Sub–Riemannian Geodesics on Lie Groups

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Abstract. We investigate the space of abnormal extremals of nonlinear deterministic control systems which are linear in the controls. Our main results concern the abnormal curves for a class of systems whose state spaces are nilpotent Lie groups.

1. Introduction

Sub–Riemannian geometry is concerned with the study of a nonintegrable k–plane distribution D, endowed with a Riemannian metric, on an n–dimensional manifold M. The case $k = n$ corresponds to classical Riemannian geometry, and each geodesic is the projection of an integral curve of the Hamiltonian vector field associated with the metric. When $k < n$, a geodesic is still a locally minimizing curve for the length functional, but in this case it is well known that geodesics that are not the projection of Hamiltonian curves do exist [6]. These curves are called strictly abnormal geodesics, the other geodesic curves are called normals (see Sect. 2). These geometries arise among other places as the limits of Riemannian geometries.

A basic question is "are all sub–Riemannian geodesics smooth" ? Since normal extremals are automatically smooth [5], this is equivalent to the question "are all abnormal minimizers smooth ?". Hamenstät suggested the following idea to investigate this last question: try to associate to every abnormal curve c some smooth submanifold $M(c)$ of M containing c and such that c is normal as an integral curve for the restriction of D to $M(c)$ (the submanifold $M(c)$ is called a characteristic manifold for c). In general, $M(c)$ does not always exist [6]. This question motivated our work.

As in classical geometry, in sub–Riemannian geometry the case of Lie groups is especially important and interesting (see Sections 3 and 4). If $M = G$ is a Lie group, it is natural to assume that both the distribution D and the metric are left invariant. In the case when G is a compact connected Lie group and D is the distribution orthogonal to its maximal torus, all sub–Riemannian geodesics are smooth [7]. In this paper, we study sub–Riemannian geodesics in nilpotent Lie groups.

The present paper has the following structure. In Section 2, the basic notations, definitions and examples are introduced. In Section 3, we introduce Carnot groups, flat distributions and Cartan system. A flat distribution is a local approximation of a sub–Riemannian structure at regular points. Carnot groups are to sub–Riemannian geometry as Euclidean geometry is to Riemannian geometry. In Section 4, we give examples of graded nilpotent Lie groups with strictly abnormal geodesics, nilpotent Lie group G with left invariant distribution D and subgroup H whose geodesics, for the induced structure, are not geodesics in (G, D).

2. Definitions

A sub–Riemannian structure on an n–dimensional manifold M consists of a vector subbundle $D \subset TM$ of the tangent bundle of M together with a fiber inner product $\langle .,. \rangle$ on this subbundle. A curve γ is said to be *horizontal* (or a D–*curve*) if its derivative $\dot{\gamma}(t)$ exists almost everywhere and lies in D when it exists. We measure the length $l(\gamma)$ of a horizontal curve as in Riemannian geometry:

$$l(\gamma) = \int \|\dot{\gamma}(t)\| \, dt.$$

In this formula, $\|\dot{\gamma}(t)\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$ is computed using the inner product on $D(\gamma(t))$ and the integral is taken over the domain of the curve. The sub–Riemannian distance $d(x, y)$ between two points x and y is taken over all horizontal paths which connect x and y. This distance is infinite if there is no path joining both points. A horizontal path is said to be a *minimizer* if it realizes the distance between its endpoints. A path $\gamma : I \subset \mathbb{R} \to M$ is said to be a *geodesic* if it is locally a minimizer.

The distribution $D \subset TM$ is *bracket generating at* p if it admits a local frame (X_i) in a neighborhood of p for which the iterated Lie brackets $[X_i, X_j]$, $[X_i, [X_j, X_k]]$ ... , together with the X_i span the tangent space at every point of this neighborhood. Given two points in a sub–Riemannian manifold $(M, D, \langle .,. \rangle)$, are there any horizontal paths that join them? A corollary to a classical theorem of Chow asserts that, if D is bracket generating and M is connected, then any two points of M can be joined by a horizontal path. Is there a minimizer joining both points? If M is connected, bracket generating and complete relative to the sub–Riemannian distance function, then any two points of M can be joined by a minimizer.

Normal curves and geodesics equations The cotangent bundle T^*M of M has a natural symplectic structure determined by the 2–forms $\Omega = d\omega$, where ω is the 1–form given by $\omega(x, \lambda)(v) = \langle \lambda, d\pi^*(v) \rangle$ for $v \in T_{(x,\lambda)}(T^*M)$. To each vector field X on M, we associate the function $H_X : T^*M \to \mathbb{R}$ given by $H_X(q, \lambda) = \langle \lambda, X(q) \rangle$ for $\lambda \in T_q^*M$. Then X is of class $\mathcal{C}^k$ if and only if H_X is $\mathcal{C}^k$. We use $\vec{H}_X$ to denote the Hamilton vector field associated to H_X (by definition, $\vec{H}_X$ is the vector field on T^*M such that $\Omega(Y, \vec{H}_X) = dH_X(Y)$ for every vector field Y on T^*M [2]).

Let $(M, D, \langle .,. \rangle)$ be a sub–Riemannian manifold. If $(p, \lambda) \in T^*M$, then the restriction $\lambda\Big|_{D(p)}$ of λ to the subspace $D(p)$ of T_pM has a well defined norm, since $D(p)$ is an inner product space. We will use $\|\lambda\|$ to denote this norm. The function $H : T^*M \to \mathbb{R}$ given by $H(\lambda, p) = -\|\lambda\|^2/2$ is the *Hamiltonian of the sub–Riemannain structure* $(D, \langle .,. \rangle)$. The Hamiltonian H is a smooth function on T^*M. Indeed, if $\{X_i\}_{1 \le i \le k}$ is a local orthonormal frame for the distribution

D (here, k is the rank of D), then

$$H(x,\lambda) = -\frac{1}{2}\sum_{i=1}^{k} H^2_{X_i}(x,\lambda).$$

The flow induced by H on T^*M is called the *sub–Riemannian geodesic flow.* Let $\xi(t) = (\gamma(t), p(t))$ be a solution to Hamilton's equation on T^*M for the sub–Riemannain Hamiltonian H. Then [5] every sufficiently short arc of γ is a sub–Riemannian minimizer. The projection γ is called a *normal curve. Unlike the Riemannian situation, not every geodesic is normal [6].* There may exist "abnormal" geodesics unrelated to our Hamiltonian.

Abnormal curves An abnormal curve is a horizontal curve (*i.e.* a D–curve) which is the projection onto M of an absolutely continuous curve in the annihilator $D^\perp \subset T^*M$ of D, which does not intersect the zero section and whose derivative, whenever it exists, lies in the kernel of Ω restricted to $D^\perp$. The condition that a curve is abnormal depends only on the distribution and not at all on the inner product on the distribution. Contrarily to the normal curves, the abnormal ones need not be minimizing. If a curve is abnormal but not normal, we say that it is strictly abnormal.

Examples..

1) If $D = TM$, the abnormal curves are trivial.

2) If D is a contact distribution, the abnormal curves are trivial.

3) *Heisenberg Group.* Consider $\mathbb{R}^3$ with the multiplication

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1y_2 - x_2y_1).$$

The left invariant contact distribution D on $\mathbb{R}^3$ is spanned by

$$X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

A curve $\gamma(t) = (x(t), y(t), z(t))$ is horizontal if and only if $\dot{z} = y\dot{x} - x\dot{y}$ (constraint equation). D is bracket generating because $X, Y, [X, Y]$ are linearly independent at each point. It then follows from Chow's theorem that, given any two points p_1, p_2, there exists a finite concatenation of integral curves of X and Y that goes from p_1 to p_2. We define a Riemannain metric $\langle .,. \rangle$ on D by declaring $\{X, Y\}$ to be an orthonormal basis of sections of D. The Hamiltonian $H : T^*\mathbb{R}^3 \to \mathbb{R}$ is $H(p,\lambda) = -\frac{1}{2}(H^2_X(p,\lambda) + H^2_Y(p,\lambda))$; then $H(x, y, z, \xi, \eta, \mu) = -\frac{1}{2}((\xi - y\mu)^2 + (\eta + x\mu)^2)$. The normal curves are projection of integral curves of $\vec{H}$. Let $\pi(x, y, z) = (x, y)$ be the projection. Given a sufficiently smooth curve $c(t) = (x(t), y(t))$ in the xy plane, and a starting height z_0, there is a unique horizontal curve which starts at $p_0 = (x(0), y(0), z_0)$ and projects onto $(x(t), y(t))$. It is called the horizontal lift of c passing through p_0. THe geodesics for the Heisenberg Group are exactly the horizontal lifts of arcs of circles and every minimizer is normal.

4) The Heisenberg Group is a principal bundle over the plane, with projection $(x, y, z) \mapsto (x, y)$ and the form $\alpha = dz - (ydx - xdy)$ is a connection for this

bundle. To generalize, replace the Heisenberg Group with a general principal bundle P over a Riemannian base, and put a connection on this principal bundle. The space P then inherits a sub–Riemannian structure. ITs geodesics, projected to the base, satisfy the non–Abelian Lorentz equations. In the spacial case when $P = G$ is a Lie group and the projection is that for a homogeneous space $G \to G/H$ then, with an additional assumption regarding the inner product, we can explicitly write down all the normal sub–Riemannian geodesics in Lie theoretical terms.

3. Carnot groups and nilpotentization

Let G be a connected simply connected nilpotent Lie group. Suppose that its Lie algebra g is graded:

$$g = g_1 \oplus g_2 \oplus \cdots \oplus g_r, \quad [g_i, g_j] \subset g_{i+j}, \quad g_s = \{0\} \text{ if } s > r,$$

and generated by its component of degree 1, *i.e.*, $\mathrm{Lie}(g_1) = g$. Then $D = g_1$ can be considered as a nonintegrable left invariant distribution on the Lie group G. We say that the distribution D is *flat*. If D is equipped with a left invariant inner product in g_1, then $(D, \langle .,. \rangle)$ is called a *flat sub–Riemannian distribution on* G.

A Lie group whose Lie algebra is graded nilpotent, endowed with an inner product on the space g_1, with the property that this subspace Lie–generates the whole Lie algebra, is called a *Carnot group*.

Examples.

1) Let g be the 3–dimensional Heisenberg algebra, *i.e.*, the unique 3–dimensional 2–step nilpotent Lie algebra:

$$\dim g = 3, \quad \dim[g, g] = 1, \quad \dim[g, [g, g]] = 0.$$

Let G be the corresponding connected simply connected Lie group. A flat rank two distribution D on G is just any rank two nonintegrable left invariant distribution on G:

$$D \subset g, \quad \dim D = 2, \quad \mathrm{Lie}(D) = g.$$

To obtain a flat sub–Riemannian structure on G, one has to add any left invariant inner product $\langle .,. \rangle$ in D. Choose an orthonormal frame:

$$D = span(u_1, u_2), \quad \langle u_i, u_j \rangle = \delta_{i,j}, \quad 1 \le i, j \le 2.$$

D is nonintegrable. Then $u_3 = [u_1, u_2] \notin D$ and $g = \mathrm{span}(u_1, u_2, u_3)$, $[u_1, u_3] = 0$, $[u_2, u_3] = 0$, $g = g_1 \oplus g_2$ with $g_1 = \mathrm{span}(u_1, u_2)$, $g_2 = \mathrm{span}(u_3)$. "Up to isomorphism there exists exactly one flat distribution on G, and the same is true for flat sub–Riemannian structures".

2) *The Cartan case [4].* Let g be the 5–dimensional nilpotent 3–step Lie algebra with multiplication rules in some base $(u_1, u_2, u_3, u_4, u_5)$:

$$[u_1, u_2] = u_3, \quad [u_1, u_3] = u_4, \quad [u_2, u_3] = u_5$$

(all other brackets beeing equal to zero); $(u_1, u_2, u_3, u_4, u_5)$ is called a standard frame on g. The Lie algebra g is graded:

$$g = g_1 \oplus g_2 \oplus g_3, \quad g_1 = \mathrm{span}(u_1, u_2), \quad g_2 = \mathrm{span}(u_3), \quad g_3 = \mathrm{span}(u_4, u_5).$$

Denote by G the simply connected Lie group corresponding to g. Any flat distribution or flat sub–Riemannian structure of rank two on the Lie group G is isomorphic to the following ones defined *via* a standard frame in g:

$$D = \mathrm{span}(u_1, u_2), \quad \langle u_i, u_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq 2.$$

Indeed, take an arbitrary flat sub–Riemannian structure $(D, \langle ., . \rangle)$ on the Lie group G corresponding to a graduation of g:

$$\begin{aligned} g = h_1 \oplus h_2 \oplus h_3, \quad D = h_1, \quad \dim h_1 = 2, \\ h_2 = [h_1, h_1], \quad \dim h_2 = 1, \quad h_3 = [h_1, h_2], \quad \dim h_3 = 2. \end{aligned}$$

Choose any base of D such that $D = \mathrm{span}(u_1, u_2)$, $\langle u_i, u_j \rangle = \delta_{ij}$. Then $u_3 = [u_1, u_2]$ spans h_2, and the vectors $u_4 = [u_1, u_3]$ and $u_5 = [u_1, u_2]$ span h_3. Thus $\{u_1, u_2\}$ generates a standard frame in g. This proves the uniqueness of flat sub–Riemannian structures on G up to isomorphism.

Why flat distributions and Carnot groups are important in SR geometry? We are going to show how to associate a Carnot group of dimension n to a regular point q of a sub–Riemannian manifold M of dimension n. This Carnot group G_q will be called a *nilpotentization of M at q [8]*. For a distribution $D \subset TM$, its Lie flag is defined as follows:

$$D = D^1 \subset D^2 \cdots \subset D^j \subset \ldots$$

where $D^2 = [D^1, D^1], \ldots D^{j+1} = [D, D^j]$ (here D also denotes the $\mathcal{C}^\infty(M)$–module of vector fields on M which are tangent to the distribution D). The *growth vector* of D at point q is the vector $(n_1, n_2, \ldots)$, $n_j = \dim D^j(q)$. The distribution is said to be *regular near* q if the growth vector is constant in a neighborhood of q. Let $\langle ., . \rangle$ be a Riemannian metric on D and q a regular point. Write

$$\mathrm{Gr}(D)_q = \overset{r}{\underset{i=1}{\oplus}} V_i(q), \quad V_i(q) = D^i(q)/D^{i-1}(q),$$

so that $V_1(q) = D(q)$. This $\mathrm{Gr}(D)_q$ is a graded vector space whose total dimension is that of the manifold M. It inherits a Lie algebra structure from the Lie bracket of vector fields. $\mathrm{Gr}(D)_q$ becomes a graded nilpotent r–step (here, r is the smallest integer such that $D^r(q) = T_qM$). The corresponding simply connected Lie group G_q inherits a left invariant sub–Riemannian structure defined by taking the distribution to be $V_1 = D(q) \subset T_eG_q$ with its given inner product and then left–translating this on G_q.

The nilpotentization at q has the same relationship to M as the euclidean tangent space of a Riemannian manifold has to that manifold. THis statement was made precise by Mitchell, who showed that G_q is isometric to the metric tangent cone to M.

Examples..

1) For a flat distribution $D \subset TG$, $D^i(q) = \overset{i}{\underset{j=1}{\oplus}} g^j(q)$ and the growth vector takes the form $(n_1, n_2, \ldots)$, $n_i = \sum_{j=1}^{i} \dim g^j$.

2) For the Cartan case, the growth vector is $(2, 3, 5)$.

3) $G = \mathbb{R}^5$ endowed with the multiplication rule

$$(x_1, y_1, z_1, u_1, v_1)(x_2, y_2, z_2, u_2, v_2)$$
$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2, u_1 + u_2 + z_1 y_2 + x_1 y_2^2/2, v_1 + v_2 + 2x_1 z_2 + x_1^2 y_2)$$

becomes a nilpotent Lie group with the standard left invariant frame

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + z\frac{\partial}{\partial u} + x^2\frac{\partial}{\partial v},$$
$$X_3 = [X_1, X_2], \quad X_4 = [X_1, X_3], \quad X_5 = [X_2, X_3].$$

Set $D = \mathrm{span}(X_1, X_2)$, $\langle X_i, X_j\rangle = \delta_{ij}$. The growth vector is $(2, 3, 5)$ and $\mathbb{R}^5$ is a Carnot group.
Remark.. A Cartan distribution is a 2–dimensional distribution on a 5–manifold with growth vector $(2, 3, 5)$ [4]. If we consider two solid bodies in $\mathbb{R}^3$ that roll without slipping and twisting, the main questions are as follows: what motions of the bodies are realized and which of them are optimal in a certain sense? The geometric model for this problem is a Cartan distribution [3]. The authors of [3] construct a canonical nilpotent approximation (nilpotentization) and express extremals of the corresponding optimal control problem *via* elliptic functions. But the geometric model is not flat.
Problem.. When is a Cartan distribution D flat, *i.e.* isomorphic to a G_q, where q is a regular point of D? In [1], I begin the analysis of this problem in terms of G–structures (Cartan's method of equivalence).

4. Minimizers on Lie groups

Consider now a Lie group G with left invariant distribution $D \subset TG$, $TG \cong G \times g$. We shall identify every left invariant distribution with a subspace of the Lie algebra g of G. More specifically, $D \cong G \times D_e$ and its annihilator is $D^\perp \cong G \times D_e^\perp$. The identifications are made through left translations, e denotes the identity element of G. We choose a left invariant frame $\{e_1, \ldots, e_r\}$ for D and complete it to a basis $\{e_1, \ldots, e_n\}$ of g. Let $(\theta_1, \ldots, \theta_n)$ be the dual coframe. We write the vector field in G as $\sum_i \gamma_i e_i$ where $\gamma_1, \ldots, \gamma_n$ are the coordinate functions on the fiber of TG. The structure constants C_{ij}^k of g are defined by $[e_i, e_j] = \sum_k C_{ij}^k e_k$ and $\{H_{e_i}, H_{e_j}\} = \sum_k C_{ij}^k H_{e_k}$ (Poisson brackets of functions on g^*).

The abnormal equations Let ω be the canonical 1–form on T^*G and $\Omega = d\omega$. Let $\Gamma(t) = (x(t), \lambda(t)) \in T^*G - \{0\}$ where $\lambda(t) = \sum_{i=1}^n \lambda_i(t)\theta_i \in g^*$, $x'(t) = \sum_{i=1}^r \gamma_i(t)e_i$ ($x(t)$ is a D–curve). $x(t)$ is abnormal if and only if $\lambda(t) \neq 0$, $\lambda(t) \in D^\perp$ and $\Omega(\dot\lambda(t), \psi) = 0$ for all $\psi \in T_{\lambda(t)}D^\perp$. We obtain the equations:

$$(I)\begin{cases} \lambda_1(t) = \cdots = \lambda_r(t) = 0 \\ \sum_{j=1}^r \sum_{k=r+1}^n C_{ij}^k \gamma_j(t)\lambda_k(t) = 0 \quad (i = 1, \ldots, r) \\ \lambda_i'(t) = -\sum_{j=1}^r \sum_{k=r+1}^n C_{ij}^k \gamma_j(t)\lambda_k(t), \quad (i = r+1, \ldots, n). \end{cases}$$

The normal equations Let $(e_1, \ldots, e_r, \ldots, e_n)$ be a basis of g, $D = \mathrm{span}(e_1, \ldots, e_r)$, $\langle e_i, e_j\rangle = \delta_{ij}$ $(1 \leq i, j \leq r)$, $H = -\dfrac{1}{2}\sum_{i=1}^r H_{e_i}^2$ the SR–Hamiltonian. Let $x(t)$ be a D–curve, $x'(t) = \sum_{i=1}^r \gamma_i(t)e_i$ and $\Gamma(t) = (x(t), h(t)) \in T^*G - \{0\}$ where $h(t) = \sum_{i=1}^n h_i(t)\theta_i$. A normal curve is the projection onto G of a solution of the Hamiltonian system in T^*G with Hamiltinian H (here $h_i = H_{e_i}$). We can write the Hamiltonian system

$$(II)\left\{\begin{array}{lll} \gamma_i(t) & = -h_i(t) & (i = 1, \ldots, r), \\ h_i'(t) & = -\sum_{j=1}^r \sum_{k=1}^n C_{ij}^k h_j h_k & (i = 1, \ldots, n). \end{array}\right.$$

Some results..
1) From (I) and (II), we obtain the following results.
a) Take the Engel algebra $g = \mathrm{span}(e_1, e_2, e_3, e_4)$, $D = (e_1, e_2)$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$. Then the only abnormal curves are tangent to e_2. They are also normal.
b) Let $g = \mathrm{span}(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)$ with the following relations:

$$\begin{array}{llll} [e_1, e_2] = e_3, & [e_1, e_3] = e_4, & [e_2, e_3] = e_5, & \\ & [e_1, e_4] = e_6, & [e_1, e_5] = e_7, & [e_2, e_4] = e_7, \quad [e_2, e_5] = e_8. \end{array}$$

g is nilpotent 4–step and D is flat (the Lie group of g is a Carnot group). Any curve obtained by integrating $x'(t) = -\sin te_1 + \cos te_2$ is not left invariant, but $x(t)$ is both normal and abnormal.
2) Consider now $(G, D, \langle ., .\rangle)$ as a sub–Riemannian manifold, where G is a Lie group and D (resp. $\langle ., .\rangle$) a left invariant distribution (resp. metric). For every Lie subgroup H of G, let $D_H = D \cap TH$ and let $\langle ., .\rangle_H$ be the restriction of $\langle ., .\rangle$ to D_H. Let c be an abnormal path in $(G, D, \langle ., .\rangle)$. If $c \subset H$ is a geodesic in G which is not an abnormal path of $(H, D_H, \langle ., .\rangle)$, c is normal in $(H, D_H, \langle ., .\rangle_H)$.

Lemma Let H be a Lie subgroup of G having the following properties: for each point $p \in H$, there exists locally
(i) a distribution $D_1 \subset D$ such that $D_1\Big|_H = D_H$,
(ii) a distribution D_2 containing the orthogonal of D_1 in D and transvers to H (*i.e.* $TG = TH \oplus D_2$ along H),
(iii) an orthonormal basis $X_1, \ldots, X_r$ of D_1 and a basis $Y_1, \ldots, Y_s$ of D_2 such that each $[X_i, Y_j]$ is tangent to D_2 along c.
Then each normal curve of $(H, D_H, \langle ., .\rangle_H)$ is normal for $(G, D, \langle ., .\rangle)$. In particular, a geodesic c of G having H as characteristic group is normal in G.

Theorem Let G be a 2–step nilpotent Lie group and D a subspace of g which is supplementary to the center $z(g)$. Let H be a Lie subgroup of G and h its Lie algebra. Suppose hat H has the following property: for all $X \in D_H = h \cap D$ and for all Y in the orthogonal of D_H, $[X, Y]$ is orthogonal to h. Then each normal geodesic of $(H, D_H, \langle ., .\rangle_H)$ is a normal geodesic of $(G, D, \langle ., .\rangle)$.

Proof. Let D^0 be the orthogonal of D in g. We suppose that $\langle ., .\rangle$ is such that $D^0 = z(g)$. Set $D_1 = (h + D)^0$ and $D_2 = D_H^0 + D_1$. Then $[X, Y] \in h^0 \cap z(g)$. The distributions D_1 and D_2 satisfy the hypothesis of the Lemma. ∎

3) On $\mathbb{R}^6$, consider the vector fields

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2}+x_1\frac{\partial}{\partial x_5},\quad X_3=\frac{\partial}{\partial x_3}+x_1\frac{\partial}{\partial x_6},\quad X_4=\frac{\partial}{\partial x_4}+x_2\frac{\partial}{\partial x_5}.$$

Let $X_5=[X_1,X_2]$, $X_6=[X_1,X_3]$ and $g=\operatorname{span}(X_1,X_2,X_3,X_4,X_5,X_6)$; g is a 2–step nilpotent Lie algebra. Consider the left invariant distribution $D=\operatorname{span}(X_1,X_2,X_3,X_4)$. Endow this distribution with the metric such that (X_1,X_2,X_3,X_4) is orthonormal. The fields X_1,X_2,X_5 define a Lie subalgebra h whose Lie subgroup H is isomorphic to the Heisenberg group. In the subgroup H, we consider the induced structure $(H,D_H,\langle .,.\rangle_H)$. From Equations (I) and (II), the normal geodesics in H are not geodesics in $(G,D,\langle .,.\rangle)$.
4) We give an example of a strictly abnormal curve in the Carnot group which is contained in no proper subgroup. Let $g=\operatorname{span}(e_1,e_2,e_3,e_4,e_5,e_6)$ be defined by the relations

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=e_4,\quad [e_2,e_3]=e_5,\quad [e_1,e_4]=e_6.$$

Let $D=\operatorname{span}(e_1,e_2)$. D is flat. Let G be the Lie group associated with g. G is a Carnot group. We equip D with the metric that makes the frame (e_1,e_2) orthonormal. We are looking for an abnormal curve, say $x(t)=(x_1(t),\dots,x_6(t))$ which cannot be normal for (G,D). Let $x'(t)=\gamma_1(t)e_1+\gamma_2(t)e_2$. The abnormal equations (I) are

$$(I)\left\{\begin{array}{l}\gamma_2\lambda_3=0,\quad -\gamma_1\lambda_3=0,\quad \lambda_3'-\gamma_1\lambda_4-\gamma_2\lambda_5=0,\\ \lambda_4'-\gamma_1\lambda_6=0,\quad \lambda_5'=\lambda_6'=0=\lambda_1=\lambda_2.\end{array}\right.$$

Here $\lambda_3=0$. We shall assume that $x(t)$ is parametrized by arc length, *i.e.* $\gamma_1^2+\gamma_2^2=1$. We look for a solution $x(t)$ with λ_5 and λ_6 nonzero. From

$$\gamma_1=\frac{\lambda_4'}{\lambda_6},\quad \gamma_2=\gamma_1\frac{\lambda_4}{\lambda_5}=-\frac{\lambda_4\lambda_4'}{\lambda_5\lambda_6}\text{ and }\gamma_1^2+\gamma_2^2=1,$$

we get

$$(\lambda_4')^2(\lambda_4^2+\lambda_5^2)=\lambda_5^2\lambda_6^2. \tag{1}$$

Proposition Let λ_4 be a solution of (1) with λ_5 and λ_6 nonzero. Then any solution to $x'(t)=\gamma_1(t)e_1+\gamma_2(t)e_2$ with $x(0)=0$, where γ_1,γ_2 satisfy $\gamma_1=\dfrac{\lambda_4'}{\lambda_6}$, $\gamma_2=-\dfrac{\lambda_4\lambda_4'}{\lambda_5\lambda_6}$, is a strictly abnormal minimizer. Such a curve cannot be normal in any subgroup of G.

In this example, the normal equations are

$$(II)\ \gamma_1'+\gamma_2h_3=0,\ \gamma_2'-\gamma_1h_3=0,\ h_3'-\gamma_1h_4-\gamma_2h_5=0,\ h_4'-\gamma_1h_6=0,\ h_5'=h_6'=0.$$

Let $x(t)$ be an abnormal curve which is also normal, parametrized by arc length, $x(t)=\gamma_1(t)e_1+\gamma_2(t)e_2\in D$. If $\lambda_5,\lambda_6\neq 0$ we get a contradiction $(\lambda_4'=0)$. Then $\lambda_5=0$ or $\lambda_6=0$. From Equation (1), λ_4' must be also zero and therefore λ_4 is constant. This gives the linear equation $\gamma_1\lambda_4+\gamma_2\lambda_5=0$ (with λ_4,λ_5 being constant). But $\lambda_4\neq 0$ and $\gamma_1^2+\gamma_2^2=1$ implies that γ_1 and γ_2 are constant. *Then $x(t)$ is a left invariant curve.*

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