Stochastic Versions of Hartman-Grobman Theorems

Edson A. Coayla-Teran[1]
Departamento de Matemática, Universidade Federal de Ouro Preto, 35.400-000 - Ouro Preto - MG, Brazil. e-mail: eact@iceb.ufop.br

Paulo R. C. Ruffino[2]
Departamento de Matemática, Universidade Estadual de Campinas, 13.081-970 - Campinas - SP, Brazil. e-mail: ruffino@ime.unicamp.br

Abstract

We present versions of Hartman-Grobman theorems for random dynamical systems (RDS) in the discrete and continuous case. We apply the same random norm used by Wanner [23], but instead of using difference equations, we perform an apropriate generalization of the deterministic arguments in an adequate space of measurable homeomorphisms to extend his result with weaker hypotheses and simpler arguments.

Key words: Hartman-Grobman theorems, random diffeomorphisms, stochastic flows, hyperbolic fixed point, stochastic differential equation, local conjugacy.

MSC2000 subject classification: 93E99, 37D40, 34A34, (37H15).

1. Introduction

The celebrated Hartman-Grobman theorems play a fundamental rule in the theory of dynamical systems once one can obtain properties of a certain deterministic system around a hyperbolic fixed point via a conjugation with a linearised system. Precisely, consider the dynamical system in $\mathbb{R}^m$ generated by the following differential equation:

$$\dot{x} = f(x), \tag{1.1}$$

where f is a C^1 vector field with a singularity at $p \in \mathbb{R}^m$. Let $(\varphi_t)_{t\in\mathbb{R}}$, be the solution flow of this equation. Consider the system generated by the linearization of the vector field f at p, i.e., calling $A = Df(p)$, take the linear equation:

[1]Research supported by CAPES, Brazilian Foundation.

[2]Currently visiting fellow at the Mathematical Institute, University of Oxford, UK. Research supported by FAPESP grant no. 00/04591-3.

$$\dot{x}= Ax \tag{1.2}$$

and denote by $\Phi_t = e^{At}$ its fundamental linear solution. Hartman [12] and Grobman [11] have proved that if the singularity p is a hyperbolic fixed point of the system, that is, if the eigenvalues of A have non-vanishing real part then there exists an open neighbourhood $U \subset \mathbb{R}^m$ of p and a homeomorphism $h : U \to h(U) \subset \mathbb{R}^m$ with $h(p) = 0$ which is a topological conjugacy of the trajectories of these systems. Namely, for all $x \in U$ we have that:

$$e^{At}h(x) = h \circ \varphi_t(x).$$

The proof is obtained performing first a demonstration to the discrete case for local diffeomorphisms in the following sense, let f be a C^1-diffeomorphism in $\mathbb{R}^m$ such that $p \in \mathbb{R}^m$ is a hyperbolic fixed point (the modulo of the eigenvalues of $A = Df(p)$ are different from one), then there exists an open neighbourhood U of p and a homeomorphism $h : U \to h(U) \subset \mathbb{R}^m$ with $h(p) = 0$, such that for x in U we have the following conjugacy:

$$A \circ h(x) = h \circ f(x).$$

Later, in 1960, Hartman [13] proved that if f has continuous second derivative and the real part of the eigenvalues have all the same sign (all positive or all negative), then the trajectories can be conjugated by a C^1-diffeomorphism (cf. Section 4).

The main motivation for this article is to find a version of the Hartman-Grobman theorem for continuous random dynamical systems, particularly for stochastic flows generated by an Stratonovich stochastic differential equation (SDE):

$$dx_t = f_0(x_t)\ dt + \sum_{i=1}^{n} f_i(x_t)\ \circ dB_t^i \tag{1.3}$$

where $(B_t^1, \dots, B_t^n)$ is a Brownian motion in $\mathbb{R}^n$ based on a certain complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f_i : \mathbb{R}^m \to \mathbb{R}^m$, $i = 0, 1, \dots, n$, are smooth vector fields such that there exists a stochastic solution flow of diffeomorphisms $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ (see e.g. Kunita [16] or [17]).

We shall assume that all vector fields f_i have a singularity at $p \in \mathbb{R}^m$, hence $\varphi(t, \omega, p) = p$ for all $(t, \omega) \in \mathbb{R} \times \Omega$. We say that p is a stochastic hyperbolic fixed point of φ if the Lyapunov spectrum of the system at point p does not contain zero (see e.g. Arnold [1] or Carverhill [5], [6]).

Let $\Phi(t, \omega) = D\varphi(t, \omega, p)$ be the solution flow of the linearized SDE:

$$dx_t = Df_0(p)x\ dt + \sum_{i=1}^{n} Df_i(p)x\ \circ dB_t.$$

In this context, we are looking for a random measurable local homeomorphism $h(\omega)$ such that

$$\Phi(t,\omega) \circ h(\omega,x) = h(\theta_t(\omega),\cdot) \circ \varphi(t,\omega,x); \tag{1.4}$$

for x in a neighbourhood of p, t in a certain interval containing zero and $\theta_t : \Omega \to \Omega$ is the canonical shift on the Wiener space.

One of the first results concerning this kind of generalisation of the Hartman-Grobman theorems was given by T. Wanner [23], where the arguments were based in random difference equation to get initially the discrete generalised version ω-wise. His proof is completed showing that the random homeomorphism is in fact measurable. Our approach in this article is a proper random adaptation of the well-known deterministic proof, which here is extended to an appropriate Banach space of measurable homeomorphisms $\mathrm{Homeo}(\Omega,\mathbb{R}^m)$, with respect to an adequate norm (see next section). We shall follow mainly the deterministic arguments presented in Palis and Melo [19] and Sternberg [21].

Our hypothesis turn out to be weaker than Wanner's hypothesis in [23], basically: integrability instead of boundedness. Nevertheless, like him, we also could not disregard the random norm in the Euclidean space. We emphasise that, although apparently this norm leads to artificial hypotheses, it is rather an intrinsic parameter for this problem; it comes directly from the multiplicative ergodic theorem, which one has to deal whenever dealing with linearised random systems.

The article is organised in the following way: in Section 2 we introduce the basic spaces, the norms and properties which we are going to work with in the following sections. In Section 3 we present global and local versions of the Hartman-Grobman theorem for random diffeomorphisms. Roughly speaking: let f be an element of $C^1_0(\Omega,\mathbb{R}^m)$, the space of measurable C^1-maps $f(\cdot,\cdot) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ which fix the origin. Suppose that the origin is a hyperbolic point, i.e. the Lyapunov exponents

$$\lambda = \lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\|$$

are all different from zero, for all $v \in \mathbb{R}^m, v \neq 0$, where

$$\Phi(n,\omega) = \begin{cases} A(\theta_{n-1}\omega)...A(\omega), & n > 0, \\ I, & n = 0, \\ A^{-1}(\theta_n\omega)...A^{-1}(\theta_{-1}\omega), & n < 0, \end{cases}$$

and $A(\omega) = Df(\omega,0)$. We present necessary conditions on the non-linear component $\Psi(\omega,x) := f(\omega,x) - Df(\omega,0)x$, which guarantees that for almost every $\omega \in \Omega$ there exists a random neighbourhood $U(\omega)$ of the origin and a map $h \in \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ with $h(\omega,0) = 0$, such that if $x \in U(\omega)$ then

$$Df(\omega,0)h(\omega)(x) = h(\theta_1\omega)(\cdot)\circ f(\omega,x). \tag{1.5}$$

In the Section 4, motivated by the mentioned regularity results of Hartman [13], we study the differentiability of the homeomorphism h which performs the conjugation. In Section 5 we show a random global version of the Hartman-Grobman theorem for a continuous C^1-random dynamical systems. The local case was explored for SDE. The existence of the conjugation does not depend directly on the vector fields of the SDE, but rather on the parameter $B_\epsilon(\omega)$ (defined in the Proposition 2.1) which establishes the equivalence between the random and the Euclidean norm.

Finally, in the appendix we show how to extend our results to hyperbolic random fixed point.

2. Basic Framework

Consider $(\Omega,\mathcal{F},\mathbb{P})$ a complete probability space. For $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$ we shall denote a group of ergodic transformations on Ω by $\theta_t:\Omega\to\Omega$, for all $t\in\mathbb{T}$ (in the discrete case sometimes we will find the notation θ^k clearer than θ_k). We shall recall some basic structures and results (for details, see e.g. L. Arnold [1]).

Definition 2.1 *A measurable map $\varphi:\mathbb{T}\times\Omega\times\mathbb{R}^m\to\mathbb{R}^m$, $(t,\omega,x)\mapsto\varphi(t,\omega,x)$ is called a random dynamical system or (cocycle) of C^k-maps, $k\geq 1$, on $\mathbb{R}^m$ (abbreviated C^k-RDS) over θ_t if there exists a measurable subset $\Omega_0\subseteq\Omega$, $\mathbb{P}(\Omega_0)=1$ which is θ_t-invariant for all $t\in\mathbb{T}$ and such that for $\omega\in\Omega_0$ the following properties are satisfied:*

i) *$\varphi(\cdot,\omega,\cdot)$ is continuous;*

ii) *$\varphi(t,\omega,\cdot)$ is a C^k-diffeomorphisms for all $t\in\mathbb{T}$.;*

iii) *For $t,s\in\mathbb{T}$, $x\in\mathbb{R}^m$; $\varphi(t+s,\omega,x)=\varphi(s,\theta_t\omega,.)\circ\varphi(t,\omega,x)$ (cocycle property).*

If $\varphi(t,\omega,\cdot)$ is linear, the structure (φ,θ) is called a linear cocycle.

We introduce the well known Osseledec's multiplicative ergodic theorem (MET), a crucial result on the linear algebra of RDS.

Theorem 2.1 (MET) *Let Φ a linear RDS on $\mathbb{R}^m$ over the ergodic group of ergodic transformations $(\theta_t)_{t\in\mathbb{T}}$ and assume the following integrability conditions:*

$$\log^+\sup_{0\leq t\leq 1}\|\Phi(t,\cdot)\| \;+\; \log^+\sup_{0\leq t\leq 1}\left\|\Phi^{-1}(t,\cdot)\right\|\in L^1(\Omega,\mathcal{F},\mathbb{P}).$$

Then, there exists a θ_t-invariant set $\Omega_0\in\mathcal{F}$ with $\mathbb{P}(\Omega_0)=1$, and real numbers $\lambda_1>\lambda_2>\ldots>\lambda_p$ (the Lyapunov exponents) with multiplicities d_i, $\sum_{i=1}^p d_i=m$, such that for every $\omega\in\Omega_0$ the following holds:

i) There is a splitting of $\mathbb{R}^m$ into random subspaces $E_i(\omega)$

$$\mathbb{R}^m = E_1(\omega) \oplus \cdots \oplus E_p(\omega),$$

where each $E_i(\omega)$ depends measurably on ω, with non-random dimensions $\dim E_i(\omega) = d_i$ and

$$\Phi(t,\omega)E_i(\omega) = E_i(\theta_t\omega) \tag{2.1}$$

for $t \in \mathbb{T}$ and $i = 1, 2, \cdots, p$ (the subspaces $E_i(\omega)$ are called the Oseledec's subspaces);

ii) for every $v \in \mathbb{R}^m$ we have:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)v\| = \lambda_i \Longleftrightarrow v \in E_i(\omega) \setminus \{0\}\,.$$

Proof: See, e.g., Osseledts [20], Ruelle [22] or L. Arnold [1] and the references therein.
□

Definition 2.2 *In the situation of MET, we say that the linear RDS Φ is hyperbolic if none of the Lyapunov exponents is zero.*

In the hyperbolic case, for each $\omega \in \Omega_0$ we can define the stable and unstable subspaces: $E_s(\omega)$ and $E_u(\omega)$ respectively by:

$$E_s(\omega) := \bigoplus_{\lambda_i<0} E_i(\omega) \qquad \text{and} \qquad E_u(\omega) := \bigoplus_{\lambda_i>0} E_i(\omega).$$

Hence, for every $t \in \mathbb{T}$ and every $\omega \in \Omega_0$ we have the splitting of the Euclidean space:

$$\mathbb{R}^m = E_s(\theta_t\omega) \oplus E_u(\theta_t\omega),$$

For each $x \in \mathbb{R}^m$ we will write $x = x_s(\omega) + x_u(\omega)$; where $x_s(\omega) = \pi_{s,\omega}(x)$, with $\pi_{s,\omega} : \mathbb{R}^m \to E_s(\omega)$ a projection on $E_s(\omega)$ along $E_u(\omega)$. Analogously, $x_u(\omega) = \pi_{u,\omega}(x)$.

We introduce a random norm which satisfies the same nice properties (with respect to a linear RDS) of the Euclidean norm (with respect to a deterministic linear system):

Definition 2.3 (Random scalar product) *Fix an arbitrary constant $a > 0$. In the same context of the MET, for $\omega \in \Omega_0$ and for $x = \oplus_{i=1}^p x_i$ and $y = \oplus_{i=1}^p y_i$ with $x_i, y_i \in E_i(\omega)$ define*

$$\langle x, y\rangle_\omega := \sum_{i=1}^p \langle x_i, y_i\rangle_\omega\,,$$

where for $u_i, v_i \in E_i(\omega)$ *we set*

$$\langle u_i, v_i \rangle_\omega := \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{\langle \Phi(t,\omega)u_i, \Phi(t,\omega)v_i \rangle}{e^{2(\lambda_i t + a|t|)}} dt, & \text{for } \mathbb{T} = \mathbb{R}, \\ \displaystyle\sum_{n\in\mathbb{Z}} \frac{\langle \Phi(n,\omega)u_i, \Phi(n,\omega)v_i \rangle}{e^{2(\lambda_i n + a|n|)}}, & \text{for } \mathbb{T} = \mathbb{Z}, \end{cases}$$

and $\langle u_i, v_j \rangle_\omega := 0$ *if* $u_i \in E_i(\omega)$ *and* $v_j \in E_j(\omega)$ *for* $i \neq j$. *For* $\omega \notin \Omega_0$ *we set* $\langle x, y \rangle_\omega := \langle x, y \rangle$.

Proposition 2.1 *The following properties hold:*

a) $\langle x, y \rangle_\omega$ *is a random scalar product in* $\mathbb{R}^m$ *which depends measurably on* ω*;*

b) For each $\varepsilon > 0$ *there exists a measurable map* $B_\varepsilon(.) : \Omega \to [1, +\infty)$ *such that for all* $x \in \mathbb{R}^m$*:*

$$\frac{1}{B_\varepsilon(\omega)} \|x\| \leq \|x\|_\omega \leq B_\varepsilon(\omega) \|x\| \tag{2.2}$$

and $B_\varepsilon(\theta_t\omega) \leq B_\varepsilon(\omega)e^{\varepsilon|t|}$ *where* $\|x\|_\omega^2 = \langle x, x \rangle_\omega$, *is the random norm;*

c) For each $\omega \in \Omega_0$, $i = 1..., p,$, $x_i \in E_i(\omega)$ *and* $t \in \mathbb{T}$*:*

$$e^{\lambda_i t - a|t|} \|x_i\|_\omega \leq \|\Phi(t,\omega)x_i\|_{\theta_t\omega} \leq e^{\lambda_i t + a|t|} \|x_i\|_\omega . \tag{2.3}$$

Proof: See L. Arnold [1, Thm. 3.7.4] or P. Boxler [4]. □

The following inequalities come straightforward from equation (2.3). They will play an essential rule in the prove of the main results of next section.

Corollary 2.2 *Fix a constant* a *such that the intervals* $[\lambda_i - a, \lambda_i + a]$ *are disjoint and* $0 \notin [\lambda_i - a, \lambda_i + a]$ *for all* $i = 1, \cdots, p$.

1) *Let* $\lambda_s = \max\{\lambda_i < 0\}$, $\lambda_u = \min\{\lambda_i > 0\}$ *and* β *any number in the interval* $(0, \min\{-\lambda_s - a, \lambda_u - a\})$ *then:*

$$\|\Phi(t,\omega)x_s\|_{\theta_t\omega} \leq e^{-\beta t} \|x_s\|_\omega \quad \text{for all } t \geq 0, \tag{2.4}$$

and

$$\left\|\Phi(t,\omega)^{-1}x_u\right\|_\omega \leq e^{-\beta t} \|x_u\|_{\theta_t\omega} \quad \text{for all } t \geq 0. \tag{2.5}$$

2) *Taking* $\Lambda := \max\{\lambda_1, -\lambda_p\}$ *we have that*

$$e^{-(\Lambda+a)|t|}\left\|x\right\|_\omega \le \left\|\Phi(t,\omega)x\right\|_{\theta_t\omega} \le e^{(\Lambda+a)|t|}\left\|x\right\|_\omega \quad \textit{for } t\in\mathbb{T}, \tag{2.6}$$

and

$$e^{-(\Lambda+a)|t|}\left\|x\right\|_{\theta_t\omega} \le \left\|\Phi(t,\omega)^{-1}x\right\|_\omega \le e^{(\Lambda+a)|t|}\left\|x\right\|_{\theta_t\omega} \quad \textit{for } t\in\mathbb{T}. \tag{2.7}$$

The following definitions will introduce the spaces and norms which we are going to work with along this article. Next definition states the natural ambient where the conjugacy which we are looking for lives.

Definition 2.3 $\mathrm{Homeo}(\Omega,\mathbb{R}^m)$ *is the space of random homeomorphisms given by measurables* $h(\cdot,\cdot):\Omega\times\mathbb{R}^m\to\mathbb{R}^m$ *such that for each* $\omega\in\Omega$, $h(\omega,\cdot):\mathbb{R}^m\to\mathbb{R}^m$ *is a homeomorphism.*

Definition 2.4 *Let* k *be a positive integer.* $C^k(\Omega,\mathbb{R}^m)$ *is the space of random* C^k*-maps given by measurables* $u(\cdot,\cdot):\Omega\times\mathbb{R}^m\to\mathbb{R}^m$*, such that for each* $\omega\in\Omega$, $u(\omega,\cdot)\in C^k(\mathbb{R}^m)$*. In particular,* $C(\Omega,\mathbb{R}^m):=C^0(\Omega,\mathbb{R}^m)$ *is the space of random continuous maps. The notation* $C_0^k(\Omega,\mathbb{R}^m)$ *will refer to the subspace such that all* u *in this subspace fix the origin, i.e.* $u(\omega,0)=0$ *for almost all* $\omega\in\Omega$*.*

It will be convenient to establish the following decomposition in the space $C_0^k(\Omega,\mathbb{R}^m)$. Consider, as before, the projections on the stable and unstable subspace $\pi_{s,\omega}:\mathbb{R}^m\to E_s(\omega)$ and $\pi_{u,\omega}:\mathbb{R}^m\to E_u(\omega)$ respectively, each random C_0^k-map u has a unique decomposition:

$$u(\omega)=u_s(\omega)+u_u(\omega)$$

where $u_s(\omega)=\pi_{s,\omega}\circ u(\omega)$ and $u_u(\omega)=\pi_{u,\omega}\circ u(\omega)$. We shall denote this direct sum decomposition by

$$C_0^k(\Omega,\mathbb{R}^m)=C_{0,s}^k(\Omega,\mathbb{R}^m)\oplus C_{0,u}^k(\Omega,\mathbb{R}^m).$$

Definition 2.5

a) $C_b(\Omega,\mathbb{R}^m)$ *is the space of random bounded continuous maps* $u(\omega,\cdot)$ *such that:*

$$\left\|u\right\|_{C_b(\Omega,\mathbb{R}^m)}=\mathbb{E}\left[\sup_{x\in\mathbb{R}^m}\left\|u_s(\omega,x)\right\|_\omega\right]+\mathbb{E}\left[\sup_{x\in\mathbb{R}^m}\left\|u_u(\omega,x)\right\|_\omega\right]<+\infty.$$

Note that $\left\|\cdot\right\|_{C_b(\Omega,\mathbb{R}^m)}$ *is a norm.*

b) $C_{0,b}(\Omega,\mathbb{R}^m)$ *is the subspace of random bounded continuous maps which fix the origin, i.e.* $u\in C_{0,b}(\Omega,\mathbb{R}^m)\subset C_b(\Omega,\mathbb{R}^m)$ *if* $u(\omega,0)=0$ *for almost all* ω*. We shall denote the norm in* $C_b(\Omega,\mathbb{R}^m)$ *restricted to this subspace by* $\left\|\cdot\right\|_{C_{0,b}(\Omega,\mathbb{R}^m)}$.

c) $C^1_{0,b}(\Omega,\mathbb{R}^m)$ *is the space of random bounded differentiable maps which fix the origin given by* $u(\omega,\cdot)$ *such that:*

$$\begin{aligned}\|u\|_{C^1_b(\Omega,\mathbb{R}^m)} &= \|u\|_{C_{0,b}(\Omega,\mathbb{R}^m)} + \mathbb{E}\left[\sup_{x\in\mathbb{R}^m}\sup_{\|v\|_\omega\le 1}\|(Du(\omega,x)v)_s\|_\omega\right]\\ &\quad +\mathbb{E}\left[\sup_{x\in\mathbb{R}^m}\sup_{\|v\|_\omega\le 1}\|(Du(\omega,x)v)_u\|_\omega\right] < +\infty.\end{aligned}$$

The last space stated in item (c) will be used in section 4, where we will discuss regularity of the conjugation. The decomposition mentioned above of $C^k(\Omega,\mathbb{R}^m)$ restricts naturally to a direct sum decomposition of each of these spaces according to the projection on stable and unstable random subspace. Hence we shall denote by $C_{b,s}(\Omega,\mathbb{R}^m)$, $C_{0,b,s}(\Omega,\mathbb{R}^m)$ and $C^1_{0,b,s}(\Omega,\mathbb{R}^m)$ the stable components and by $C_{b,u}(\Omega,\mathbb{R}^m)$, $C_{0,b,u}(\Omega,\mathbb{R}^m)$ and $C^1_{0,b,u}(\Omega,\mathbb{R}^m)$ the unstable component of each of the three spaces of Definition 2.5.

Proposition 2.4 *The spaces* $C_b(\Omega,\mathbb{R}^m)$, $C_{0,b}(\Omega,\mathbb{R}^m)$ *and* $C^1_{0,b}(\Omega,\mathbb{R}^m)$ *are Banach spaces in theirs respective norms.*

Proof: We prove the result for $C_b(\Omega,\mathbb{R}^m)$. One easily checks that $C_{0,b}(\Omega,\mathbb{R}^m)\subset C_b(\Omega,\mathbb{R}^m)$ is a closed subspace. For $C^1_{0,b}(\Omega,\mathbb{R}^m)$ the proof goes with the same arguments, only adapting to the norm in $C^1_b(\mathbb{R}^m)$

Let $\{h_k\}_{k=1}\subset C_b(\Omega,\mathbb{R}^m)$ be a sequence whose series converges absolutely. We have to prove that the series itself converges in $C_b(\Omega,\mathbb{R}^m)$. We have, by definition, that

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|(h_k)_s(\omega,\cdot)\|_\omega\right] < +\infty$$

and

$$\sum_{k=1}^{\infty}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|(h_k)_u(\omega,\cdot)\|_\omega\right] < +\infty$$

To fix the ideas, we concentrate the calculations on the stable component. Let $G_n(\omega)_s = \sum_{k=1}^n \sup\limits_{\mathbb{R}^m}\|(h_k)_s(\cdot)\|_\omega$, and $G(\omega)_s = \sum_{k=1}^\infty \sup\limits_{\mathbb{R}^m}\|(h_k)_s(\cdot)\|_\omega$, then, by the Beppo-Levi convergence theorem $G(\omega)_s < +\infty$ for all $\omega\in\Omega^1\subseteq\Omega$ such that $\mathbb{P}(\Omega^1)=1$. Moreover, by Proposition 2.1:

$$\sum_{k=1}^{\infty}\sup_{\mathbb{R}^m}\|(h_k)_s(\omega,\cdot)\| \le B(\omega)\left\{\sum_{k=1}^{\infty}\sup_{\mathbb{R}^m}\|(h_k)_s(\omega,\cdot)\|_\omega\right\}$$

Since $C_b(\mathbb{R}^m)$ is a Banach space with the supremum norm , there exists $H(\omega,\cdot) \in C_b(\mathbb{R}^m)$ for each $\omega \in \Omega^1$ and $\sum_{k=1}^{\infty}(h_k)_s(\omega,\cdot) \to H(\omega,\cdot)$, moreover $H(\omega,\cdot)$ is measurable since this is a limit of measurable map (assume without lost of generality that $H(\omega) = 0$ if $\omega \notin \Omega^1$). Since

$$\left\| \sum_{k=1}^{n}(h_k)_s(\omega,\cdot) - H(\omega,\cdot) \right\|_\omega \leq B(\omega) \left\| \sum_{k=1}^{n}(h_k)_s(\omega,\cdot) - H(\omega,\cdot) \right\|$$

then, fixed ω, we have that $\sum_{k=1}^{n}(h_k)_s(\omega,\cdot) \to H(\omega,\cdot)$ in the $\sup\limits_{\mathbb{R}^m} \|\cdot\|_\omega$ norm. Thus

$$\sup_{\mathbb{R}^m} \left\| \sum_{k=1}^{n}(h_k)_s(\cdot) \right\|_\omega \leq \sum_{k=1}^{n} \sup_{\mathbb{R}^m} \|(h_k)_s(\cdot)\|_\omega ,$$

therefore, when $n \to \infty$ we have that $\sup\limits_{\mathbb{R}^m} \|H(\omega,\cdot)\|_\omega \leq G(\omega)_s$, moreover

$$\sup_{\mathbb{R}^m} \left\| H(\omega,\cdot) - \sum_{k=1}^{n}(h_k)_s(\omega,\cdot) \right\|_\omega \leq 2G(\omega)_s,$$

then, by the dominated convergence theorem

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\| H - \sum_{k=1}^{n}(h_k)_s(\omega,\cdot) \right\|_\omega \right] = 0.$$

We claim that $H(\omega,\cdot) \in C_{s,b}(\Omega,\mathbb{R}^m)$. In fact, if $\omega \in \Omega^1 \subseteq \Omega$, and $x \in \mathbb{R}^m$, we have that $\sum_{k=1}^{n}(h_k)_s(\omega,x) \in E_s(\omega)$ for all n; since $\sum_{k=1}^{n}(h_k)_s(\omega,x) \to H(\omega,x)$ then $H(\omega,x) \in \overline{E_s(\omega)} = E_s(\omega)$. For $\omega \notin \Omega^1$ we have $H(\omega,x) = 0 \in E_s(\omega)$.

Analogously, for the unstable part, there exists a random continuous map $K \in C_{b,u}(\Omega,\mathbb{R}^m)$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\| K(\omega,\cdot) - \sum_{k=1}^{n}(h_k)_u(\omega,\cdot) \right\|_\omega \right] = 0.$$

The proof finishes taking $h = H + K$. □

In what follows, in order to assure the existent of the Lyapunov exponents for the linear cocycle Φ, we shall always assume the hypothesis of the MET that $\log^+ \|A\|$, $\log^+ \|A^{-1}\| \in L^1(\Omega)$.

3. Discrete Case

In this section we show a global and a local version of the Hartman-Grobman theorem (HGT) for a random map $f \in C_0^1(\Omega,\mathbb{R}^m)$. We start with the global version.

3.1. Global version in $\mathbb{R}^m$

Let $f \in \mathrm{Dif}_0^1(\Omega, \mathbb{R}^m)$, the space of random global diffeomorphisms of $\mathbb{R}^m$. We shall denote by $\Psi(\omega, x)$ the non-linear part of f, i.e.:

$$\Psi(\omega, x) := f(\omega, x) - A(\omega)x,$$

where $A(\omega) = Df(\omega, 0)$.

Besides assuming that the origin is a hyperbolic fix point, we shall assume the following hypotheses on the non-linear part Ψ:

(H1) (Globally Lipschitz in the random norm): there exists a constant $L < \frac{1-e^{-\beta}}{2\sqrt{2}e^{\Lambda+a}}$ such that for a.a. $\omega \in \Omega$, and all $x, y \in \mathbb{R}^m$ we have

$$\begin{cases} \|\Psi_s(\omega, x) - \Psi_s(\omega, y)\|_{\theta\omega} \leq L \|x - y\|_\omega , \\ \\ \|\Psi_u(\omega, x) - \Psi_u(\omega, y)\|_{\theta\omega} \leq L \|x - y\|_\omega , \end{cases}$$

where β, Λ, a are chosen as in Corollary 2.2.

(H2) (Integrability of the supremum in the random norm): there exists a constant $M > 0$ such that

$$\mathbb{E}(\sup_{\mathbb{R}^m} \|\Psi_s(\omega)(.)\|_{\theta\omega}) + \mathbb{E}(\sup_{\mathbb{R}^m} \|\Psi_u(\omega)(.)\|_{\theta\omega}) \leq M.$$

We recall that Wanner [23, Thm 3.3] assumes, besides (H1), the almost sure bounded hypothesis:

$$\|\Psi_s(\omega)(x)\|_{\theta\omega} \leq M \qquad \text{and} \qquad \|\Psi_u(\omega)(x)\|_{\theta\omega} \leq M;$$

Our technique consists basically in decomposing the homeomorphism $h(\omega, \cdot)$ which we are looking for into $h = (I + u)$, then find u in $C_{0,b}(\Omega, \mathbb{R}^m)$ using a fixed point argument.

Theorem 3.1 (HGT, global discrete case) *Let $f \in \mathrm{Dif}_0^1(\omega, \mathbb{R}^m)$ such that the origin is a hyperbolic fixed point. Assume that the non-linear part Ψ satisfies (H1) and (H2). Then there exists a unique $h \in \mathrm{Homeo}(\Omega, \mathbb{R}^m)$ such that*

$$A(\omega)(\cdot) = h(\theta\omega)^{-1} \circ f(\omega, \cdot) \circ h(\omega, \cdot).$$

Proof: Write $h(\omega, \cdot) = I + u(\omega, \cdot)$ with $u \in C_{0,b}(\Omega, \mathbb{R}^m)$ where $I(\omega, x) = x$ for all $(\omega, x) \in \Omega \times \mathbb{R}^m$. The first step is to solve the following equation:

$$(I + u(\theta\omega))A(\omega) = f(\omega)(I + u(\omega))$$

which is equivalent to:

$$A(\omega)u(\omega) - u(\theta\omega)A(\omega) = -\Psi(\omega)(I + u(\omega)). \tag{3.1}$$

The second step is to prove that its unique solution $(I + u(\omega, \cdot))$ is invertible with continuous inverse. We split the proof into the following five lemmas.

Lemma 3.1 *There exists a unique solution u of equation (3.1) in $C_{0,b}(\Omega, \mathbb{R}^m)$.*

Proof: Define the linear operator $\mathcal{L} : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ given by

$$\mathcal{L}u(\omega) := A(\theta^{-1}\omega)u(\theta^{-1}\omega) - u(\omega)A(\theta^{-1}\omega).$$

We claim that $\mathcal{L}$ is well defined and is invertible. Factorize $\mathcal{L} = \overline{A} \circ \mathcal{L}^*$ where $\overline{A}, \mathcal{L}^* : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ are given by

$$\overline{A}u(\omega) := A(\theta^{-1}\omega)u(\theta^{-1}\omega),$$

and

$$\mathcal{L}^*u(\omega) := u(\omega) - A(\omega)^{-1}u(\theta\omega)A(\omega).$$

Now, we shall work separately with the operators $\overline{A}$ and $\mathcal{L}^*$ to show that they are well defined and invertible. We recall the invariance of the stable and unstable subspaces $E_s(\omega)$ and $E_u(\omega)$ from the MET, equation (2.1):

$$A(\theta^{-1}\omega) : E_{s,u}(\theta^{-1}\omega) \to E_{s,u}(\omega).$$

By the inequalities (2.4) and (2.6) in Corollary (2.2) we have that:

$$\begin{aligned}
\left\|\overline{A}u\right\|_{C_{0,b}(\Omega,\mathbb{R}^m)} &= \mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|(\overline{A}u)_s(\omega)\right\|_\omega + \sup_{\mathbb{R}^m}\left\|(\overline{A}u)_u(\omega)\right\|_\omega\right] \\
&= \mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|(\overline{A}u_s)(\omega)\right\|_\omega + \sup_{\mathbb{R}^m}\left\|(\overline{A}u_u)(\omega)\right\|_\omega\right] \\
&= \mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|A(\theta^{-1}\omega)u_s(\theta^{-1}\omega)\right\|_\omega + \sup_{\mathbb{R}^m}\left\|A(\theta^{-1}\omega)u_u(\theta^{-1}\omega)\right\|_\omega\right] \\
&\leq e^{-\beta}\mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|u_s(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega}\right] + e^{\Lambda+a}\mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|u_u(\theta^{-1}\omega, \cdot)\right\|_{\theta^{-1}\omega}\right] \\
&\leq c\left\|u\right\|_{C_{0,b}(\Omega,\mathbb{R}^m)},
\end{aligned}$$

where c is a positive constant. Hence $\overline{A}$ is a continuous operator.

One easily checks that the inverse of $\overline{A}$ is given by $(\overline{A})^{-1}u(\omega) = A(\omega)^{-1}u(\theta\omega)$. Moreover we have that $\|(\overline{A})^{-1}\| \leq e^{\Lambda+a}$; in fact, let $u \in C_{0,b}(\Omega,\mathbb{R}^m)$ then:

$$\begin{aligned}
\left\|(\overline{A})^{-1}u\right\|_{C_{0,b}(\Omega,\mathbb{R}^m)} &= \mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|A(\omega)^{-1}u_s(\theta\omega)\right\|_\omega + \sup_{\mathbb{R}^m}\left\|A(\omega)^{-1}u_u(\theta\omega)\right\|_\omega\right] \\
&\leq e^{\Lambda+a}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_s(\theta\omega)(\cdot)\|_{\theta\omega}\right] + e^{-\beta}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_u(\theta\omega,\cdot)\|_{\theta\omega}\right] \\
&\leq e^{\Lambda+a}\|u\|_{C_{0,b}(\Omega,\mathbb{R}^m)}.
\end{aligned}$$

since, by definition in Corollary 2.2, $0 < \beta < \Lambda + a$.

Analogously, for $\mathcal{L}^*$, by inequalities (2.5), (2.7) and using again the invariance of the subspaces: $A(\omega)^{-1} : E_{s,u}(\theta\omega) \to E_{s,u}(\omega)$, we have that:

$$\begin{aligned}
\|\mathcal{L}^*u\|_{C_{0,b}(\Omega,\mathbb{R}^m)} &= \mathbb{E}\left[\sup_{\mathbb{R}^m}\|(\mathcal{L}^*u)_s(\omega)\|_\omega + \sup_{\mathbb{R}^m}\|(\mathcal{L}^*u)_u(\omega)\|_\omega\right] \\
&\leq \mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_s(\omega)\|_\omega + \sup_{\mathbb{R}^m}\left\|A(\omega)^{-1}u_s(\theta\omega)A(\omega)\right\|_\omega\right] \\
&\quad + \mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_u(\omega)\|_\omega + \sup_{\mathbb{R}^m}\left\|A(\omega)^{-1}u_u(\theta\omega)A(\omega)\right\|_\omega\right] \\
&\leq \mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_s(\omega)\|_\omega\right] + e^{(\Lambda+a)}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_s(\theta\omega)A(\omega)\|_{\theta\omega}\right] \\
&\quad + \mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_u(\omega)\|_\omega\right] + e^{-\beta}\mathbb{E}\left[\sup_{\mathbb{R}^m}\|u_u(\theta\omega)A(\omega)\|_{\theta\omega}\right] \\
&\leq c'\|u\|_{C_{0,b}(\Omega,\mathbb{R}^m)},
\end{aligned}$$

for some constant c', hence $\mathcal{L}^*$ is continuous. Now, we show that $\mathcal{L}^*$ is invertible.

Initially note that

$$\left[u_s(\omega) - A(\omega)^{-1}u_s(\theta\omega)A(\omega)x\right] \in E_s(\omega) \tag{3.2}$$

and

$$\left[u_u(\omega) - A(\omega)^{-1}u_u(\theta\omega)A(\omega)x\right] \in E_u(\omega) \tag{3.3}$$

for all $x \in \mathbb{R}^m$, hence $\mathcal{L}^*$ preserves each component of the decomposition $C_{0,b}(\Omega,\mathbb{R}^m) = C_{0,b,s}(\Omega,\mathbb{R}^m) \oplus C_{0,b,u}(\Omega,\mathbb{R}^m)$. We shall consider the decomposition $\mathcal{L}^* = \mathcal{L}^*_s \oplus \mathcal{L}^*_u$ where $\mathcal{L}^*_s := \mathcal{L}^*\Big|_{C_{0,b,s}(\Omega,\mathbb{R}^m)}$ and $\mathcal{L}^*_u := \mathcal{L}^*\Big|_{C_{0,b,u}(\Omega,\mathbb{R}^m)}$.

Going further in the decomposition, we will write $\mathcal{L}^*_s = (I + \mathcal{T})$ and $\mathcal{L}^*_u = (I + \mathcal{S})$, where $\mathcal{T} : C_{0,b,s}(\Omega,\mathbb{R}^m) \to C_{0,b,s}(\Omega,\mathbb{R}^m)$ and $\mathcal{S} : C_{0,b,u}(\Omega,\mathbb{R}^m) \to C_{0,b,u}(\Omega,\mathbb{R}^m)$ are given by

$$\mathcal{T}u_s(\omega) := -A(\omega)^{-1}u_s(\theta\omega)A(\omega)$$

and

$$\mathcal{S}u_u(\omega) := -A(\omega)^{-1}u_u(\theta\omega)A(\omega).$$

Equations (3.2) and (3.3) guarantee that the operators $\mathcal{S}$ and $\mathcal{T}$ are well defined. One easily checks that $\mathcal{T}$ is invertible with

$$\mathcal{T}^{-1}u_s(\omega) := -A(\theta^{-1}\omega)u_s(\theta^{-1}\omega)A(\theta^{-1}\omega)^{-1}.$$

Moreover, by the inequalities (2.5) and (2.7) of Corollary 2.2 we have:

$$\begin{aligned}
\left\|\mathcal{T}^{-1}\right\| &= \sup_{\|u_s\|=1} \left\|\mathcal{T}^{-1}u_s(\omega)\right\| \\
&= \sup_{\|u_s\|=1} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|A(\theta^{-1}\omega)u_s(\theta^{-1}\omega)A(\theta^{-1}\omega)^{-1}\right\|_\omega\right] \\
&\leq e^{-\beta} \sup_{\|u_s\|=1} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|u_s(\theta^{-1}\omega)A(\theta^{-1}\omega)^{-1}\right\|_{\theta^{-1}\omega}\right] \\
&= e^{-\beta} \sup_{\|u_s\|=1} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|u_s(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega}\right] \\
&= e^{-\beta},
\end{aligned}$$

Therefore, $-\mathcal{T}^{-1}$ is a contraction, and by the von Neumann theorem (see e.g. Hutson e Pym [14, p. 86]) $\mathcal{L}_s^* = I + \mathcal{T}$ is invertible with $\left\|(I+\mathcal{T})^{-1}\right\| \leq \frac{e^{-\beta}}{1-e^{-\beta}}$.

Considering the unstable component, the same argument shows that $\|\mathcal{S}\| \leq e^{-\beta}$. Therefore, $\mathcal{L}_u^* = (I+\mathcal{S})$ is an isomorphism with $\left\|(\mathcal{L}_u^*)^{-1}\right\| \leq \frac{1}{1-e^{-\beta}}$. Hence, $\mathcal{L}^* = \mathcal{L}_s^* \oplus \mathcal{L}_u^*$ is invertible with $\left\|(\mathcal{L}^*)^{-1}\right\| \leq \frac{1}{1-e^{-\beta}}$.

Going back to the original operator $\mathcal{L} = \overline{A} \circ \mathcal{L}^*$ we have that

$$\left\|\mathcal{L}^{-1}\right\| \leq \frac{e^{\Lambda+a}}{1-e^{-\beta}}.$$

Consider the operator $\mathcal{P}_\Psi : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ given by

$$\mathcal{P}_\Psi u(\omega) := -\Psi(\theta^{-1}\omega)(I+u)(\theta^{-1}\omega).$$

The integrability hypothesis (H2) is used to guarantee that the operator $\mathcal{P}_\Psi$ is well defined, in fact:

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^m} \left\|\Psi_s(\theta^{-1}\omega)(I+u)(\theta^{-1}\omega))x\right\|_\omega\right] \leq \mathbb{E}\left[\sup_{x\in\mathbb{R}^m} \left\|\Psi_s(\theta^{-1}\omega)x\right\|_\omega\right] \leq M,$$

and

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^m} \left\|\Psi_u(\theta^{-1}\omega)(I+u)(\theta^{-1}\omega))x\right\|_\omega\right] \leq \mathbb{E}\left[\sup_{x\in\mathbb{R}^m} \left\|\Psi_u(\theta^{-1}\omega)x\right\|_\omega\right] \leq M.$$

Finally, consider the operator $\mu : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ given by the composition:

$$\mu(u) := \mathcal{L}^{-1} \circ \mathcal{P}_\Psi(u).$$

Note that a fixed point of μ satisfies

$$A(\theta^{-1}\omega)u(\theta^{-1}\omega) - u(\omega)A(\theta^{-1}\omega) = -\Psi(\theta^{-1}\omega)(I+u)(\theta^{-1}\omega)$$

which is equivalent to the conjugation equation (3.1).

The Lipschitz hypothesis (H1) is used in this proof exclusively to guarantee that μ is a contraction. In fact: Let u_1 and u_2 be arbitrary elements in $C_{0,b}(\Omega, \mathbb{R}^m)$, then:

$$\|\mu(u_1) - \mu(u_2)\|_{C_{0,b}(\Omega,\mathbb{R}^m)} \leq \left\|\mathcal{L}^{-1}\right\| \|\mathcal{P}_\Psi(u_1) - \mathcal{P}_\Psi(u_2)\|_{C_{0,b}(\Omega,\mathbb{R}^m)}.$$

The second factor on the right hand side is:

$$\begin{aligned}
& \mathbb{E}\left[\sup_{\mathbb{R}^m} \|\mathcal{P}_\Psi(u_1) - \mathcal{P}_\Psi(u_2))_s(\omega)\|_\omega + \sup_{\mathbb{R}^m} \|(\mathcal{P}_\Psi(u_1) - \mathcal{P}_\Psi(u_2))_u(\omega)\|_\omega\right] \\
\leq\ & \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-\Psi_s(\theta^{-1}\omega)(I+u_1)(\theta^{-1}\omega) + \Psi_s(\theta^{-1}\omega)(I+u_2)(\theta^{-1}\omega)\right\|_\omega\right] \\
& + \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-\Psi_u(\theta^{-1}\omega)(I+u_1)(\theta^{-1}\omega) + \Psi_u(\theta^{-1}\omega)(I+u_2)(\theta^{-1}\omega)\right\|_\omega\right] \\
\leq\ & L\mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-u_1(\theta^{-1}\omega) + u_2(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega} + \sup_{\mathbb{R}^m} \left\|-u_1(\theta^{-1}\omega) + u_2(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega}\right] \\
\leq\ & 2\sqrt{2}L \|u_1 - u_2\|_{C_{0,b}(\Omega,\mathbb{R}^m)},
\end{aligned}$$

hence

$$\|\mu(u_1) - \mu(u_2)\|_{C_{0,b}(\Omega,\mathbb{R}^m)} \leq \frac{e^{\Lambda+a}}{1-e^{-\beta}} 2\sqrt{2}L \|u_1 - u_2\|_{C_{0,b}(\Omega,\mathbb{R}^m)}.$$

Therefore, by hypothesis (H1), the map μ is a contraction. The proof is completed by the Banach fixed point theorem. $\square$

Lemma 3.2 *There exists a unique $v \in C_{0,b}(\Omega, \mathbb{R}^m)$ such that*

$$A(\omega)((I + v(\omega)) = (I + v(\theta\omega))(A(\omega) + \Psi(\omega)).$$

Proof: This equation is equivalent to

$$A(\omega)v(\omega) - v(\theta\omega)(A + \Psi)(\omega) = \Psi(\omega).$$

Define the linear operator $\mathcal{H} : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ by

$$\mathcal{H}(v)(\omega) = A(\theta^{-1}\omega)v(\theta^{-1}\omega) - v(\omega)(A + \Psi)(\theta^{-1}\omega).$$

Using the same kind of calculations which we did with the operator $\mathcal{L}$ of the last lemma, we conclude that $\mathcal{H}$ is continuous and invertible with

$$\left\|\mathcal{H}^{-1}\right\| \leq \frac{e^{\Lambda+a}}{1-e^{-\beta}}.$$

We define the operator $\mathcal{D}$ in the space of random applications by $\mathcal{D}v(\omega) := v(\theta\omega)$. Hypothesis (H2) says that $\mathcal{D}(\Psi) \in C_{0,b}(\Omega, \mathbb{R}^m)$. Hence, there exists a unique $v \in C_{0,b}(\Omega, \mathbb{R}^m)$ such that $v = \mathcal{H}^{-1} \circ \mathcal{D}(\Psi)$. □

We remark that Hypothesis (H1) does not appear in the proof of Lemma 3.2. We will refer to this lemma again in the proof of Theorem 5.2.

Lemma 3.3 *There exists a unique solution $w \in C_{0,b}(\Omega, \mathbb{R}^m)$ to the equation*

$$(I+w)(\theta\omega)(A+\Psi)(\omega) = (A+\Psi)(\omega)(I+w)(\omega),$$

which is the trivial $w \equiv 0$ a.s..

Proof: The equation is equivalent to

$$A(\omega)w(\omega) - w(\theta\omega)(A+\Psi)(\omega) = -\Psi(\omega)w(\omega).$$

Define the operator $\mathcal{Q}_\Psi : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ by

$$\mathcal{Q}_\Psi(w)(\omega) := -\Psi(\theta^{-1}\omega)w(\theta^{-1}\omega).$$

Hypothesis (H2) over Ψ guarantees that $\mathcal{Q}_\Psi$ is well defined. Consider the map $\beta : C_{0,b}(\Omega, \mathbb{R}^m) \to C_{0,b}(\Omega, \mathbb{R}^m)$ given by

$$\beta(w) := \mathcal{H}^{-1} \circ \mathcal{Q}_\Psi(w),$$

where $\mathcal{H}$ is the operator defined in the proof of Lemma 3.2. We show that β is a contraction. Let $u_1, u_2 \in C_{0,b}(\Omega, \mathbb{R}^m)$, from Hypothesis (H1) we have

$$\begin{aligned}
\|\beta(u_1) - \beta(u_2)\|_{C_{0,b}(\Omega,\mathbb{R}^m)} &\leq \left\|\mathcal{H}^{-1}\right\| \|\mathcal{Q}_\Psi u_1(\omega) - \mathcal{Q}_\Psi u_2(\omega)\|_{C_{0,b}(\Omega,\mathbb{R}^m)} \\
&\leq \frac{e^{\Lambda+a}}{1-e^{-\beta}} \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-\Psi_s(\theta^{-1}\omega)(u_1)(\theta^{-1}\omega) + \Psi_s(\theta^{-1}\omega)(u_2)(\theta^{-1}\omega)\right\|_\omega\right] \\
&\quad + \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-\Psi_u(\theta^{-1}\omega)(u_1)(\theta^{-1}\omega) + \Psi_u(\theta^{-1}\omega)(u_2)(\theta^{-1}\omega)\right\|_\omega\right] \\
&\leq \frac{e^{\Lambda+a}}{1-e^{-\beta}} L(\mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-u_1(\theta^{-1}\omega) + u_2(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega}\right] \\
&\quad + \mathbb{E}\left[\sup_{\mathbb{R}^m} \left\|-u_1(\theta^{-1}\omega) + u_2(\theta^{-1}\omega)\right\|_{\theta^{-1}\omega}\right] \\
&\leq \frac{L2\sqrt{2}e^{\Lambda+a}}{1-e^{-\beta}} \|u_1 - u_2\|_{C_{0,b}(\Omega,\mathbb{R}^m)}.
\end{aligned}$$

Hence β is a contraction. Therefore there exists a unique fixed point $\beta(w) = w$ which implies that $w \equiv 0$ a.s. $\square$

Lemma 3.4 *There exists a unique solution $z \in C_{0,b}(\Omega, \mathbb{R}^m)$ to the equation*

$$(I + z)(\theta\omega)A(\omega) = A(\omega)(I + z)(\omega),$$

which is the trivial $z \equiv 0$ a.s..

Proof: It is a particular case of Lemma 3.3 with $\Psi \equiv 0$. $\square$

Lemma 3.5 *Consider the elements u and v from Lemmas 3.1 and 3.2. Then $(I + u) \in$ $\mathrm{Homeo}(\Omega, R^m)$ moreover $(I + u)^{-1} = (I + v)$.*

Proof: In fact, from Lemmas 3.1 and 3.2 we have that

$$\begin{aligned}(I + u(\theta\omega))(I + v(\theta\omega))(A(\omega) + \Psi(\omega)) &= (I + u(\theta\omega)A(\omega)(I + v(\omega)) \\ &= (A(\omega) + \Psi(\omega))(I + u(\omega))(I + v(\omega)),\end{aligned}$$

and by Lemma 3.3 we find $(I + u)(I + v) = I$.

On the other hand:

$$\begin{aligned}(I + v(\theta\omega))(I + u(\theta\omega))A(\omega) &= (I + v(\theta\omega))(A + \Psi)(\omega)(I + u(\omega)) \\ &= A(\omega)(I + v(\omega))(I + u(\omega)),\end{aligned}$$

and by Lemma 3.4 we find $(I + v)(I + u) = I$. It concludes the proof of the theorem. $\square$

The next example illustrates a simple application of this result.

Example 3.1. Consider a map $f \in C^1(\Omega, \mathbb{R}^2)$ given by

$$f(\omega, x, y) = (c(\omega)x, d(\omega)y + \frac{h(x)}{eB(\omega)^2}),$$

where $h \in C^1(\mathbb{R})$ is bounded and such that $h(0) = h'(0) = 0$. Assume that h is Lipschitz with constant $L < (\frac{2\sqrt{2}\exp(\Lambda + a)}{1 - e^{-\beta}})^{-1}$. The random variable $B(\omega)$ is given by Proposition 2.1 with $\varepsilon = 1$ and the constants β, Λ and a are chosen as in Corollary 2.2. We shall assume that the random variables $c, d : \Omega \to \mathbb{R}$ are such that $\log|c|$, $\log|d|$ are in $L^1(\Omega)$ and $c(\omega) \neq 0$, $d(\omega) \neq 0$ a.a.. Call $\gamma := \mathbb{E}\log|c|$ and $\delta := \mathbb{E}\log|d|$. The linearization of f at the origin is given by

$$Df(\omega, 0) = \begin{bmatrix} c(\omega) & 0 \\ 0 & d(\omega) \end{bmatrix};$$

and the Lyapunov exponents are given by $\lambda_1 = \gamma$ and $\lambda_2 = \delta$. Assume that $\lambda_1 > 0 > \lambda_2$, i.e. the origin is a hyperbolic fixed point. The Osseledec's subspaces are simply $E_s(\omega) = \mathbb{R}(e_1)$ and $E_u(\omega) = \mathbb{R}(e_2)$. Note that f is invertible with

$$f^{-1}(\omega, x, y) = (\frac{x}{c(\omega)}, \frac{y - h(x/c(\omega))}{ed(\omega)B(\omega)^2}).$$

In our notation, the non-linear part of f is given by:

$$\Psi(\omega, x, y) = (0, \frac{h(x)}{eB(\omega)^2}).$$

We claim that Ψ satisfies the global Lipschitz in the random norm (Hypothesis (H1)). In fact, given $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$:

$$\begin{aligned} \|\Psi_u(\omega, x_1, y_1) - \Psi_u(\omega, x_2, y_2)\|_{\theta\omega} &\leq eB(\omega) \left| \frac{h(x_1)}{eB(\omega)^2} - \frac{h(x_2)}{eB(\omega)^2} \right| \\ &\leq \frac{L}{B(\omega)} |x_1 - x_2| \\ &\leq \frac{L}{B(\omega)} \|(x_1, y_1) - (x_2, y_2)\| \\ &\leq L \|(x_1, y_1) - (x_2, y_2)\|_{\omega}. \end{aligned}$$

Hypothesis (H2) also holds once:

$$\|\Psi_u(\omega, x, y)\|_{\theta\omega} \leq eB(\omega) \left| \frac{h(x)}{eB(\omega)^2} \right| \leq \sup_{\mathbb{R}} |h(x)|.$$

Analogous estimate also holds for the stable component Ψ_s. Hence, by Theorem 3.1 there exists a random global homeomorphism $h = (I + u)$ with $u \in C_{0,b}(\Omega, \mathbb{R}^2)$ such that

$$\operatorname{diag}[c(\omega) \quad d(\omega)] = (I + u(\theta\omega))^{-1} f(\omega)(I + u(\omega)).$$

In some simple cases it is possible to calculate explicitly the random homeomorphism $h(\omega, \cdot)$.

Example 3.2. Consider the discrete probability space $\Omega = \{a, b\}$ with $\mathbb{P}(a) = \mathbb{P}(b) = \frac{1}{2}$ and the ergodic transformation $\theta : \Omega \to \Omega$ given by $\theta(a) = b$, $\theta(b) = a$. We shall consider the mapping

$$f(\omega, x, y) = (\alpha(\omega)x + cy - y^2, \frac{1}{2}y)$$

where c is a constant in $\mathbb{R}$ and α is defined by

$$\alpha(\omega) := \begin{cases} 2 & , \text{ if } \omega = a \\ -2 & , \text{ if } \omega = b. \end{cases}$$

Therefore,

$$Df(\omega,0)=\begin{bmatrix}\alpha(\omega) & c\\ 0 & \frac{1}{2}\end{bmatrix}.$$

In this case the random conjugation is performed by $H:\Omega\to \mathrm{Homeo}(\mathbb{R}^2)$ given by

$$H(a,x,y)=(x+\frac{28}{65}y^2,y)\quad \text{with inverse}\quad H^{-1}(a,x,y)=(x-\frac{28}{65}y^2,y);$$

and

$$H(b,x,y)=(x-\frac{36}{65}y^2,y)\quad \text{with inverse}\quad H^{-1}(b,x,y)=(x+\frac{36}{65}y^2,y).$$

One easily checks the conjugation property, for all $\omega\in\Omega$ and $(x,y)\in\mathbb{R}^2$. As in Arnold [1, Example 3.6.1] or in Furstenberg and Kifer [9], the Lyapunov exponents are $\lambda_1=\log 2>\lambda_2=-\log 2$ and the Osseledets spaces are $E_1=\mathbb{R}\cdot\ e_1$ and

$$E_2(\omega)=\mathbb{R}\begin{bmatrix}u(\omega)\\ 1\end{bmatrix}$$

with

$$u(\omega)=-c\sum_{k=0}^{\infty}\frac{(\frac{1}{2})^k}{\alpha_{k+1}(\omega)}.$$

One checks that $u(a)=-\frac{6c}{17}$ and $u(b)=\frac{10c}{17}$.

3.2. Local discrete version

The approach for the local version of the HGT will start with a result which again extends the deterministic arguments (see, e.g. Palis and Melo [19, Lemma II.4.4]).

Lemma 3.6 *Let f be in $\in C^1(\Omega,\mathbb{R}^m)$ and consider the mapping A, and Ψ as defined above. If the origin is a hyperbolic fixed point of the systems (f,θ) then for $\mathbb{P}$-almost all $\omega\in\Omega$ there exists a neighbourhood $U(\omega)$ of the origin and a function $\widetilde{\Psi}\in C^1(\Omega,\mathbb{R}^m)$ which satisfies Hypothesis (H1) and (H2) such that $\widetilde{f}(\omega,\cdot):\equiv A(\omega)(\cdot)+\widetilde{\Psi}(\omega,\cdot)$ is in $\mathrm{Dif}_0^1(\Omega,\mathbb{R}^m)$ and if $x\in U(\omega)$ then,*

$$f(\omega,x)=\widetilde{f}(\omega,x).$$

Proof: Consider a C^∞ function $\alpha:\mathbb{R}\to\mathbb{R}$ such that

$$\alpha(t)=\begin{cases}1, & |t|\le\frac{1}{2}\\ 0, & |t|\ge 1\end{cases}$$

and $|\alpha'(t)| \leq k$ with $k > 2$. Let L be a positive constant and $B(\widetilde{l}(\omega))$ a ball with centre at the origin and radius $\widetilde{l}(\omega)$. The random variable $\widetilde{l}(\omega)$ will be defined such that

$$\|D\Psi_{s,u}(\omega,x)\| \leq \frac{L}{eB(\omega)^2 2k}$$

for all $\|x\| \leq \widetilde{l}(\omega)$ where the term $B(\omega)$ is the same of Proposition 2.1 with $\varepsilon = 1$. We have, for $x, y \in B(\widetilde{l}(\omega))$ that:

$$\|\Psi(\omega,x) - \Psi(\omega,y)\| \leq \frac{L}{eB(\omega)^2 2k} \|x - y\|$$

and

$$\|\Psi(\omega,x)\| \leq \frac{L}{eB(\omega)^2 2k} \|x\|.$$

Let $l(\omega) = \min\left\{\widetilde{l}(\omega), 1\right\}$. The function in the statement of this lemma $\widetilde{\Psi} \in C^1(\Omega, \mathbb{R}^m)$ can be defined now by:

$$\widetilde{\Psi}(\omega,x) = \alpha\left(\frac{\|x\|}{l(\omega)}\right)\Psi(\omega,x).$$

Moreover, the neighbourhood of the statement can be defined by

$$U(\omega) = B\left(\frac{l(\omega)}{2}\right).$$

Then, naturally, $\Psi(\omega,x) = \widetilde{\Psi}(\omega,x)$ if $x \in U(\omega)$. We claim that $\widetilde{\Psi}$ satisfies Hypothesis (H1). In fact, considering first the unstable part, by Proposition 2.1:

$$\begin{aligned}
\left\|\widetilde{\Psi}_u(\omega,x) - \widetilde{\Psi}_u(\omega,y)\right\|_{\theta\omega} &\leq B(\theta\omega)\left\|\widetilde{\Psi}_u(\omega,x) - \widetilde{\Psi}_u(\omega,y)\right\| \\
&\leq B(\omega)e\left\|\widetilde{\Psi}_u(\omega,x) - \widetilde{\Psi}_u(\omega,y)\right\|.
\end{aligned}$$

If $x, y \in B(l(\omega))$, by Proposition 2.1 again we have:

$$\begin{aligned}
\left\|\widetilde{\Psi}_u(\omega,x) - \widetilde{\Psi}_u(\omega,y)\right\|_{\theta\omega} &\leq B(\omega)e\left\{\left|\alpha(\frac{\|x\|}{l(\omega)}) - \alpha(\frac{\|y\|}{l(\omega)})\right| \|\Psi_u(\omega,x)\|\right. \\
&\qquad \left. + \alpha(\frac{\|y\|}{l(\omega)}) \|\Psi_u(\omega,x) - \Psi_u(\omega,y)\|\right\} \\
&\leq B(\omega)e\frac{k}{l(\omega)}\|x - y\| \frac{L}{eB(\omega)^2 2k}\|x\| \\
&\qquad + \frac{k}{l(\omega)}\|y\| \frac{L}{eB(\omega)^2 2k}\|x - y\| \\
&\leq \frac{L}{B(\omega)}\|x - y\| \leq L\|x - y\|_\omega.
\end{aligned}$$

Suppose now that $x \in B(l(\omega))$ and $y \notin B(l(\omega))$, then

$$\begin{aligned}\left\|\widetilde{\Psi}_u(\omega,x)-\widetilde{\Psi}_u(\omega,y)\right\|_{\theta\omega} &\leq B(\omega)e\left\{\left|\alpha(\frac{\|x\|}{l(\omega)})-\alpha(\frac{\|y\|}{l(\omega)})\right|\|\Psi_u(\omega,x)\|\right\}\\ &\leq B(\omega)e\left\{\frac{k}{l(\omega)}\|x-y\|\frac{L}{eB(\omega)^2 2k}\|x\|\right\}\\ &\leq \frac{L}{2B(\omega)}\|x-y\| \leq L\|x-y\|_{\omega}.\end{aligned}$$

Finally, if $x, y \notin B(l(\omega))$ the Lipschitz property is trivial once

$$\left\|\widetilde{\Psi}_u(\omega,x)-\widetilde{\Psi}_u(\omega,y)\right\|_{\theta\omega}=0.$$

The Lipschitz property for the stable part is proved using the same kind of arguments. Now we show that $\widetilde{\Psi}$ satisfies Hypothesis (H2), more specifically we will show that, by our construction, its random norm for the stable or unstable component (at fibre $\theta\omega$) is bounded by positive constant (which coincides with the Lipschitz constant L). In fact, if $x \in B(l(\omega))$ we have:

$$\begin{aligned}\left\|\widetilde{\Psi}_s(\omega,x)\right\|_{\theta\omega} &\leq B(\omega)e\alpha(\frac{\|x\|}{l(\omega)})\|\Psi_s(\omega,x)\| \leq B(\omega)e\frac{k}{l(\omega)}\|x\|\frac{L}{eB(\omega)^2 2k}\|x\|\\ &\leq \frac{L}{2B(\omega)}\|x\| \leq \frac{L}{2B(\omega)}l(\omega) \leq L.\end{aligned}$$

And if $x \notin B(l(\omega))$ then $\left\|\widetilde{\Psi}_s(\omega,x)\right\|_{\theta\omega}=0$.
The proof for the unstable part follows using the same arguments. □

Theorem 3.2 (HGT: local discrete case) *Let f be in $C_0^1(\Omega,\mathbb{R}^m)$ and consider the mapping A, and Ψ as defined above. If the origin is a hyperbolic fixed point of the systems (f,θ) then for $\mathbb{P}$-almost all $\omega \in \Omega$ there exists a neighbourhood $U(\omega)$ of the origin and a local homeomorphism $h \in \mathrm{Homeo}(\Omega, U(\omega); h(U(\omega)))$ such that:*

$$f(\omega,x)=h^{-1}(\theta\omega)A(\omega)h(\omega)(x),$$

for all x in the domain of the composition.

Proof: Define $\widetilde{f}(\omega,x) := A(\omega,x)+\widetilde{\Psi}(\omega,x)$ as in Lemma 3.6. By Theorem 3.1 there exists a global homeomorphism $\widetilde{h} \in \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ such that $\widetilde{h}(\theta\omega)\widetilde{f}(\omega,\cdot) = D\widetilde{f}(\omega,0)\widetilde{h}(\omega)(\cdot)$. Take the restriction $h=\widetilde{h}|_{U(\omega)}$. □

We finish this section presenting the natural extension of the discrete random version of the Hartman-Grobman theorem to mappings on a differentiable manifold M of dimension m. Here $C^1(\Omega, M)$ is the space of measurable applications $f : \Omega \times M \to M$, with $f(\omega) \in C^1(M)$ a.s..

Corollary 3.2 *Let $f \in C^1(\Omega, M)$ where M is a differentiable manifold. Let $p \in M$ be a hyperbolic fixed point for the random dynamical system generated by (f, θ). Calling $A(\omega) = T_p f(\omega) : T_p M \to T_p M$, then, for each $\omega \in \Omega$ there exists a neighbourhood $V(\omega, p) \subset M$ of p and $U(\omega) \subset T_p M$, a neighbourhood of the origin and a homeomorphism $h(\omega) : U(\omega) \to V(\omega, p)$ such that:*

$$h(\theta\omega)A(\omega)(x) = f(\omega)h(\omega)(x).$$

for all x in the domain of the composition.

Proof: Consider a local chart $\psi : W \subset M \to Z \subset \mathbb{R}^m$ with $p \in W$ such that $\psi(p) = 0$. Consider the random map $\widetilde{f}(\omega, x) = \psi \circ f(\omega, \cdot) \circ \psi^{-1} : Z \to \mathbb{R}^m$. The result follows by the HGT, local discrete case. □

4. Regularity of the conjugation

In this section we are going to extend the results of Hartman [13]. We establish a theorem which guarantees the existence of a random diffeomorphism which perform the conjugation between a discrete random dynamical systems and its linearization. We shall deal now with random mappings $f \in C^2(\mathbb{R}^m)$ and we keep the same notations and hypotheses as before: the origin is a hyperbolic fixed point and $f(\omega, \cdot) = A(\omega) + \Psi(\omega, \cdot)$ where $A(\omega) = Df(\omega, 0)$. We shall assume, besides the hypothesis (H1) and (H2) of last section, the following extra conditions:

(H3) There exist constants k, $k_1 > 1$ such that $k_1 e^{-\beta} < 1$, $ke^{-\beta} < \dfrac{1}{3 - 2e^{-\beta}} < 1$,

$$\frac{1}{1 - k_1 e^{-\beta}} < \frac{ke^{-\beta}}{1 - ke^{-\beta}},$$

and for all $v \in \mathbb{R}^m$, $\omega \in \Omega$ we have

a) $\|A(\omega)v\|_{\theta\omega} \le k_1 \|v\|_\omega$ and $\left\|A(\omega)^{-1}v\right\|_\omega \le k \|v\|_{\theta\omega}$;

b) $\|(A(\omega) + \Psi(\omega))v\|_{\theta\omega} \le k_1 \|v\|_\omega$ and $\left\|(A(\omega) + \Psi(\omega))^{-1}v\right\|_\omega \le k \|v\|_{\theta\omega}$.

Here, the constant β is the same which appears in Corollary 2.2.

(H4) (The derivative is Lipschitz) There exists a constant $L > 0$ such that

$$1 - e^{-\beta} + 4\sqrt{2}e^{\Lambda+a}L < \frac{1 - ke^{-\beta}}{2ke^{-\beta}},$$

and for all $x \in \mathbb{R}^m, v \in \mathbb{R}^m$, $\omega \in \Omega$ we have:

$$\begin{cases} \|((D\Psi(\omega,x) - D\Psi(\omega,y))v)_s\|_{\theta\omega} \leq L\,\|x-y\|_\omega\,; \\ \\ \|((D\Psi(\omega,x) - D\Psi(\omega,y))v)_u\|_{\theta\omega} \leq L\,\|x-y\|_\omega\,. \end{cases}$$

(H5) (The derivative is bounded in the random norm) There exists a constant $L_1 > 0$ with $L_1 < \dfrac{1 - ke^{-\beta}}{4e^{\Lambda+a}\sqrt{2}ke^{-\beta}}$ where Λ, a are the constants which appear in Corollary 2.2, such that for all $x, v \in \mathbb{R}^m$ and $\omega \in \Omega$ we have:

$$\|(D\Psi(\omega,x)v)_s\|_{\theta\omega} \leq L_1\,\|v\|_\omega\,, \qquad \|(D\Psi(\omega,x)v)_u\|_{\theta\omega} \leq L_1\,\|v\|_\omega$$

Remark 4.1

a) *In the Hypothesis (H3), the restriction on k_1 is given only by inequality $k_1e^{-\beta} < 1$, once we already know that $\|A(\omega)v\|_{\theta\omega} \leq e^{\Lambda+a}\,\|v\|_\omega$;*

b) *Again, in the Hypothesis (H3), the restriction on k is given only by the inequality $\frac{1}{2-k_1e^{-\beta}} < ke^{-\beta}$, once we already know that $\left\|A(\omega)^{-1}v\right\|_\omega \leq e^{\Lambda+a}\,\|v\|_{\theta\omega}$;*

c) *In Hypothesis (H4), the set of possibilities for the constant L is non-empty once k satisfies $1 - e^{-\beta} < \dfrac{1-ke^{-\beta}}{2ke^{-\beta}}$.*

Theorem 4.1 (HGT, differentiable global discrete case) *Let $f \in \mathrm{Dif}^2_0(\Omega, \mathbb{R}^m)$ such that the origin is a hyperbolic fixed point. Writing $f(\omega) = A(\omega) + \Psi(\omega)$, with $A(\omega) = Df(\omega, 0)$, assume that its non-linear part satisfies the Hypothesis (H1) till (H5). Then, there exists a unique random diffeomorphism $h \in \mathrm{Dif}^1_0(\Omega, \mathbb{R}^m)$ such that*

$$h(\theta\omega) \circ A(\omega) = f(\omega) \circ h(\omega).$$

Proof: The technique is exactly the same as in the proof of Theorem 3.1, but with longer calculations. We shall only show the main steps.

Again, we are looking for a solution of the form $h = (I + u)$ with $u \in C^1_{0,b}(\Omega, \mathbb{R}^m)$. The proof is performed by the following two lemmas:

Lemma 4.1 *In the conditions of Theorem 4.1 there exists a unique homeomorphism* $h \in C_0^1(\Omega,\mathbb{R}^m) \cap \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ *such that*

$$h(\theta\omega) \circ A(\omega) = f(\omega) \circ h(\omega).$$

Proof: Imitating Lemma 3.1, define the linear operators:

$$\begin{array}{cccc} \mathcal{L}^*: & C_{0,b}^1(\Omega,\mathbb{R}^m) & \longrightarrow & C_{0,b}^1(\Omega,\mathbb{R}^m) \\ & u\,(\omega) & \longmapsto & \mathcal{L}^* u(\omega) = u(\omega) - A(\omega)^{-1} u(\theta\omega) A(\omega) \end{array}$$

and the non-linear operator:

$$\begin{array}{cccc} (\overline{A})^{-1} \circ \mathcal{P}_\Psi: & C_{0,b}^1(\Omega,\mathbb{R}^m) & \longrightarrow & C_{0,b}^1(\Omega,\mathbb{R}^m) \\ & u\,(\omega) & \longmapsto & (\overline{A})^{-1} \mathcal{P}_\Psi u(\omega) = -A(\omega)^{-1} \Psi(\omega)(I+u)(\omega). \end{array}$$

Applying Hypothesis (H3), following similarly the calculations of Lemma 3.1 one finds that $\mathcal{L}^*$ and $(\overline{A})^{-1} \circ \mathcal{P}_\Psi$ are well defined. Again we consider the decomposition:

$$\mathcal{L}_s^* = \mathcal{L}^* \Big|_{C_{b,s}^1(\Omega,\mathbb{R}^m)} \qquad \text{and} \qquad \mathcal{L}_u^* = \mathcal{L}^* \Big|_{C_{b,u}^1(\Omega,\mathbb{R}^m)}$$

Next step is to show that $\mathcal{L}_s^*$ and $\mathcal{L}_u^*$ are invertible. We define $\mathcal{T} : C_{0,b,s}^1(\Omega,\mathbb{R}^m) \to C_{0,b,s}^1(\Omega,\mathbb{R}^m)$ and $\mathcal{S} : C_{0,b,u}^1(\Omega,\mathbb{R}^m) \to C_{0,b,u}^1(\Omega,\mathbb{R}^m)$ operators like in Lemma 3.1. With similar calculations in the norm of $C^1(\mathbb{R}^m)$ eventually one concludes that $\mathcal{L}^*$ is invertible with

$$\left\|(\mathcal{L}^*)^{-1}\right\| \le \max\left\{\frac{1}{1-k_1 e^{-\beta}}, \frac{ke^{-\beta}}{1-ke^{-\beta}}\right\} = \frac{ke^{-\beta}}{1-ke^{-\beta}},$$

by Hypothesis (H3).

Finally we introduce the non-linear operator $\mu : C_{0,b}^1(\Omega,\mathbb{R}^m) \to C_{0,b}^1(\Omega,\mathbb{R}^m)$ given by

$$\mu(u) = (\mathcal{L}^*)^{-1} \circ (\overline{A})^{-1} \circ \mathcal{P}_\Psi(u).$$

We claim that μ is a contraction. In fact, given $u_1, u_2 \in C_{0,b}^1(\Omega,\mathbb{R}^m)$ then, by straightforward calculations we have:

$$\|\mu(u_1) - \mu(u_2)\|_{C_{0,b}^1(\Omega,\mathbb{R}^m)}$$

$$\le \frac{ke^{-\beta}}{1-ke^{-\beta}} \left\|(\overline{A})^{-1} \circ \mathcal{P}_\Psi(u_1) - (\overline{A})^{-1} \circ \mathcal{P}_\Psi(u_2)\right\|_{C_{0,b}^1(\Omega,\mathbb{R}^m)}$$

$$\leq \quad (1 - e{-}\beta + 4e\Lambda + aL\sqrt{2}\|u_1 - u_2\|_{C_{0,b}(\Omega,\mathbb{R}^m)}$$

$$+ (e^{\Lambda+a}L_1 2\sqrt{2})\mathbb{E}\left[\sup_{x\in\mathbb{R}^m}\sup_{\|v\|_\omega\leq 1}\|(D_x(u_1)(\omega)v - D_x(u_2)(\omega)v)_s\|_\omega\right.$$

$$\left. + \sup_{x\in\mathbb{R}^m}\sup_{\|v\|_\omega\leq 1}\|(D_x(u_1)(\omega)v - D_x(u_2)(\omega)v)_u\|_\omega\right].$$

Hence, by Hypothesis (H4) L is such that $(1 - e^{-\beta} + 4e^{\Lambda+a}L\sqrt{2}) < \dfrac{1-ke^{-\beta}}{2ke^{-\beta}}$ and by Hypothesis (H5) $L_1 < \dfrac{1-ke^{-\beta}}{4e^{\Lambda+a}\sqrt{2}ke^{-\beta}}$. Therefore μ is a contraction. The result follows by Banach fixed point theorem. □

Lemma 4.2 *In the conditions of Theorem 4.1, there exists a unique homeomorphism $g \in C^1(\Omega,\mathbb{R}^m)\cap \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ such that*

$$A(\omega)\circ g(\theta\omega) = g(\theta\omega)\circ f(\omega).$$

Proof: We repeat the same arguments as in Lemma 3.2. Note that in the previous Lemma 4.1 we only used the Hypotesis (H3.a). For the proof of this lemma one will need to assume Hypothesis (H3.b) instead. The calculations are again straightforward. □

End of the Proof of Theorem 4.1: Considering the homeomorfisms h and g of the previous two lemmas, the result follows by Lemma 3.5, which guarantees that $h = g^{-1}$. □

4.1. Local version

For the local version, we shall first introduce the following lemma (similarly to the approach of last section):

Lemma 4.3 *Let f be in $\in C^2(\Omega,\mathbb{R}^m)$ and consider the mapping A, and Ψ as defined aboved. Assume the the second derivative $D^2\Psi(x)$ is bounded and vanishes at $x=0$. If the origin is a hyperbolic fixed point of the systems (f,θ) then for $\mathbb{P}$-almost all $\omega\in\Omega$ there exists a neighbourhood $U(\omega)$ of the origin and a function $\widetilde{\Psi}\in C^1(\Omega,\mathbb{R}^m)$ which satisfies Hypothesis (H1), till (H5) such that $\widetilde{f}(\omega,\cdot) :\equiv A(\omega)(\cdot) + \widetilde{\Psi}(\omega,\cdot)$ is invertible and if $x\in U(\omega)$ then,*

$$\Psi(\omega,x) = \widetilde{\Psi}(\omega,x).$$

Proof: Take the same function α of the proof of Lemma 3.6. Let $\widetilde{l_1}(\omega)$ be defined such that

$$\|D\Psi(\omega,x)\| \leq \frac{L}{eB(\omega)^2 2k}$$

for all $\|x\| \leq \widetilde{l_1}$. And let $\widetilde{l_2}(\omega)$ be defined such that

$$\left\|D^2\Psi(\omega,x)\right\| \leq \frac{L}{eB(\omega)^2 2k}$$

Define $l(\omega) = \min\left\{\widetilde{l_1}(\omega), \widetilde{l_2}(\omega), 1\right\}$. Then, as in Lemma 3.6 the function of the statement can be defined by

$$\widetilde{\Psi}(\omega,x) = \alpha\left(\frac{\|x\|}{l(\omega)}\right)\ \Psi(\omega,x).$$

And the neighbourhood can be defined by

$$U(\omega) = B(\frac{l(\omega)}{2}).$$

□

Theorem 4.2 (HGT: differentiable local discrete case) *Let $f \in C^2(\Omega,\mathbb{R}^m)$ with the mapping A, and Ψ as defined aboved. Assume that the second derivative $D^2\Psi(x)$ is bounded and vanishes at $x = 0$. If the origin is a hyperbolic fixed point of the systems (f,θ) then for $\mathbb{P}$-almost all $\omega \in \Omega$ there exists a neighbourhood $U(\omega)$ of the origin and a local diffeomorphism $h \in \mathrm{Dif}_0^1(\Omega, U(\omega); h(U(\omega)))$ such that:*

$$f(\omega,x) = h^{-1}(\theta\omega)A(\omega)h(\omega)(x)$$

for all x in the domain of the composition.

Proof: Define $\widetilde{f}(\omega,x) := A(\omega,x) + \widetilde{\Psi}(\omega,x)$ like in the Lemma 4.3. By Theorem 4.1 there exists a global random diffeomorphism $\widetilde{h} \in \mathrm{Dif}_0^1(\Omega,\mathbb{R}^m)$ such that $\widetilde{h}(\theta\omega)\widetilde{f}(\omega,\cdot) = D\widetilde{f}(\omega,0)\widetilde{h}(\omega)(\cdot)$. Take the restriction $h = \widetilde{h}|_{U(\omega)}$. □

Before we close this section we mention that the results presented here can be extended to higher degrees of differentiability, just adapting the norms in each appropriate space.

5. Continuous versions

In this section we deal with continuous random dynamical systems, more specifically, we will concern mainly with systems generated by stochastic differential equations. Particularly, the local version will be proved for this last case. We shall deal with perfect cocycles once in this case every crude cocycle is indistinguishable from a perfect cocycle (see Arnold and Scheutzow [3]).

Let $\varphi(t,\omega)$ be an stochastic flow such that $p=0$ is a hyperbolic fixed point. As before, we separate the linear and non-linear part:

$$\varphi(t,\omega,\cdot)=\Phi(t,\omega,\cdot)+\Psi(t,\omega,\cdot)$$

where

$$\Phi(t,\omega):=D_0\varphi(t,\omega)$$

and $\Psi(t,\omega,\cdot)$ is the corresponding non-linear part.

5.1. Global version

The assumptions for the following global version of HGT rest only upon the time-one random diffeomorphism $\varphi(1,\omega,\cdot)$; essentially it has to satisfy the hypotheses (H1) and (H2) stated for the global discrete HGT in section 3.

Theorem 5.1 (HGT, global continuous case) *Assume that the time-one non-linear part $\Psi(1,\omega,\cdot)$ satisfies the hypotheses (H1) and (H2). Hence, there exists a unique $H\in \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ such that for all $t\in\mathbb{R}$*

$$\varphi(t,\omega,\cdot)=H(\theta_t\omega)^{-1}\Phi(t,\omega)H(\omega)(\cdot).$$

Proof: By Theorem 3.1 there exists a unique $h\in \mathrm{Homeo}(\Omega,\mathbb{R}^m)$ such that

$$h(\theta\omega)\varphi(1,\omega,\cdot)=\Phi(1,\omega)h(\omega)(\cdot).$$

Let $k\in\mathbb{Z}$, then, by induction:

$$h(\theta_k\omega)(\cdot)=\Phi(k,\omega)h(\omega)\varphi(-k,\theta_k\omega,\cdot).$$

The proof follows essentially as a random adaptation of the deterministic arguments. We follow S. Sternberg [21, Lemma 4]. Define:

$$H(\omega,x)=\int_0^1\Phi(-s,\theta_s\omega)h(\theta_s\omega)\varphi(s,\omega,x)ds. \tag{5.1}$$

Before we show that H is in fact the homeomorphism of the statement, we prove the existence of the integral. Initially, note that $\Phi(-s, \theta_s\omega)$ is continuous in s once it corresponds to the inverse $\Phi(s,\omega)^{-1}$. Secondly, using the continuity of φ and the fact that $h(\theta_s\omega) = I + u(\theta_s\omega)$ with $u \in C_{0,b}(\Omega, \mathbb{R}^m)$, it only remains to proof that $\Phi(-s, \theta_s\omega)u(\theta_s\omega)\varphi(s,\omega,x)$ is integrable in the interval $s \in [0,1]$. By Proposition 2.1 and its corollary we have that

$$\|\Phi(s,\omega,\cdot)u(\theta_s\omega,\cdot)\| \leq B(\omega)e^{\Lambda+a}\left\|u(\theta_s\omega,\cdot)\right\|_{\theta_s\omega},$$

for all $s \in [0,1]$. Finally, by $\mathbb{P}$-invariance of θ_s and Tonelli-Fubini Theorem:

$$\mathbb{E}\int_0^1 \sup_{\mathbb{R}^m}\left\|u(\theta_s\omega,\cdot)\right\|_{\theta_s\omega}\ ds = \int_0^1 \mathbb{E}\left[\sup_{\mathbb{R}^m}\left\|u(\omega,\cdot)\right\|_{\omega}\right] ds < +\infty.$$

Hence, for a.a. ω the integral of equation 5.1 makes sense.

Now, we show that H conjugates the flows Φ and φ for t in the interval $[-1,1]$. One sees by the definition that

$$\Phi(t,\omega)H(\omega)(x) = \int_0^1 \Phi(t-s,\theta_s\omega)h(\theta_s\omega)\varphi(s-t,\theta_t\omega,\cdot)ds\ \varphi(t,\omega,x).$$

With the change of variable $r = s-t$, we have:

$$\begin{aligned}
\Phi(t,\omega)H(\omega)(x) &= \int_{-t}^{1-t} \Phi(-r,\theta_{r+t}\omega)h(\theta_{r+t}\omega)\varphi(r,\theta_t\omega,.)dr\ \varphi(t,\omega,x) \\
&= \int_{-t}^{0} \Phi(-r,\theta_{r+t}\omega)h(\theta_{r+t}\omega)\varphi(r,\theta_t\omega,\cdot)dr\ \varphi(t,\omega,x) \\
&\quad + \int_{0}^{1-t} \Phi(-r,\theta_{r+t}\omega)h(\theta_{r+t}\omega)\varphi(r,\theta_t\omega,\cdot)dr\ \varphi(t,\omega,x).
\end{aligned}$$

The first integral is:

$\int_{-t}^0 \Phi(-r,\theta_{r+t}\omega)h(\theta_{r+t}\omega)\varphi(r,\theta_t\omega,\cdot)dr$

$= \int_{-t}^0 \Phi(-r-1,\theta_{r+t+1}\omega)\Phi(1,\theta_{t+r}\omega)h(\theta_{r+t}\omega)\varphi(-1,\theta_{r+t+1}\omega,\cdot)\varphi(r+1,\theta_t\omega,\cdot)dr$

$= \int_{-t}^0 \Phi(-r-1,\theta_{r+t+1}\omega)h(\theta(\theta_{r+t}\omega))\varphi(r+1,\theta_t\omega,\cdot)dr,$

once by Theorem 3.1, we have that

$$\Phi(1,\theta_{t+r}\omega)h(\theta_{r+t}\omega)\varphi(-1,\theta_{r+t+1}\omega,.) = h(\theta(\theta_{r+t}\omega)). \tag{5.2}$$

Now, changing the variable $s = r + 1$, we have:

$$\int_{1-t}^{1} \Phi(-s, \theta_{s+t}\omega) h(\theta_{s+t}\omega)) \varphi(s, \theta_t \omega, \cdot) ds.$$

Hence,

$$\begin{aligned}
\Phi(t,\omega)H(\omega)(x) &= \left[\int_{1-t}^{1} \Phi(-s, \theta_{s+t}\omega) h(\theta_{s+t}\omega) \varphi(s, \theta_t\omega, \cdot) ds \right. \\
&\quad \left. + \int_{0}^{1-t} \Phi(-r, \theta_{r+t}\omega) h(\theta_{r+t}\omega) \varphi(r, \theta_t\omega, \cdot) dr \right] \varphi(t,\omega,x) \\
&= \left[\int_{0}^{1} \Phi(-s, \theta_{s+t}\omega) h(\theta_{s+t}\omega)) \varphi(s, \theta_t\omega, \cdot) ds \right] \varphi(t,\omega,x) \\
&= H(\theta_t\omega)\varphi(t,\omega,x).
\end{aligned}$$

Finally, let $t \in \mathbb{R}$ and write $t = k + s$ with $k \in \mathbb{Z}$ and $s \in [-1, 1]$. Then

$$\begin{aligned}
\Phi(t,\omega)H(\omega) &= \Phi(k+s,\omega)H(\omega) \\
&= \Phi(k, \theta_s\omega) H(\theta_s\omega) \varphi(s, \omega, .) \\
&= H(\theta_t\omega)\varphi(t, \omega, .).
\end{aligned}$$

Note that by the uniqueness established by Theorem 3.1, $H(\omega) = h(\omega)$ a.s. hence it is guaranteed the invertibility of H.

□

We remark that for a fixed ω and $x \in \mathbb{R}^m$ the map $t \mapsto H(\theta_t\omega, x)$ is continuous once $H(\theta_t\omega, x) = \Phi(t,\omega) \circ H(\omega) \circ (\varphi(t,\omega,\cdot))^{-1}(x)$.

5.2. Local version for SDE

In this section we present a local version of the Hartman-Grobman theorem for stochastic dynamical systems generated by stochastic Stratonovich differential equations. In order to fix our terminology, consider the following SDE in $\mathbb{R}^m$:

$$dx_t = f_0(x_t)dt + \sum_{i=1}^{k} f_i(x_t) \circ dB_t^i \tag{5.3}$$

where $(B_t^1, \cdots, B_t^k)$ is a Brownian motion in $\mathbb{R}^k$, $f_0, \ldots, f_k : \mathbb{R}^m \to \mathbb{R}^m$ are C^1-vector fields. We shall denote by $\varphi(t, \omega, \cdot)$ the solution flow. Negative time is obtained considering independent copies of Brownian motions for $t \geq 0$ and for $t \leq 0$, as in Boxler [4] or Arnold and Imkeller [2]. We assume that the origin is a hyperbolic fixed point.

We shall denote the linearised vector fields at the origin by $L_i = Df_i(0)$, for $i = 0, 1, \ldots, k$. As before, $\Phi(t, \omega)$ will denote the linear part of the flow φ, it is the solution of the linear SDE

$$dv_t = L_0(v_t)dt + \sum_{i=1}^{k} L_i(v_t) \ \circ dB_t^i.$$

As before, the non-linear part of the flow will be called $\Psi = \varphi - \Phi$.

The localisation argument starts fixing a positive radius $l \geq 0$. Take the C^1-vector fields $\tilde{f}_i$ which satisfies the conditions: $\tilde{f}_i(x) = f_i(x)$ for all $x \in B(0, l/2)$, the ball of centre in the origin and radius $l/2$ and $\tilde{f}_i(x) = L_i(x)$ for $x \notin B(0, l)$, with $i = 0, 1, \ldots, k$. We shall denote by $\tilde{\varphi}(t, \omega, x)$ the solution flow of the SDE

$$dx_t = \tilde{f}_0(x_t)dt + \sum_{i=1}^{k} \tilde{f}_i(x_t) \ \circ dB_t^i. \tag{5.4}$$

Obviously the linear part of φ and $\tilde{\varphi}$ coincides to Φ. The non-linear part of the flow $\tilde{\varphi}$ we shall denote by $\tilde{\Psi}(t, \omega, x) = \tilde{\varphi}(t, \omega, x) - \Phi(t, w)x$.

As before, in the next theorem the random variable $B(\omega)$ denotes the variable B_ε with $\varepsilon = 1$ defined in Proposition 2.1.

Theorem 5.2 (HGT, local case for SDE) *Let φ be the C^1-RDS generated by the SDE (5.3) such that the origin is a hyperbolic fixed point. If $B(\omega) \in L^1(\Omega)$ then there exists a random homeomorphism $H(\omega) : V(\omega) \to W(\omega)$, where $V(\omega)$ and $W(\omega)$ are random neighbourhoods of the origin, such that:*

$$H(\theta_t \omega)\varphi(t, \omega, x) = \Phi(t, \omega)H(\omega, x)$$

for $\mathbb{P}$-almost all ω and $t = t(x)$ in a random interval containing zero, such that x is in the domain of the composition.

Proof: The proof is based mainly in the discrete arguments of section 3. Let $x \in B(0, l/2)$ and consider the following stopping times:

$$\begin{aligned} T(\omega, x) &:= \inf \{t \geq 0, \varphi(t, \omega, x) \notin B(0, l/2)\}, \\ S(\omega, x) &:= \sup \{t \leq 0, \varphi(t, \omega, x) \notin B(0, l/2)\}; \end{aligned}$$

Once φ and $\tilde{\varphi}$ coincide in $B(0, l/2)$ in the random interval $t \in (S, T)$, we only have to prove that there exists a local conjugation $H(\omega)$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$ we have $H(\theta_t \omega)\tilde{\varphi}(t, \omega, x) = \Phi(t, \omega)H(\omega)(x)$.

The time-one diffeomorphism $\tilde{\varphi}(1, \omega, \cdot)$ (or more precisely, its nonlinear part $\tilde{\Psi}(1, \omega, \cdot)$) satisfies the hypothesis (H2) of section 3. That is, we claim that there exists a constant

$M > 0$ such that

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^k}\left\|\tilde{\Psi}_s(1,\omega,x)\right\|_{\theta\omega}\right] \le M, \qquad \text{and} \qquad \mathbb{E}\left[\sup_{x\in\mathbb{R}^k}\left\|\tilde{\Psi}_s(1,\omega,x)\right\|_{\theta\omega}\right] \le M.$$

First, with a fixed $\omega \in \Omega$, assume that $x \in \mathbb{R}^m$ is far enough from the origin, more precisely, assume that:

$$\|x\|_\omega > e^{(1+\Lambda+a)}\ B(\omega)\ l$$

where Λ and a are defined as in Corollary 2.2. Hence, by Proposition 2.1 and its Corollary again, for $0 \le t \le 1$ we have:

$$\|\Phi(t,\omega)x\| \ge \frac{\|\Phi(t,\omega)x\|_{\theta_t\omega}}{B(\theta_t\omega)} \ge \frac{e^{-(\Lambda+a)}\ \|x\|_\omega}{B(\theta_t\omega)} > l$$

Hence $\|\tilde{\varphi}(t,\omega,x)\| > l$ for all $0 \le t \le 1$. Since $\tilde{\varphi}$ and Φ coincide outside $B(0,l)$, we have that, in this case $\tilde{\Psi}(1,\omega,x) = 0$.

Secondly, assume that

$$\|x\|_\omega \le e^{(1+\Lambda+a)}\ B(\omega)\ l.$$

In this case the trajectory of x by $\tilde{\varphi}$ can pass through the ball $B(0,l)$, and can not follow the linear trajectory of Φ anymore. Moreover, one can not estimate the last exit time from $B(0,l)$ because it is not a stopping time. Nevertheless, in any case, one can guarantee that, for $0 \le t \le 1$:

$$\|\tilde{\varphi}_s(1,\omega,x)\|_{\theta\omega} \le \sup_{0\le t,r\le 1}\ \sup_{\|x\|=l} \|\Phi_s(r,\theta_t\omega)x\|_{\theta_{r+t}\omega},$$

and since $\|\Phi_s(r,\theta_t\omega)x\|_{\theta_{r+t}\omega} \le e^{(1+\Lambda+a)}B(\omega)\,\|x\|$ with $0 \le t, r \le 1$, we conclude that:

$$\|\tilde{\varphi}_s(1,\omega,x)\|_{\theta\omega} \le e^{2(\Lambda+a)+1}B(\omega)\ l.$$

On the other hand $\|\Phi_s(1,\omega)x\|_{\theta\omega} \le e^{(\Lambda+a)}\ \|x\|_\omega \le e^{2(\Lambda+a)+1}B(\omega)\ l$. Therefore:

$$\mathbb{E}\left[\sup_{\mathbb{R}^k}\left\|\tilde{\Psi}_s(1,\omega,\cdot)\right\|_{\theta\omega}\right] \le 2e^{2(\Lambda+a)+1}\ \mathbb{E}[B(\omega)]\ l < +\infty.$$

Analogously to the unstable part one finds that:

$$\mathbb{E}\left[\sup_{\mathbb{R}^k}\left\|\tilde{\Psi}_u(1,\omega,\cdot)\right\|_{\theta\omega}\right] \le e^{2(\Lambda+a)+1}\ (\mathbb{E}[B(\omega)] + e\ \mathbb{E}[B(\omega)]\ l)\ l < +\infty.$$

Now, applying Lemma 3.2 to the random (discrete in time) C^1-diffeomorphism $\tilde{\varphi}(1,\omega,\cdot)$ we conclude that there exists a unique continuous application $h = I + u$, with $u \in C_{0,b}(\Omega,\mathbb{R}^k)$ (h is not necessarily invertible) such that

$$h(\theta\omega)\tilde{\varphi}(1,\omega,x) = \Phi(1,\omega)h(\omega)(x). \tag{5.5}$$

The continuous dynamics are obtained applying the same calculations we did in the proof of Theorem 5.1 defining:

$$H(\omega,x)=\int_0^1 \Phi(-s,\theta_s\omega)h(\theta_s\omega)\tilde{\varphi}(s,\omega,x)ds.$$

We point out that although at this point we can not guarantee the invertibility of h (neither of H), the property of Lemma 3.2 stated in equation (5.5) is enough to carry on the calculations we did with H in the proof of Theorem 5.1, particularly equation (5.2).

The local invertibility of $H(\omega)$ follows by uniqueness of the conjugacy and the local invertibility of $h(\omega)$ in some neighbourhood $U(\omega)$ of the origin, guaranteed by Theorem 3.2. Finally, define the neighbourhood stated in the theorem by $V(\omega)=U(\omega)\cap B(0,l/2)$. □

Before we present an example, we show that it is possible to weaken the hypothesis of the last theorem substituting the random variable $B(\omega)$ by another variable which would satisfies the same basic properties of $B(\omega)$. Precisely, assume that there exists a strictly positive real random variable $C\in L^2(\Omega)$ such that $\|\cdot\|_\omega\le C(\omega)\|\cdot\|$ and $C(\theta_t\omega)\le e^{k|t|}\eta(t,\omega)C(\omega)$ where k is a positive constant and η is a continuous process such that $\sup_{0\le t\le 1}\eta(t,\omega)\in L^2(\Omega)$.

Corollary 5.1 *In the context of the last theorem, the local conjugacy described still holds if instead of integrability of $B(\omega)$ we assume that there exists a random variable $C(\omega)$ as described above.*

Proof: It is enough to prove that we still have $\tilde{\Psi}(1,\omega,\cdot)$ satisfying hypothesis (H2) of section 3.

Firstly, assume that $x\in\mathbb{R}^m$ is far enough from the origin, precisely:

$$\|x\|_\omega > l\ e^{\Lambda+a+k}\ C(\omega)\ \sup_{0\le t\le 1}\eta(t,\omega).$$

Then

$$\|\Phi(t,\omega)x\|\ge\frac{\|\Phi(t,\omega)x\|_{\theta_t\omega}}{C(\theta_t\omega)}\ge\frac{e^{-(\Lambda+a+k)}\|x\|_\omega}{C(\omega)\sup_{0\le t\le 1}\eta(t,\omega)}>l,$$

for all $0\le t\le 1$, hence $\tilde{\varphi}(1,\omega,x)=0$.

Now, consider the case

$$\|x\|_\omega\le l\ e^{\Lambda+a+k}\ C(\omega)\ \sup_{0\le t\le 1}\eta(t,\omega).$$

As in the proof of the last theorem, either if the trajectory of $\tilde{\varphi}(x)$ passes through the ball $B(0, l/2)$ or not,

$$\|\tilde{\varphi}_s(1,\omega,x)\|_{\theta\omega} \leq \sup_{0\leq t,r\leq 1} \sup_{\|x\|=l} \|\Phi_s(r,\theta_t\omega,x)\|_{\theta_{t+r}\omega},$$

and for $0 \leq t, r \leq 1$

$$\begin{aligned}\|\Phi_s(r,\theta_t\omega,x)\|_{\theta_{r+t}\omega} &\leq e^{(\Lambda+a)(r)}e^{kt} \sup_{0\leq t\leq 1} \eta(t,\omega)C(\omega)\,\|x\| \\ &\leq e^{2(\Lambda+a)+k} \sup_{0\leq t\leq 1} \eta(t,\omega)C(\omega)l.\end{aligned}$$

Hence

$$\|\tilde{\varphi}_s(1,\omega,x)\|_{\theta\omega} \leq e^{2(\Lambda+a)+k}\ C(\omega)\ l \sup_{0\leq t\leq 1} \eta(t,\omega),$$

and

$$\|\Phi_s(1,\omega,x)\|_{\theta\omega} \leq e^{2(\Lambda+a)+k}\ C(\omega)\ l \sup_{0\leq t\leq 1} \eta(t,\omega).$$

Therefore,

$$\mathbb{E}\left[\sup_{x\in\mathbb{R}^k} \left\|\tilde{\Psi}_s(1,\omega,x)\right\|_{\theta\omega}\right] \leq 2e^{2(\Lambda+a)+k} \left\|\sup_{0\leq t\leq 1} \eta(t,\omega)\right\|_{L^2} \cdot \|C(\omega)\|_{L^2}.$$

For the unstable part one calculates an analogous estimate. □

We present an example which illustrates this last corollary.

Example 5.1. Consider the following SDE in $\mathbb{R}^2$

$$d(x_t,y_t) = f_0(x,y)\ dt + f_1(x,y)\ dB_t^1 + f_2(x,y)\ dB_t^2 \tag{5.6}$$

where (B_t^1, B_t^2) is a Brownian motion on $\mathbb{R}^2$,

$$f_0(x,y) = \begin{bmatrix} -\alpha_1 + \beta_1^2\ x(1-x^2) \\ \\ -\alpha_2 + \beta_2^2\ y(1-y^2) \end{bmatrix}, \qquad f_1(x,y) = \begin{bmatrix} -\beta_1^2\ (1-x^2) \\ \\ 0 \end{bmatrix},$$

and

$$f_2(x,y) = \begin{bmatrix} 0 \\ \\ -\beta_2^2\ (1-y^2) \end{bmatrix},$$

with α_i and β_i, $i = 1, 2$, real constants such that $\alpha_1, \alpha_2 \neq 0$, and $8\beta_i^2 < a$, where a is the constant of Proposition 2.1.

Note that the points $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ are singularities of the vector fields f_0, f_1 and f_2. We shall focus our attention at the point $x_0=(-1,-1)$. The solution of equation (5.6) is given by the decoupled flow $\varphi(t,\omega,x,y)=(\varphi_1(t,\omega,x,y),\varphi_2(t,\omega,x,y))$ where

$$\varphi_1(t,\omega,x,y)=\frac{(1+x)\exp(-2\alpha_1 t+2\beta_1 B_t)+x-1}{(1+x)\exp(-2\alpha_1 t+2\beta_1 B_t)+1-x}$$

and

$$\varphi_2(t,\omega,x,y)=\frac{(1+y)\exp(-2\alpha_2 t+2\beta_2 B_t)+y-1}{(1+y)\exp(-2\alpha_2 t+2\beta_2 B_t)+1-y}.$$

(See Kloeden and Platen [15, Pag. 124]). The linearization at our point x_0 is given by

$$d\varphi(t,\omega,(-1,-1))=\begin{bmatrix}\exp(-2\alpha_1 t+2\beta_1 B_t^1) & 0\\ 0 & \exp(-2\alpha_2 t+2\beta_2 B_t^2)\end{bmatrix}.$$

The Osseledet's subspaces are deterministic and correspond to the canonical axes $E_i(\omega)=\mathbb{R}\cdot e_i$ for $i=1,2$. If $x\in E_i(\omega)$ then the Lyapunov exponent λ_i are

$$\lambda_i=\lim_{t\to\infty}\frac{1}{t}\log\|d\varphi(t,\omega,(-1,-1))x\|=-2\alpha_i\neq 0,$$

hence $(-1,-1)$ is hyperbolic. Let $x=(x_1,x_2)\in\mathbb{R}^2$, by the very definition of the random norm (Definition 2.3):

$$\begin{aligned}\|x\|_\omega^2 &= \int_{-\infty}^{+\infty}\frac{(x_1)^2 e^{4(-\alpha_1 t+\beta_1 B_t^1)}}{e^{2(\lambda_1 t+a|t|)}}\,dt+\int_{-\infty}^{+\infty}\frac{(x_2)^2 e^{4(-\alpha_2 t+\beta_2 B_t^2)}}{e^{2(\lambda_2 t+a|t|)}}dt\\ &= \int_{-\infty}^{+\infty}(x_1)^2 e^{4\beta_1 B_t^1-2a|t|}\,dt+\int_{-\infty}^{+\infty}(x_2)^2 e^{4\beta_2 B_t^2-2a|t|}\,dt.\end{aligned}$$

Define

$$c_1(\omega):=\int_{-\infty}^{+\infty}e^{4\beta_1 B_t^1-2a|t|}\,dt,$$

and

$$c_2(\omega):=\int_{-\infty}^{+\infty}e^{4\beta_2 B_t^2-2a|t|}\,dt.$$

We claim that the measurable functions c_1, c_2 are square integrable. In fact

$$\begin{aligned}\mathbb{E}\left[\int_{-\infty}^{+\infty}\exp(4\beta_1 B_t^1-2a\,|t|)dt\right]^2 &= \mathbb{E}\left[\int_0^{+\infty}\exp(4\beta_1 B_t^1-2at)dt\right.\\ &\quad\left.+\,\mathbb{E}(\int_{-\infty}^0\exp(4\beta_1 B_t^1+2at)dt\right]^2. \qquad (5.7)\end{aligned}$$

We show that each integral in the right hand side is in $L^2(\Omega)$. In fact,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{+\infty} \exp(4\beta_1 B_t^1 - 2at)dt\right]^2 \\
= \;& \mathbb{E}\left[\int_0^{+\infty}\int_0^{+\infty} \exp(4\beta_1 B_s^1 - 2as)ds \exp(4\beta_1 B_t^1 - 2at)dt\right] \\
= \;& \int_0^{+\infty} \mathbb{E}\left[\int_0^{+\infty} \exp(4\beta_1 B_s^1 - 2as)ds \exp(4\beta_1 B_t^1 - 2at)dt\right] \\
\leq \;& \int_0^{+\infty} \left(\mathbb{E}\left[\int_0^{+\infty} \exp(4\beta_1 B_s^1 - 2at)ds\right]^2\right)^{1/2} \\
& \cdot \left(\mathbb{E}\left[\exp(4\beta_1 B_t^1 - 2at)\right]^2\right)^{1/2} dt \\
= \;& M\left(\mathbb{E}\left[\int_0^{+\infty} \exp(4\beta_1 B_s^1 - 2as)ds\right]^2\right)^{1/2},
\end{aligned}$$

where $M = \int_0^{+\infty} \left(\mathbb{E}\left[\exp(4\beta_1 B_t^1 - 2at)\right]^2\right)^{1/2} dt$. Sumarizing, we have that:

$$\left(\mathbb{E}\left[\int_0^{+\infty} \exp(4\beta_1 B_t^1 - 2at)dt\right]^2\right)^{1/2} \leq M.$$

Note that the integrand in the definition of M is a martingale, hence $\mathbb{E}\left[\exp(8\beta_1 B_t^1)\right] = \exp(32\beta_1^2 t)$ for all $t \geq 0$, so,

$$M = \int_0^{+\infty} \exp 2t(8\beta_1^2 - a)dt$$

which converges once $8\beta_1^2 - a < 0$ Analogously for the second integral of equation (5.7), one finds:

$$\mathbb{E}\left[\int_{-\infty}^0 \exp(4\beta_1 B_t^1 + 2at)dt\right]^2 \leq \int_{-\infty}^0 \exp 2t(8\beta_1^2 + a)dt$$

which converges once $8\beta_1^2 + a > 0$. By our calculations and Cauchy- Schwarz inequality we have $c_1 \in L^2(\Omega)$. In the same way one checks that $c_2 \in L^2(\Omega)$, hence $\|x\|_\omega \leq kC(\omega)\|x\|$ where k is a constant and $C(\omega) = \max\{c_1(\omega), c_2(\omega)\} \in L^2(\Omega)$. By construction, we have $c_1(\theta_t\omega) \leq e^{-4\beta_1 B_t^1} e^{2a|t|} c_1(\omega)$. To fulfill the hypotheses of Corollary 5.1 it only remains to prove that $a_1(\omega) = \sup_{0\leq t\leq 1} e^{-4\beta B_t^1}$ is square integrable. By Ito formula:

$$e^{-4\beta_1 B_t^1} = 1 - 4\beta_1 \int_0^t e^{-4\beta_1 B_s^1} dB_s^1 + 8\beta_1^2 \int_0^t e^{-4\beta_1 B_s^1} ds,$$

hence, by the Burkholder-Doob inequality:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le 1} e^{-4\beta_1 B_t^1}\right]^2 &\le k\left[1+16\ \beta_1^2\ \mathbb{E}[\int_0^1 e^{-4\beta_1 B_s^1}\ dB_s^1\ ]^2 \right.\\
&\quad \left. +64\ \beta_1^4\ (\int_0^1 (\mathbb{E}\left[e^{-8\beta_1 B_s^1}\right])^{1/2} ds)^2\right]\\
&\le k\left[1+16\beta_1^2\mathbb{E}[\int_0^1 e^{-4\beta_1 B_s^1} dB_s^1]^2 + 64\beta_1^4(\int_0^1 (e^{16\beta_1 s} ds)^2\right]\\
&\le k\left[1+16\beta_1^2\mathbb{E}[\int_0^1 e^{-8\beta_1 B_s^1} ds] + 64\beta_1^4(\int_0^1 e^{16\beta_1^2 s} ds)^2\right]\\
&\le k\left[1+16\beta_1^2\int_0^1 e^{32\beta_1^2 s} ds + 64\beta_1^4(\int_0^1 e^{16\beta_1^2 s} ds)^2\right].
\end{aligned}$$

Analogously, one checks that

$$c_2(\theta_t\omega) \le e^{-4\beta_2 B_t^2} e^{2a|t|} c_2(\omega)$$

and the random variable $a_2(\omega) = \sup_{0\le t\le 1} e^{-4\beta_2 B_t^2}$ is square integrable as well. Hence,

$$C(\theta_t\omega) \le e^{|t|}\eta(\omega,t)C(\omega)$$

where $\eta(\omega,t) = \max\left\{e^{-4\beta_1 B_t^1}, e^{-4\beta_2 B_t^2}\right\}$ and $\sup_{0\le t\le 1}\eta(\omega,t) \in L^2(\Omega)$. It follows by Corollary 5.1 that indeed there exists a local random conjugation of the system (5.6) with its linearization.

References

[1] L. Arnold, *Random Dynamical Systems.* Springer, Berlin (1998).

[2] L. Arnold and P. Imkeller, Stratonovich calculus with spatial parameters and anticipative problems in multiplicative ergodic theory, *Stochastic Processes and their Applications,* **62 (**1996) 19-54.

[3] Arnold, L. and Scheutzow, M. – Perfect cocycles through stochastic differential equations. *Prob. Th. Rel. Fields,* **101**, p. 65–88, 1995 .

[4] P. Boxler, A Stochastic Version of Center Manifold Theory. *Probability Theory and Related Fields.* **83,** (1989), 509-545.

[5] A. P. Carverhill, Flows of stochastic dynamical systems: ergodic theory . *Stochatics*, **14** (1985), 273-317.

[6] A. P. Carverhill, A nonrandom Lyapunov spectrum for nonlinear stochastic dynamical systems. *Stochatics*, **17** (1986), 253-287.

[7] N. D. Cong, Topological classification of linear hyperbolic cocycles, *Journal of Dynamics and Differential Equations*, **8,** no. 3 (1996), 427-467.

[8] K. D. Elworthy, Geometric aspects of diffusions on manifolds. In: *Ecole d'Eté de Probabilitiés de Saint-Flour*, XV-XVII, 1985-1987 (P.L. Hennequin, ed.) 276-425. Lectures Notes Math. **1362**, Springer, 1987.

[9] H. Furstenberg and Y. Kifer, Random matrix products and measures on projective spaces. *Israel Journal of Mathematics,* **46,** nos. 1-2 (1983), 12- 32.

[10] G. B. Folland, *Real Analysis - Modern Techniques and their Applications.* Jonh Wiley & Sons. Inc., 1980.

[11] D. M. Grobman, Homeomorphisms of systems of differential equations, *Dokl. Akad. Nauk SSSR*, **128 (**1959), 880-881.

[12] P. Hartman, A lemma in the theory of structural stability of diferential equations. *Proc. A.M.S.* **11** (1960), 610-620.

[13] P. Hartman, On local homeomorphisms of Euclidean spaces, *Bol. Soc. Math. Mexicana,* **5** (1960), 220-241.

[14] V. Hutson and J. S. Pym, *Applications of Functional Analysis and Operator Theory.* Academic Press Inc (London) LTD, 1980.

[15] P. E. Kloeden and E. Platen *Numerical Solutions of Stochastic Differrential Equations.* Springer (1995).

[16] H. Kunita, Stochastic differential equations and stochastic flows of diffeomorphisms, In: *Ecole d'Eté de Probabilitiés de Saint-Flour*, XII, 1982, Lectures Notes Math. **1097**, (1984) 143-303.

[17] H. Kunita, *Stochastic flows and stochastic differential equations,* Cambridge Univ. Press, 1990.

[18] S.-E. A. Mohammed and M. Scheutzow, The stable manifold theorem for stochastic differential equations, *Annals of Prob.*, Vol. **27**, no. 2, pp. 615-652, 1999.

[19] J. Palis Jr. and Melo, W. *Geometric theory of Dynamical systems: an introduction.* Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[20] V. I. Oseledets, A multiplicative ergodic theorem. Lyapunov characteristics numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19**, (1968) 197-231.

[21] S. Sternberg, Local contractions and a theorem of Poincaré. *Amer. J. Math.* vol. **79** (1957), 809-824.

[22] D. Ruelle, Ergodic theory of differentiable dynamical systems, *Publications IHES*, **50**, (1979) 275-305.

[23] T. Wanner, Linearization of random dynamical systems. In John, C., Kirchgraaber, U., and Walter, H. O. (Eds.), Dynamics Reported, *Vol.* **4**, Springer, Berlin/Heidelberg/New York (1994).