\Omega)}+\|\mathbf{w}_0\|^2_{W_q^{2-2/q}(\Omega)}\right. \\ &\quad\left.+\|\mathbf{f}\|_{L_q(Q_T)}+\|\mathbf{g}\|_{L_q(Q_T)}+2\right) \end{aligned}$$

and we define

$$T_1=\min\left\{A_1^{-1(1-1/p)^{-1}(1-3/p)^{-1}},A_1^{-2/\delta},A_1^{-1/\delta_1},(C_{1S}A_1)^{-(1-1/p)^{-1}}\right\}.$$

Then, $\phi^{(k)}(T_1)\leq A_1$, holds provided that $\phi^{(k-1)}(T_1)\leq A_1$.
Since

$$\begin{aligned} \phi^{(1)}(T_1) &\leq K(M+\sqrt{3}\|\nabla\rho_0\|_{L_\infty(\Omega)},T_1)\Big\{\|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} \\ &\quad+\|\mathbf{w}_0\|_{W_q^{2-2/q}(\Omega)}+M(\|\mathbf{f}\|_{L_q(Q_T)}+\|\mathbf{g}\|_{L_q(Q_T)})\Big\} \\ &\leq A_1, \end{aligned}$$

the assertion of the lemma follows.

5 Proof of the theorem

Setting $\rho^{(n,k)}=\rho^{(n+k)}-\rho^{(n)},\mathbf{u}^{(n,k)}=\mathbf{u}^{(n+k)}-\mathbf{u}^{(n)},p^{(n,k)}=p^{(n+k)}-p^{(n)}$ and $\mathbf{w}^{(n,k)}=\mathbf{w}^{(n+k)}-\mathbf{w}^{(n)}$, we have

$$\begin{aligned} \frac{\partial\rho^{(n,k)}}{\partial t}+(\mathbf{u}^{(n+k-1)}\cdot\nabla)\rho^{(n,k)} &= -(\mathbf{u}^{(n-1,k)}\cdot\nabla)\rho^{(n)}, \\ \rho^{(n,k)}(0) &= 0, \end{aligned}$$

$$\begin{aligned} \rho^{(n+k)}\frac{\partial\mathbf{u}^{(n,k)}}{\partial t}-(\mu+\mu_r)\triangle\mathbf{u}^{(n,k)}+\nabla p^{(n,k)} &= F^{(n,k)}, \\ \text{div }\mathbf{u}^{(n,k)} &= 0, \\ \mathbf{u}^{(n,k)}|_{\sum_T} &= 0, \\ \mathbf{u}^{(n,k)}(0) &= 0, \end{aligned}$$

where $F^{(n,k)} = 2\mu_r$ rot $\mathbf{w}^{(n-1,k)} - \rho^{(n,k)}[\mathbf{f} - \mathbf{u}_t^{(n)} - (\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1)}]$
$-\rho^{(n+k)}[(\mathbf{u}^{(n-1,k)} \cdot \nabla)\mathbf{u}^{(n-1+k)} - (\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,k)}]$ and

$$\rho^{(n+k)}\frac{\partial \mathbf{w}^{(n,k)}}{\partial t} - (c_a + c_d)\triangle \mathbf{w}^{(n,k)} - (c_0 + c_d - c_a)\nabla \text{ div } \mathbf{w}^{(n,k)} + 4\mu_r \mathbf{w}^{(n,k)} = G^{(n,k)}$$

$$\begin{aligned} \mathbf{w}^{(n,k)}|_{\sum_T} &= 0, \\ \mathbf{w}^{(n,k)}(0) &= 0, \end{aligned}$$

where $G^{(n,k)} = 2\mu_r$ rot $\mathbf{u}^{(n-1,k)} - \rho^{(n,k)}[\mathbf{g} - \mathbf{w}_t^{(n)} - (\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{w}^{(n-1)}]$
$-\rho^{(n+k)}[(\mathbf{u}^{(n-1+k)} \cdot \nabla)\mathbf{w}^{(n-1+k)} - (\mathbf{u}^{(n-1,k)} \cdot \nabla)\mathbf{w}^{(n-1)}]$.

Let

$$\Psi^{(n,k)}(t) = \|\mathbf{u}^{(n,k)}\|_{W_q^{2,1}(Q_t)} + \|\mathbf{w}^{(n,k)}\|_{W_q^{2,1}(Q_t)} + \|\nabla p^{(n,k)}\|_{L_q(Q_t)}.$$

Then, from , it follows that for $t \in (0, T_1]$,

$$\begin{aligned} \|F^{(n,k)}\|_{L_q(Q_t)} \leq\ & c(\|\nabla \mathbf{w}^{(n-1,k)}\|_{L_q(Q_t)} + \|\rho^{(n,k)}\|_{L_\infty(Q_t)}\{\|\mathbf{f}\|_{L_q(Q_t)} + \|\mathbf{u}_t^{(n)}\|_{L_q(Q_t)} \\ & + \|(\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1)}\|_{L_q(Q_t)}\} \\ & + \|\rho^{(n+k)}\|_{L_\infty(Q_t)}\{\|(\mathbf{u}^{(n-1,k)} \cdot \nabla)\mathbf{u}^{(n-1+k)}\|_{L_q(Q_t)} \\ & + \|(\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,k)}\|_{L_q(Q_t)}\}. \end{aligned}$$

We observe that

$$\begin{aligned} \|\nabla \mathbf{w}^{(n-1,k)}\|_{L_q(Q_t)}^q &\leq \int_0^t \|\nabla \mathbf{w}^{(n-1,k)}(\tau)\|_{L_q(\Omega)}^q d\tau \\ &\leq \int_0^t \|\mathbf{w}^{(n-1,k)}(\tau)\|_{W_q^1(\Omega)}^q d\tau \\ &\leq c\int_0^t \|\mathbf{w}^{(n-1,k)}(\tau)\|_{W_q^{2,1}(Q_\tau)}^q d\tau \\ &\leq c\int_0^t \Psi^{(n-1,k)}(\tau) d\tau. \end{aligned}$$

By other hand

$$\|(\mathbf{u}^{(n-1,k)} \cdot \nabla)\mathbf{u}^{(n-1+k)}\|_{L_q(Q_t)}^q \leq \int_0^t \|\nabla \mathbf{u}^{(n-1+k)}(\tau)\|_{L_q(\Omega)}^q \|\mathbf{u}^{(n-1,k)}(\tau)\|_{L_\infty(\Omega)}^q d\tau$$

$$
\begin{aligned}
&\le \sup_{0\le\tau\le t}\|\nabla\mathbf{u}^{(n-1+k)}(\tau)\|^q_{L_q(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,k)}(\tau)\|^q_{L_\infty(\Omega)}d\tau\\
&\le \sup_{0\le\tau\le t}\|\mathbf{u}^{(n-1+k)}(\tau)\|^q_{W^1_q(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,k)}(\tau)\|^q_{W^1_q(\Omega)}d\tau\\
&\le \sup_{0\le s\le t}\|\mathbf{u}^{(n-1+k)}(\tau)\|^q_{W_q^{2-\frac{2}{q}}(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,k)}(\tau)\|^q_{W_q^{2-\frac{2}{q}}(\Omega)}d\tau\\
&\le (\|\mathbf{u}^{(n-1+k)}(0)\|_{W_q^{2-\frac{2}{q}}(\Omega)}\\
&\quad +\widehat{c}\|\mathbf{u}^{(n-1+k)}(\tau)\|_{W^{2,1}_q(Q_t)})^q\int_0^t\widehat{c}^q\|\mathbf{u}^{(n-1,k)}\|^q_{W^{2,1}_q(Q_\tau)}d\tau\\
&\le c\int_0^t\Psi^{(n-1,k)}(\tau)^q d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\|(\mathbf{u}^{(n-1)}\cdot\nabla)\mathbf{u}^{(n-1,k)}\|^q_{L_q(Q_t)} &\le \int_0^t d\tau\int_\Omega|\mathbf{u}^{(n-1)}|^q|\nabla\mathbf{u}^{(n-1,k)}|^q dx\\
&\le \|\mathbf{u}^{(n-1)}\|^q_{L_\infty(Q_t)}\int_0^t\|\nabla\mathbf{u}^{(n-1,k)}\|^q_{L_q(\Omega)}d\tau\\
&\le \sup_{0\le\tau\le t}\|\mathbf{u}^{(n-1)}\|^q_{W_q^{2-\frac{2}{q}}(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,k)}\|^q_{W^1_q(\Omega)}d\tau\\
&\le \sup_{0\le\tau\le t}\|\mathbf{u}^{(n-1)}\|^q_{W_q^{2-\frac{2}{q}}(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,k)}\|^q_{W_q^{2-\frac{2}{q}}(\Omega)}d\tau\\
&\le (\|\mathbf{u}^{(n-1)}(0)\|_{W_q^{2-\frac{2}{q}}(\Omega)}\\
&\quad +\widehat{c}\|\mathbf{u}^{(n-1)}\|_{W^{2,1}_q(Q_t)})^q\widehat{c}^q\int_0^t\|\mathbf{u}^{(n-1,k)}\|^q_{W^{2,1}_q(Q_\tau)}d\tau\\
&\le c\int_0^t\Psi^{(n-1,k)}(\tau)^q d\tau.
\end{aligned}
$$

Also from Lemma 3.4, we have

$$
\begin{aligned}
\|\rho^{(n,k)}(t)\|_{L_\infty(\Omega)} &\le \int_0^t\|\mathbf{u}^{(n-1,k)}(\tau)\|_{L_\infty(\Omega)}\|\nabla\rho^{(n)}(\tau)\|_{L_\infty(\Omega)}d\tau\\
&\le c\int_0^t\|\mathbf{u}^{(n-1,k)}\|_{W^{2,1}_q(Q_\tau)}d\tau\\
&\le c\int_0^t\Psi^{(n-1,k)}(\tau)d\tau.
\end{aligned}
$$

Also, by the above estimates and the hypothesis on $\mathbf{f}$, we have

$$
\|\rho^{(n,k)}\|_{L_\infty(Q_t)}\{\|\mathbf{f}\|_{L_q(Q_t)}+\|\mathbf{u}^{(n)}_t\|_{L_q(Q_t)}+\|(\mathbf{u}^{(n-1)}\cdot\nabla)\mathbf{u}^{(n-1)}\|_{L_q(Q_t)}\}
$$

$$
\begin{aligned}
&\leq c\|\rho^{(n,k)}\|_{L_\infty(Q_t)} \\
&\leq c(\int_0^t \Psi^{(n-1,k)}(\tau)^q d\tau)^{\frac{1}{q}}
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\|F^{(n,k)}\|_{L_q(Q_t)} &\leq c\int_0^t \Psi^{(n-1,k)}(\tau)d\tau + c(\int_0^t \Psi^{(n-1,k)}(\tau)^q d\tau)^{\frac{1}{q}} \\
&\leq c(\int_0^t \Psi^{(n-1,k)}(\tau)^q d\tau)^{\frac{1}{q}}.
\end{aligned} \tag{12}
$$

Analogously, we have

$$
\|G^{(n,k)}\|_{L_q(Q_t)} \leq c(\int_0^t \Psi^{(n-1,k)}(\tau)^q d\tau)^{\frac{1}{q}}. \tag{13}
$$

By using the estimates (5.1), (5.2) and together with Lemmas 3.1 and 3.2, we have for $t \in [0, T_1]$ and $q > 3$

$$
\Psi^{(n,k)}(t) \leq c\left(\int_0^t \Psi^{(n-1,k)}(\tau)^q d\tau\right)^{\frac{1}{q}} \tag{14}
$$

or

$$
\left[\Psi^{(n,k)}(t)\right]^q \leq c^q \int_0^t \left[\Psi^{(n-1,k)}(\tau)\right]^q d\tau,
$$

consequently $\Psi^{(n,k)}(t) \to 0$ as $n \to \infty$, $\forall\ t \in [0, T_1]$. Firstly, we observe that $W_q^{2,1}(Q_t)$ is a Banach space and consequently, we have there exist $\mathbf{u}, \mathbf{w} \in W_q^{2,1}(Q_{T_1})$, such that

$$
\begin{aligned}
\mathbf{u}^n &\to \mathbf{u} \text{ strongly in } W_q^{2,1}(Q_{T_1}), \\
\mathbf{w}^n &\to \mathbf{w} \text{ strongly in } W_q^{2,1}(Q_{T_1}).
\end{aligned}
$$

Also, from of the completeness of $L_q(Q_{T_1})$, there exist $p \in L_q(Q_{T_1})$ such that

$$
p^n \to p \text{ strongly in } L_q(Q_{T_1}).
$$

Now, the next step is to take limit. But, once the above convergences have been established, this is a standard procedure to obtain that $\mathbf{u}, \mathbf{w}$,p is a strong solution of the problem (1.1)-(1.2).

We need only to argument the uniqueness of the solution in order to complete the proof of Theorem . Suppose that there exist another solution

$\mathbf{u}_1, \mathbf{w}_1, p_1$ of (1.1) and (1.2) with the same regularity as stated in the Theorem. Define

$$U = \mathbf{u}_1 - \mathbf{u}, W = \mathbf{w}_1 - \mathbf{w}, P = p_1 - p.$$

These auxiliary functions verify a set of equations similar to (4.1)-(4.3). Repeat the argument used to obtain (5.2), we get for $\theta(t) = \|U\|^q_{W^{2,1}_q(Q_t)} + \|W\|^q_{W^{2,1}_q(Q_t)} + \|P\|^q_{L_q(Q_t)}$ an inequality of the following type

$$\theta(t) \leq c \int_0^t \theta(\tau) d\tau$$

which, by Gronwall's inequality, is equivalent to assert $U = 0, W = 0, P = 0$.

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