Decomposition of stochastic flows and rotation matrix

Paulo R. C. Ruffino[1]
Departamento de Matemática, Universidade Estadual de Campinas, 13.081-970 - Campinas - SP, Brazil. e-mail: ruffino@ime.unicamp.br

Abstract

We provide geometrical conditions on the manifold for the existence of the Liao's factorization of stochastic flows [9]. If M is simply connected and has constant curvature then this decomposition holds for any stochastic flow, conversely, if every flow on M has this decomposition then M has constant curvature. Under certain conditions, it is possible to go further on the factorization: $\varphi_t = \xi_t \circ \Psi_t \circ \Theta_t$, where ξ_t and Ψ_t have the same properties of Liao's decomposition and $(\xi_t \circ \Psi_t)$ are affine transformations on M. We study the asymptotic behaviour of the isometric component ξ_t via rotation matrix, providing a Furstenberg-Khasminskii formula for this skew-symmetric matrix.

Key words: stochastic differential equations, group of affine transformations, isometries, decomposition of flows, rotation matrix.

MSC2000 subject classification: 60D05, 53C21, (60J60).

1 Introduction

Factorization of non-linear flows in a Riemannian manifold into components which lies in specific subgroups of the group of diffeomorphisms is not only interesting by itself, but also relevant in many aspects. In particular, in the study of the long time behaviour of systems: some interesting asymptotic parameters arise from the long time behaviour of each of these components. This is the contend of Ming Liao's paper [9] where he establishes a decomposition of the flow on compact manifolds with focus on Lyapunov exponents. Here, complementing this asymptotic radial analysis, we go further on his decomposition, but with focus on the asymptotic behaviour of the angular (isometric) component.

[1]Currently a visiting fellow at the Mathematical Institute, University of Oxford, UK. Research supported by FAPESP grant no. 00/04591-3.

Liao's decomposition is a kind of Iwasawa decomposition for flows in the following sense: Let φ_t be the stochastic flow associated to a certain stochastic differential equation (sde) on a Riemannian manifold M. Fixed an orthonormal frame u in the tangent space at x_o, if the vector fields involved in the sde satisfies certain geometrical properties, then there exists a (unique) decomposition $\varphi_t = \xi_t \circ \Psi_t$, where ξ_t lies in the group of isometries, $\Psi_t(x_0) \equiv x_0$ and the derivative $D\Psi(u) = u\, s_t$ where s_t is an upper triangular matrix process.

In this paper we consider global geometrical conditions on the manifold instead of on the vector fields of the sde. We show that if the manifold M is simply connected and has constant curvature then Liao's decomposition holds for any stochastic flow. Conversely, if every flow on M has a Liao's decomposition then M has constant curvature. We also prove that under certain hypothesis it is possible to go further on his decomposition, factorizing the flow into three components:

$$\varphi_t = \xi_t \circ \Psi_t \circ \Theta_t,$$

with ξ_t as above, Ψ_t satisfying the same derivative property as above, $(\xi_t \circ \Psi_t)$ is a process in the group of affine transformations of M, $\Theta_t(x_0) \equiv x_0$ and the derivative $D\Theta_t \equiv Id_{(T_{x_0}M)}$.

Along this article we shall consider the following Stratonovich sde on a connected complete Riemannian manifold M:

$$d\,x_t = \sum_{j=0}^{m} X^j(x_t) \circ dW_t^j \qquad (1)$$

where $X^0, \ldots, X^m$ are (smooth) vector fields, $W_t^0 = dt$ and $(W^1, \ldots, W^n)$ is a Brownian motion on $\mathbb{R}^n$ with respect to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}_t$. The associated stochastic flow will be denoted by φ_t. Once we deal with asymptotic properties of this flow, in order to avoid finite explosion time, we shall assume that the derivatives of the vector fields are bounded (see e.g. Kunita [6] or [7]).

Next section provides basic results which are going to be mentioned in the following sections. Section 3 presents the decompositions of flows we mention above. The main results of this section concern geometrical conditions on the manifold instead of on the vector fields of the sde. We close this section with an example in the hyperbolic space H^n. Finally, in Section 4 we present the concept of rotation matrix and find a Furstenberg-Hasminskii formula for this skew-symmetric matrix (rather neat, compare to the one in [12]).

2 Geometric preliminaries

We shall denote the linear frame bundle over a d-dimensional smooth manifold M by $GL(M)$. It is a principal bundle over M with structural group $Gl(n,\mathbb{R})$. A Riemannian structure on M is determined by a choice of a subbundle of orthonormal frames OM with structural subgroup $O(n,\mathbb{R})$. We shall denote by $\pi : GL(M) \to M$ and by $\pi_o : O(M) \to M$ the projections of these frame bundles onto M. The canonical Iwasawa decomposition established by the Gram-Schmidt orthonormalization in the elements of a frame $u = (u^1, \ldots, u^d)$ defines a projection $\perp: u \to u^{\perp} \in OM$ such that $GL(M)$ is again a principal bundle over OM with structural group $S \subset Gl(d,\mathbb{R})$, the subgroup of upper triangular matrices. The principal bundles described above factorize as $\pi = \pi_o \circ \perp$.

We shall consider the Levi-Civita connection. We recall that for a frame u in $GL(M)$ a connection Γ determines a direct sum decomposition of the tangent spaces at u into a horizontal and vertical subspaces which will be denoted by $T_uGL(M) = HT_uGL(M) \oplus VT_uGL(M)$. Analogous decomposition holds in the tangent bundle $TOM \subset T\,GL(M)$. For $k \in OM$, we have that $HT_kOM = HT_kGL(M)$. Given a vector field X on M, we denote its horizontal lift to $GL(M)$ by $HX(u) \in T_uGL(M)$.

The covariant derivative of a vector field X at x denoted by $\nabla X(x) : T_xM \to T_xM$ will be written simply $\nabla X(Y)$ or $\nabla_Y X$ for a vector $Y \in T_xM$. Via adjoint, we can associate to ∇X an element in the structural group $Gl(n)$ of the principal bundle $GL(M)$ given by the matrix $\tilde{X}(u) = \mathrm{ad}(u^{-1})\nabla X$, which acts on the right such that $\nabla X(u) = u\tilde{X}(u)$. Note that, different from ∇X, the right action matrix $\tilde{X}(u)$ does depend on u.

The natural lift of X to $GL(M)$ is the unique vector field δX in $GL(M)$ such that $L_{\delta X(u)}\theta = 0$, where θ is the canonical $\mathbb{R}^d$-valued form on $GL(M)$ defined by $\theta(Hu(\zeta)) = \zeta$ for all $\zeta \in \mathbb{R}^d$. This natural lift is given by:

$$\delta X(u) = \frac{d}{dt}[D\,\eta_t(u)]|_{t=0}.$$

where $D\eta_t$ is the derivative of the local 1-parameter group of diffeomorphisms η_t associated to the vector field X. Naturally, δX is equivariant by the right action of $Gl(d,\mathbb{R})$ in the fibres.

Lemma 2.1 *The projection $\perp$: $GL(M) \to OM$ is invariant for the linearised flow, in the sense that, for all $u \in GL(M)$,*

$$(D\eta_t(u))^{\perp} = (D\eta_t(u^{\perp}))^{\perp}. \tag{2}$$

Proof:

This is a consequence of the commutativity of the right action of $Gl(d,\mathbb{R})$ (in particular, in this case, the action of S) on $GL(M)$ with any other linear left actions (in particular, in this case, the linearised flow). In fact, consider the Iwasawa decomposition $u = u^{\perp} \cdot s_{(u)}$ for some $s_{(u)} \in S$. Hence,

$$D\eta_t(u^{\perp} \cdot s_{(u)}) = (D\eta_t u^{\perp}) \cdot s_{(u)} = (D\eta_t(u))^{\perp} \cdot s_{(D\eta_t(u))}.$$

Equality (2) follows by the uniqueness of the Iwasawa decomposition.

□

The vertical component of $V\delta X(u)$ at $u \in \pi^{-1}(x_0)$ is given by the covariant derivative $\nabla X(u)$ (see e.g. Elworthy [3], or Kobayashi and Nomizu [5]). At the Lie algebra level, consider the canonical Cartan decomposition of matrices $\mathcal{G} = \mathcal{K} \oplus \mathcal{S}$ into a skew-symmetric and upper triangular component respectively. By projecting in each of these two components, we write $\tilde{X}(u) = [\tilde{X}(u)]_{\mathcal{K}} + [\tilde{X}(u)]_{\mathcal{S}}$. With this notation, the vertical component splits into:

$$V\delta X(u) = u[\tilde{X}(u)]_{\mathcal{K}} + u[\tilde{X}(u)]_{\mathcal{S}}. \tag{3}$$

The natural lift of X to OM, denoted by $(\delta X)^{\perp}$ is the projection of δX onto OM, i.e. for $k \in OM$,

$$(\delta X)^{\perp}(k) := \frac{d}{dt}\, [D\,\eta_t(k)]^{\perp}\, |_{t=0}.$$

Again, we have the decomposition of $(\delta X)^{\perp}(k)$ into horizontal and vertical components: $(\delta X)^{\perp}(k) = H\delta X(k) + V(\delta X)^{\perp}(k)$. In terms of the right action of $\tilde{X}(k)$, the vertical component is simply $V(\delta X)^{\perp}(k) = k[\tilde{X}(k)]_{\mathcal{K}}$. In terms of the left action of (∇X) we shall denote $V(\delta X)^{\perp}(k) = (\nabla X(k))^{\perp}k$, where $(\nabla X(k))^{\perp}$ is a skew-symmetric matrix. The characterisation of $(\nabla X(k))^{\perp}$ is the contend of the following lemma.

Lemma 2.2 *Let $k = (k^1, \ldots, k^d) \in OM$ with $\pi_o(k) = x$. The image of the j-th component k^j by the matrix $(\nabla X(k))^{\perp}$ is given by:*

$$(\nabla X(k))^{\perp}k^j = \nabla X(k^j) - \langle \nabla X(k^j), k^j\rangle k^j - \sum_{r<j} \left(\langle \nabla X(k^r), k^j\rangle + \langle \nabla X(k^j), k^r\rangle\right) k^r\ .$$

Proof:

It is the non-linear version of [12, Prop. 2.1]. For reader's convenience we re-write the main steps of the proof. If $t \in \mathbb{R} \longmapsto V_t \in \mathbf{R}^d$ is differentiable with $V_t \neq 0$ for all $t \in (-\epsilon, \epsilon)$, then:

$$\frac{d}{dt}\left(\frac{V_t}{\|V_t\|}\right)_t = \frac{\dot{V}_t}{\|V_t\|} - \frac{\langle \dot{V}_t, V_t\rangle}{\|V_t\|^3} V_t\ , \tag{4}$$

where $\dot{V}_t$ is the derivative of V_t.

For sake of simplicity, fix a basis in T_xM and denote by A the matrix which represents the linear transformation $\nabla X(x)$. Now, use formula (4) with $t = 0$ in each coordinate of

$$(e^{At}(k))^{\perp} = \left(\frac{V_t^1}{\|V_t^1\|}, \ldots, \frac{V_t^n}{\|V_t^n\|} \right) ,$$

where each component of the orthogonalisation process is given by:

$$V_t^j = e^{At}(k^j) - \sum_{r<j} \frac{\langle e^{At}(k^j) \ , \ e^{At}(k^r) \rangle}{\langle e^{At}(k^r) \ , \ e^{At}(k^r) \rangle} \ e^{At} \ (k^r) \ .$$

The derivatives satisfy:

$$\left. \frac{dV_t^j}{dt} \right|_{t=0} = A(k^j) - \sum_{r<j} \left(\langle A(k^j), k^r \rangle + \langle A(k^r), k^j \rangle \right) k^r \ ,$$

which gives, by formula (4):

$$\frac{d}{dt} \left(\frac{V_t^j}{\|V_t\|} \right)_{t=0} = A(k^j) - \langle A(k^j), k^j \rangle k^j - \sum_{r<j} \left(\langle A(k^r), k^j \rangle + \langle A(k^j), k^r \rangle \right) k^r \ .$$

□

One sees the skew-symmetry of $(\nabla X(k))^{\perp}$ checking that

$$< (\nabla X(k))^{\perp} k^j, k^i >= - < (\nabla X(k))^{\perp} k^i, k^j > .$$

2.1 Affine transformations and isometries

Let Diff(M) be the group of diffeomorphisms of the Riemannian manifold M given by the exponential of the Lie algebra $\mathcal{X}(M)$ of smooth, bounded derivative vector fields. We shall denote by $A(M)$ the Lie group of affine transformations of M whose elements are given by maps $\Delta \in$ Diff(M) such that their derivatives $D\Delta$ preserve horizontal trajectories in TM. Its Lie algebra $a(M)$ is the set of infinitesimal affine transformations characterised by vector fields X such that the Lie derivative of the connection form ω on $GL(M)$ satisfies $L_{\delta X}\omega = 0$. Yet, X is an infinitesimal transformation if for all vectors fields Y:

$$\nabla A_X(Y) = R(X, Y),$$

where the tensor $A^X = L_X - \nabla_X$ and R is the curvature (see e.g. Kobayashi and Nomizu [5, Chap. VI]).

For a fixed $u \in GL(M)$, the linear map

$$\begin{aligned} i_1 : a(M) &\rightarrow T_u GL(M) \\ X &\mapsto \delta X(u) \end{aligned} \tag{5}$$

is injective, see e.g. Kobayashi and Nomizu [5, Thm VI.2.3]. Denote by $\delta a(u)$ its image in $T_u GL(M)$.

We shall denote by $I(M)$ the Lie group of isometries of M, $I(M) \subset A(M)$. Its Lie algebra $\mathcal{I}(M)$ is the space of Killing vector fields or infinitesimal isometries, characterised by the skew-symmetry of the covariant derivative, i.e.,

$$< \nabla X(Z), W >= - < Z, \nabla X(W) >,$$

for all vectors Z, W in a tangent space $T_x M$. Note that, in this case, by Lemma 2.2 (up to change of basis), for any orthonormal frame u we have that $(\nabla X(u))^{\perp} = \nabla X$ and $(\delta X)^{\perp}(u) = \delta X(u)$.

For a fixed $u \in OM$, the linear map

$$\begin{aligned} i_2 : \mathcal{I}(M) &\rightarrow T_u OM \\ X &\mapsto \delta X(u) \end{aligned} \tag{6}$$

is just a restriction of the map i_1 defined above, hence it is also injective. Denote by $\delta \mathcal{I}(u)$ its image in $T_u OM$.

3 Decompositions of Stochastic Flows

Next theorem presents a factorization of the stochastic flow φ_t associated to the sde (1) into a component Δ_t which is a diffusion on the affine transformation group and a component Θ_t which fix the initial point and has trivial derivative. Fix an element $u \in GL(M)$ with $\pi(u) = x_0$, we shall assume the following hypothesis on the vector fields X^i, $i = 0, 1, \ldots, m$ involved in the sde (1):

H1) $\delta D\Delta(X^i)(u) \in \delta a(u)$, for all affine transformation $\Delta \in I(M)$.

Theorem 3.1 *Suppose the vector fields $X^1, \ldots, X^m$ in the sde (1) satisfy the hypothesis (H1) for a certain frame $u \in GL(M)$, with $x_0 = \pi_o(u)$. Then, the associated stochastic flow φ_t has a unique decomposition:*

$$\varphi_t = \Delta_t \circ \Theta_t,$$

where Δ_t is a diffusion in the group of affine transformations $A(M)$, $\Theta_t(x_0) = x_0$ and $D\Theta_t = Id_{(T_{x_0} M)}$ for all $t \geq 0$.

Proof:

The proof goes in a similar way of the proof of the M. Liao decomposition [9, Thm. 1] or see Theorem 3.2 below. Once the linear map i_1 of equation (5) is injective, let X^a be the unique infinitesimal affine transformation which satisfies $\delta X^a(u) = \delta X(u)$. Hence, obviously, $X^a(x_0) = X(x_0)$ and $\nabla X^a(x_0) = \nabla X(x_0)$.

Let Δ_t be the solution of the following equation in the group $A(M)$, with $\Delta_0 = Id_M$:

$$d\Delta_t = \sum_{j=0}^{n} \Delta_t [D\Delta_t^{-1}(X^j)]^a \ \circ dW_t^j, \tag{7}$$

where the elements in the Lie algebra $a(M)$ acts on the right in $A(M)$. By Itô formula in the identity $\Theta_t \Theta_t^{-1} = Id_M$ one easily finds the Stratonovich equation for the inverse Δ_t^{-1} in $A(M)$:

$$d\,\Delta_t^{-1} = -\sum_{j=0}^{m} [D\Delta_t^{-1}(X^j)]^a \Delta_t^{-1} \ \circ dW_t^j.$$

Now, write $\Theta_t = \Delta_t^{-1} \circ \varphi_t$. At the level of the group of diffeomorphism on M, by Itô formula again, we have the following equation for Θ_t:

$$\begin{aligned} d\,\Theta_t &= D\Delta_{-1}(\circ d\varphi_t) + (\circ d\Delta_t^{-1})\varphi_t \\ &= \sum_{j=0}^{m} \{D\Delta_t^{-1}(X^j(\varphi)) - [D\Delta_t^{-1}(X^j)]^a \Theta_t\} \ \circ dW_t^j \\ &= \sum_{j=0}^{m} \{D\Delta_t^{-1} X^j - [D\Delta_t^{-1}(X^j)]^a\}(\Theta_t) \ \circ dW_t^j. \end{aligned} \tag{8}$$

In the last line we use the fact that, in a Lie group, the derivative of the left action $L_g(X)(h) = L_g(X(g^{-1}h))$.

By definition of X^a and equation (8) we have that, not only $d\,\Theta_t(x_0) = 0$ but also that $\delta\{D\Delta_t^{-1} X^j - [D\Delta_t^{-1}(X^j)]^a\}(\Theta_t) = 0$, hence the derivative process $dD\Theta_t(u) = 0$. This establishes the properties of each component of the factorization of $\varphi_t = \Delta_t \circ \Theta_t$ stated in the theorem. Note that, in general, Θ_t is not a diffusion in Diff(M) once its equation involves random and time dependent vector fields.

It only remains to prove the uniqueness of the decomposition. Suppose that $\Delta'_t \circ \Theta'_t = \Delta_t \circ \Theta_t$ where Δ'_t and Θ'_t also satisfy the properties stated. It implies that $\Delta_t^{-1}\Delta'_t(x_0) = x_0$ for all $t \geq 0$. Besides, the derivative $D_{x_0}(\Delta_t^{-1}\Delta'_t) = Id$, hence the natural lift to $GL(M)$ satisfies the differential equation $d\,D(\Delta_t^{-1}\Delta'_t) = 0$. Once the map i_1 is injective, it follows that $\Delta_t^{-1} \circ \Delta'_t = Id_M$, which guarantees the uniqueness of the decomposition.

□

Note that the affine transformation process Δ_t depends on the choice of the initial frame u. In the proof of the theorem, this dependence appears in the selection of the unique $X^a \in a(M)$ such that $\delta X^a(u) = \delta X(u)$.

Now, fix an element $u \in OM$. M. Liao [9] consider the following hypothesis on the vector fields $X^1, \ldots, X^m$ of the sde (1):

H2) $[\delta D\xi(X^i)(u)]^\perp \in \delta\mathcal{I}(u)$ for every isometry $\xi \in I(M)$.

Theorem 3.2 (M. Liao [9], Thm. 1) *Suppose that for a certain frame* $u \in GL(M)$, *with* $x_0 = \pi_o(u)$, *the vector fields* $X^0, \ldots, X^m$ *of the sde (1) satisfy the hypothesis (H2). Then, the associated stochastic flow* φ_t *has a unique decomposition:*

$$\varphi_t = \xi_t \circ \Psi_t,$$

where ξ_t *is a diffusion in the group of isometries* $I(M)$, $\Psi_t(x_0) = x_0$ *and* $D_{x_0}\Psi_t(u) = u\, s_t$ *for all* $t \geq 0$, *where* s_t *is a process in the group of upper triangular matrices.*

Proof:

For details, see [9], we just recall the equation for the diffusion ξ_t, which is the component whose asymptotic behaviour we are going to explore in the next section.

Once the linear map i_2 of equation (6) is injective, we can take X^i, the unique infinitesimal isometry which satisfies $\delta X^i(u) = (\delta X)^\perp(u)$. Hence, obviously, $X^i(x_0) = X(x_0)$.

The isometry process ξ_t satisfies the following equation in the group $I(M)$, with $\xi_0 = Id_M$:

$$d\xi_t = \sum_{j=0}^{n} \xi_t [D\xi_t^{-1}(X^j)]^i \circ dW_t^j, \tag{9}$$

where the elements in the Lie algebra $\mathcal{I}(M)$ acts on the right in $I(M)$.

The uniqueness follows easily also from the fact that the map i_2 is injective (cf. proof of uniqueness in Theorem 3.1).

□

We remark that his proof holds for non-compact manifolds as well. We also emphasise that the decomposition depends on the initial orthonormal frame $u \in OM$.

Now, juxtaposing the decompositions established by Theorems 3.1 and 3.2, we have the following factorization of φ_t in three components:

Corollary 3.3 *Suppose the vector fields $X^0, \ldots, X^m$ in the sde (1) satisfy conditions (H1) and (H2) for a certain frame $u \in OM$, with $x_0 = \pi_o(u)$. Then, the associated stochastic flow φ_t has a unique decomposition:*

$$\varphi = \xi_t \circ \Psi_t \circ \Theta_t,$$

where each of the components ξ_t, Ψ_t, Θ_t have the properties stated in Theorems 3.1 and 3.2. Moreover, $(\xi_t \circ \Psi_t)$ is a diffusion in the group of affine transformations.

Proof:

By Theorem 3.1, let $\varphi_t = \Delta_t \circ \Theta_t$ be the unique decomposition where Δ_t is a diffusion in the group of affine transformations $A(M)$, $\Theta_t(x_0) = x_0$ and $D\Theta_t = Id_{T_{x_0}M}$ for all $t \geq 0$.

On the other hand, by Theorem 3.2, let $\varphi_t = \xi_t \circ \tilde{\Psi}_t$ be the unique decomposition where ξ_t is the diffusion in the group of isometries $I(M)$, $\tilde{\Psi}_t(x_0) = x_0$ and $D_{x_0}\tilde{\Psi}_t(u) = u\,\tilde{s}_t$ for a certain process $\tilde{s}_t$ in the group of upper triangular matrices.

Take the process ξ_t and Θ_t of the statement of this corollary as defined above. Define the process $\Psi_t = \xi_t^{-1}\Delta_t$. These assignments define the decomposition.

It only remains to prove the derivative property of Ψ_t, namely, that there exists a process on the group of upper triangular matrices such that $D\Psi_t(u) = u\,s_t$. By the properties asserted:

$$\begin{aligned} D\varphi_t(u) &= D\xi_t \circ D\Psi_t(u) \\ &= D\xi_t \circ D\tilde{\Psi}_t(u) \\ &= D\xi_t(u)\tilde{s}_t. \end{aligned}$$

Hence, the upper triangular matrix process $s_t = \tilde{s}_t$. Which confirms the expected fact that although, in general Ψ_t is different from $\tilde{\Psi}_t$, they have the same derivative behaviour (which carries the Lyapunov information of the system).

□

3.1 Geometrical conditions on M instead of on the vector fields:

In Liao [9] he presents an example of application of his decomposition in the sphere S^n. The purpose of this section is to characterise the spaces whose vector fields will always satisfy the conditions for the decompositions described above. It turn out that the Liao decomposition exists for any sde on M only if M has constant curvature; in particular, the further decomposition of Corollary 3.3 exists for any sde only if M is a flat space. More precisely, we have the following:

Theorem 3.4 *If M is simply connected with constant curvature (or their quotient by discrete groups) then for every sde (1) and every orthonormal frame $u_0 \in OM$, the associated stochastic flow φ_t has a unique Liao decomposition $\varphi_t = \xi_t \circ \Psi_t$. Conversely, if every flow φ_t on M has this decomposition then the space M has constant curvature.*

Proof:

If M has constant curvature and is simply connected one checks directly that the dimension of $\mathcal{I}(M)$ is maximal $n(n+1)/2$. Hence the linear map i_2 defined in equation (6) is bijective. Therefore, hypothesis (H2) is always satisfied for any set of vector fields.

Conversely, assume that for all vector field X and for every orthonormal frame $u \in OM$, the corresponding flow η_t has the Liao decomposition $\eta_t = \xi_t \circ \Psi_t$. Then, the trajectory u_t in OM induced by η_t satisfies:

$$
\begin{aligned}
u_t := [D\,\eta\,(u)]^{\perp} &= [D\xi_t \circ D\Psi_t(u)]^{\perp} \\
&= D\xi_t(u).
\end{aligned}
$$

We recall that

$$
\frac{d}{dt}\,(D\xi_t(u))\,|_{t=0} = (\delta X)^{\perp}(u) \tag{10}
$$

For any fixed $u \in GL(M)$, the linear map $\mathcal{X} \to T_u GL(M)$ given by $X \mapsto \delta X(u)$ is surjective because it concerns only local behaviour of X on M. Hence, the projection of its image by $\perp$: $T_u GL(M) \to T_{u^{\perp}} OM$ is also surjective. In other words, if now $u \in OM$, then $X \mapsto (\delta X)^{\perp}(u)$ is surjective. If there exists the decomposition, equality (10) shows that the dimension of $\mathcal{I}(M)$ equals $n(n+1)/2$ which implies that M has constant curvature (see, e.g. Kobayashi and Nomizu [5, Thm. VI.3.3]).

$\square$

As a particular case of the theorem above, we have the following conditions on M which guarantee that every sde on it will have a flow which factorizes in the three components stated in Corollary 3.3.

Corollary 3.5 *If M is flat, simply connected (or their quotient by discrete groups) then for every sde (1) and every orthonormal frame $u \in OM$, the* *associated stochastic flow φ_t has a unique decomposition $\varphi = \xi_t \circ \Psi_t \circ \Theta_t$ as described in Corollary 3.3. Conversely, if every flow φ_t have this decomposition then M is flat.*

Proof:

If M is flat and simply connected then, checking directly, we have that the dimensions of the groups $\mathcal{I}(M)$ and $A(M)$ are $n(n+1)/2$ and $n(n+1)$ respectively. This implies that the injective maps i_1 and i_2 are bijective, hence hypotheses (H1) and (H2) are satisfied for any set of vector fields on M.

Conversely, assume that for all vector field X and for every orthonormal frame $u \in OM$ the corresponding flow η_t has the decomposition $\eta_t = \xi_t \circ \Psi_t \circ \Theta_t$ with the properties asserted. Then, the trajectory u_t in $GL(M)$ induced by η_t satisfies:

$$u_t = D\Delta_t(u),$$

where $\Delta_t = \xi_t \circ \Psi_t$. We recall that

$$\frac{d}{dt} \left(D\Delta_t(u) \right) |_{t=0} = \delta X(u). \tag{11}$$

Again, for a fixed $u \in GL(M)$, the linear map $X \mapsto \delta X(u)$ is surjective because it concerns only local structure of X on M. Hence, equality (11) implies that the dimension of the group of affine transformations $A(M)$ equals $n(n+1)$, which implies that M is flat (see, e.g. Kobayashi and Nomizu [5, Thm. VI.2.3]).

□

3.2 Examples

Liao [9] illustrates his decomposition working out with a unique example in the sphere S^n. The results above enlarge the class of examples to many well known manifolds including projective spaces, hyperbolic manifolds, flat torus and many others non-compact manifolds. In this section we shall focus on the other two simply-connected case, namely the flat and hyperbolic spaces. We shall concentrate mainly on the isometric part ξ_t once this is the component which we are going to study the long time behaviour in the next section.

The Euclidean case is rather trivial, we just recall that $A(\mathbb{R}^d)$, the group of affine transformations in $\mathbb{R}^d$ (or any of its quotient space by discrete subgroup) can be represented as a subgroup of $Gl(d+1,\mathbb{R})$:

$$A(\mathbb{R}^d) = \left\{ \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix} \text{ with } g \in Gl(d,\mathbb{R}) \text{ and } v \text{ is a column vector} \right\}.$$

It acts on the left in $\mathbb{R}^d$ through its natural embedding on $\mathbb{R}^{d+1}$ given by $x \mapsto (1,x)$. Given a vector field X and an affine frame u, the calculations of the unique elements $X^a(u) \in a(\mathbb{R}^d)$ and $X^i(u) \in \mathcal{I}(\mathbb{R}^d)$ introduced in the proofs of Theorems 3.1 and 3.2 are straightforward.

Now, we shall consider a hyperbolic manifold obtained as a homogeneous space of the Lorentz group. For other examples (including hyperbolic spaces which do not satisfy our hypotheses) one can have a look in Ratcliffe [10] and the references therein. Let $S = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$. The Lorentz group $O(1,n)$ is the linear group of transformations of $\mathbb{R}^{n+1}$ which preserves the quadratic form $< Sx, x >$, where $< \cdot, \cdot >$ is the scalar product in $\mathbb{R}^{n+1}$. Once we are dealing with flows, we can restrict ourselves to the connected component of the identity which will be denoted by:

$$G = \{A \in O(1,n);\ \det A = 1 \ \text{ and } A_{11} \geq 1\}.$$

Its Lie algebra is $o(n,1) = \{A, n\times n\text{-matrix: } A^tS + SA = 0\}$ which turns out to be:

$$o(n,1) = \begin{pmatrix} 0 & v^t \\ v & B \end{pmatrix}$$

where v is a column vector and B is in the Lie algebra $so(n)$ of the orthogonal group $O(n)$. Our hyperbolic space will be $H^n = G/SO(n)$ which is diffeomorphic to:

$$H^n = \{x \in \mathbb{R}^{d+1}; < Sx, x >= -1 \text{ and } x_1 \geq 1\}.$$

The form $< v, w >_M = < Sv, w >$ defines a Riemannian metric on H^n by restricting it to its tangent spaces. With this metric, H^n is a hyperbolic manifold with constant curvature -1. Naturally, by construction, the action of G on H^n preserves this metric (see e.g. Klingenberg [4]). Given a vector field X on H^n, this example will describe the unique infinitesimal isometry X^i such that $\delta X^i(u) = (\delta X)^\perp(u)$.

Consider the point $N = (1, 0, \ldots, 0) \in H^n$ and an orthonormal frame u in $T_N M$. Denote $\partial_k = (\partial/\partial^k)$. A vector field $X(x) = a_1(x)\,\partial_1 + \ldots + a_{n+1}(x)\,\partial_{n+1}$ is tangent to $T_x H^n$ at $x = (x^1, \ldots, x^{n+1})$ if and only if $< SX(x), x >= 0$. From this expression one finds that $a_1(N) = 0$ and $\partial_k\, a_1(N) = a_k(N)$, for $k = 2, \ldots, n+1$. We recall that the map $p: G \to H^n$ given by gN determines a principal bundle with structural group $SO(n)$, hence, there is a natural identification of the oriented orthonormal bundle SOH^n with G, see e.g. Elworthy [3].

A convenient (global) parametrisation centred at N is given by the graphic of the map $x^1 = \sqrt{1+\sum_{j=2}^{n+1}(x^j)^2}$. One checks that with respect to this parametrization, the new basis for the tangent bundle is given by

$$\partial'_k = \left(\frac{x^k}{x^1}\right)\partial_1 + \partial_k,$$

with $k = 2, \ldots, n+1$. It is easy to verify that the entries of the metric tensor with respect to this parametrization is:

$$g_{ij} = \delta_{ij} - \frac{x^i x^j}{(x^1)^2},$$

where δ_{ij} are the Kronecker symbols. From this formula one sees that the Christoffel symbols vanishes at N. Moreover, for a vector field described with respect to the canonical basis in $\mathbb{R}^{n+1}$ as: $X(x) = a_1(x)\,\partial_1 + \ldots + a_{n+1}(x)\,\partial_{n+1}$, in the basis induced by the parametrization it has the same coefficients, but a_1:

$$X(x) = a_2(x)\,\partial'_2 + \ldots + a_{n+1}(x)\,\partial'_{n+1}.$$

We conclude that the covariant derivative of $X(x)$ at N with respect to these bases (which coincide at N) is simply $\nabla X = (\partial_j\, a_i(N))_{2\leq i,j\leq n+1}$.

The infinitesimal isometry $X^i(x)$ which we are looking for can be represented by an element A in the Lie algebra $so(1,n)$ such that $X^i(x) = \tilde{A}(x)$, where $\tilde{A}$ is the vector field on H^n induced by A. In what regards the horizontal component, A has to satisfy:

$$\frac{d}{dt}\,(e^{At}\,N)\,|_{t=0} = X(N). \tag{12}$$

And for the vertical component, it has to satisfy:

$$\frac{d}{dt}\left[e^{A}t\begin{pmatrix}0\\u\end{pmatrix}\right]_{t=0} = \begin{pmatrix}*\\(\nabla X(u))^{\perp}u\end{pmatrix}. \tag{13}$$

Therefore, from equations (12) and (13) we have:

$$A = \left(\begin{array}{c|ccc} 0 & a_2(N) & \ldots & a_{n+1}(N) \\ \hline a_2(N) & & & \\ \vdots & & ([\partial_j\, a_i](u))^{\perp} & \\ a_{n+1}(N) & & & \end{array}\right).$$

Note that, if u is the canonical basis in $T_N M$ them $([\partial_j\, a_i](u))^{\perp}$ is simply $[(\partial_j a_i)]_{\mathcal{K}}$.

4 Rotation matrix

Many relevant results concerning the asymptotic exponential radial behaviour of linearised random systems have been achieved since the introduction of the concept of Lyapunov exponents. In particular, the multiplicative ergodic theorem have been playing a fundamental role on the study of stability. There are hundreds of papers on the topic (we apologise in advance for omitting many outstanding contributions), I would rather suggest the reader to have a look, for example, in the comprehensive bibliography in the book of L. Arnold [1].

In contrast with the radial component, but complementing its information, in this section we study the asymptotic behaviour of the angular part: we shall consider the long time behaviour of the induced flow in the orthogonal bundle. Although it is easier to study this induced process using the decomposition of flows presented in the previous section (simply $u_t = D\xi_t(u)$), we will be working with this process, independently of the existence of this factorization.

As before, let φ_t be the solution of the sde (1). We recall that for an initial orthonormal frame $u \in OM$ with $\pi_o(u) = x_0$, the induced trajectory u_t in OM is given by the unceasing Gram-Schmidt orthonormalization of the linearised trajectory on $GL(M)$, that is $u_t = (D\varphi_t(u))^{\perp}$. Lemma 2.1 provides a direct way to verify that $(D\varphi_t(\cdot))^{\perp}$ is indeed a flow in OM.

Considering the right action of $\tilde{X}(u)$ and the Itô formula, the process u_t is the diffusion in OM associated to the following Stratonovich sde:

$$d\,u_t = \sum_{j=0}^{m} \{H\delta X^j(u_t) + u\,[\tilde{X}^j(u)]_{\mathcal{K}}\} \circ dW_t^j$$

The *matrix of rotation* of a given initial orthonormal frame is the asymptotic average of the canonical left invariant $so(d)$-value 1-form in the structural group $O(d)$ integrated along the trajectories in OM. For non-linear systems, we parallel transport back the frame u_t to the same initial fibre of u_0. In other words, considering only the vertical component of the process u_t, we have the following covariant sde:

$$D\,u_t = \sum_{j=0}^{m} u\,[\tilde{X}^j(u)]_{\mathcal{K}} \ \circ dW_t^j,$$

where Du_t means the covariant derivative $//_t^{-1} d //_t u_t$, with $//_t : T_{x_0}M \to T_{x_t}M$ denoting the parallel transport along the trajectories. (Along this section "D" will stand for the covariant derivative, from the context it will be clear the distinction between its two denotations). Then, the matrix of rotation is defined by the limit

$$\mathcal{R}(u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u_t^* \circ Du_t \ ,$$

when the limit exists. Still, in other words, this matrix measures the average rotation that the induced process u_t performs with respect to the parallel transport of the initial frame $//_t(u_0)$ inside the same tangent bundle $T_{x_t}M$.

Before we carry on with the calculations of the matrix of rotation, note that if Y is a vector field in OM, then the derivative

$$\begin{aligned} Y\,[\tilde{X}^j(u)]_{\mathcal{K}} &= \frac{d}{dt}\ [\tilde{X}^j(e^{uYt})]_{\mathcal{K}}|_{t=0} \\ &= [\frac{d}{dt}\ e^{-Yt}\,\tilde{X}^j(u)\,e^{Yt}\,|_{t=0}]_{\mathcal{K}} \\ &= [Y, \tilde{X}^j(u)]_{\mathcal{K}} \end{aligned}$$

where the last expression is the projection into the $\mathcal{K}$ Cartan component of the Lie bracket. Therefore, by Itô formula, the matrix of rotation satisfies

$$\mathcal{R}(u) = \lim_{T\to\infty} \frac{1}{T}\left\{\int_0^T [\tilde{X}^0(u_t)]_{\mathcal{K}} + \frac{1}{2}\sum_{j=1}^{m}[[\tilde{X}^j(u_t)]_{\mathcal{K}}, \tilde{X}^j(u_t)]_{\mathcal{K}}\ dt + M(t)\right\}$$

where $M(t)$ is an Itô integral of bounded integrands. It is well known that the average of this component vanishes. Therefore, by the ergodic theorem for Markov process we have:

Theorem 4.1 *If ν is an ergodic invariant probability measure on OM for the induced flow on this space then, for ν-almost all $u \in OM$:*

$$\mathcal{R}(u) = \int_{OM} [\tilde{X}^0(k)]_{\mathcal{K}} + \frac{1}{2}\sum_{j=1}^{m}[[\tilde{X}^j(k)]_{\mathcal{K}}, \tilde{X}^j(k)]_{\mathcal{K}}\ d\nu(k)$$

.

In particular, $(2,1)$-the entry of the rotation matrix of the frame $u = (u^1, \ldots, u^d)$ corresponds to the rotation number of the vector u^1 inside the plane spanned by the 2-frames (u_t^1, u_t^2) (see e.g. [11] or Arnold and Imkeller [2]). In two-dimensional systems, due to the commutativity of the orthogonal group $SO(2)$, the rotation matrix is independent of the initial orthonormal frame, besides, the rotation number measures the asymptotic average angular rotation of the stable/unstable sub-manifolds along trajectories, where the rotation is measured with respect to parallel transport, see [11, Section 6].

When, instead of taking the action of $\tilde{X}(u)$ on the right, we use the description of the vertical component $V(\delta X)^{\perp}(u)$ in terms of the left action of the skew-symmetric matrix $(\nabla X(u))^{\perp}$ (described in Lemma 2.2), we can work coordinate-wise in each entry of the matrix of rotation. The equation of the process is:

$$D\,u_t = \sum_{j=0}^{m} (\nabla X^j(u))^{\perp}(u) \ \circ dW_t^j.$$

For a certain ergodic invariant probability measure ν on OM, the entries of the matrix of rotation turn out to be

$$\mathcal{R}(u)_{i,j} = \int_{OM} r_{ij}(k)\ d\nu(k). \tag{14}$$

where the functions $r_{ij}(k)$ are given in [12, eqn. (10)], see also the appendix.

Appendix:

When we work with the vertical vector field acting on the left (represented by $(\nabla X)^{\perp}$), we have to overcome the difficulty that the Iwasawa decomposition of the product of matrices is different from the products of the Iwasawa decompositions in this particular case (cf. Lemma 2.1). This is the reason for the formulation becoming so tortuous. For example, the formulae for the entries of the rotation matrix as presented in equation (14) are, for $1 \leq i \neq j \leq d$:

$$\begin{aligned}
r_{ij}(k) \;=\; & < \nabla X^0(k^j), k^i > \\
& + \frac{1}{2}\sum_{l=1}^{m}\Big\{ < \nabla^2 X^l(X^l, k^j), k^i > - < \nabla X^l(k^j), k^j >< \nabla X^l(k^i), k^j > \\
& + < \nabla X^l(\nabla X^l(k^j)), k^i > -2 < \nabla X^l(k^j), k^j >< \nabla X^l(k^j), k^i > \\
& + \sum_{r<j}\Big[< \nabla X^l(k^r), k^j >< \nabla X^l(k^i), k^r > - < \nabla X^l(k^j), k^r >< \nabla X^l(k^r), k^i > \Big] \\
& - \sum_{j<r<i}\Big[< \nabla X^l(k^j), k^r >< \nabla X^l(k^r), k^i > + < \nabla X^l(k^i), k^r > \Big] \\
& + < \nabla X^l(k^j), \nabla X^l(k^i) > - < \nabla X^l(k^j), k^i >< \nabla X^l(k^i), k^i > \Big\}.
\end{aligned}$$

References

[1] L. Arnold – *Random Dynamical Systems.* Springer, Berlin (1998).

[2] L. Arnold and Imkeller, P. – Rotation numbers for linear stochastic differential equations. *Ann. Probab.* Vol. **27** (1999), no. 1, 130–149.

[3] K.D. Elworthy – Geometric Aspects of Diffusions on Manifolds. *In: École d'Eté de Probabilités de Saint-Flour, XV – XVII*, 1985 – 1987 (P.L. Hennequin, ed.) pp. 276 - 425. Lecture Notes Math. 1362. Springer, 1987.

[4] W. Klingenberg – *Riemannian Geometry.* Walter de Gruyter, Berlin, New York, 1982.

[5] S. Kobayashi and Nomizu, K. – *Foundations of Differential Geometry, Vol.1*, Wiley-Interscience Publication, 1963.

[6] H. Kunita – Stochastic differential equations and stochastic flows of diffeomorphisms. *In École d'Eté de Probabilités de Saint-Flour XII* - 1982, ed. P.L. Hennequin, pp. 143–303. Lecture Notes on Maths. 1097. Springer, 1984.

[7] H. Kunita – *Stochastic flows and stochastic differential equations.* Cambridge University Press, 1988.

[8] M. Liao – Liapounov exponents of stochastic flows. *Ann. Probab.*, Vol. **25**, no. 3, pp. 1241–1256, 1997.

[9] M. Liao – Decomposition of stochastic flows and Lyapunov exponents. *Prob. Th. Rel. Fields,* **117**, no. 4, pp. 589-607, 2000.

[10] J. G. Ratcliffe – On the isometry groups of hyperbolic manifolds. *The mathematical legacy of Wilhelm Magnus: groups, geometry and special functions* (Brooklyn, NY, 1992), 491–495, Contemp. Math., 169, Amer. Math. Soc., Providence, RI, 1994.

[11] P. Ruffino – Rotation numbers for stochastic dynamical systems. *Stochastics and Stochastics Reports*, vol. **60,** pp. 289-318, 1997.

[12] P. Ruffino – Matrix of rotation for stochastic dynamical systems. *Computational and Applied Maths. - SBMA*, vol. **18**, no. 2 pp. 213-226, 1999.