The Cauchy Problem for Micropolar Fluid Equations

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Abstract: We consider the Cauchy problem for equation of a nonstationary micropolar fluid in three dimensional whole space. We show the local, global existence and the asymptotic behaviours of a strong solutions.

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1. Introduction

Micropolar fluid theory was introduced by Eringen [7] in order to describe some physical systems which do not satisfy the Navier-Stokes equations. These fluids are able to describe the behavior of colloidal solutions, suspension solutions, liquid crystal, animal blood, etc. The equations governing the flow of a micropolar fluid involve a spin vector and a microinertia tensor in addition to the velocity vector.

Over the past years, many existence (weak and strong), uniqueness results has been done for the micropolar fluids. The Dirichlet problem in a bounded domain was investigated in [13], [14] (see also [15]), [19], [20], [16], [17], in an exterior domains see [6], [18], and in an unbounded domain [4].

We observe that the Cauchy problem for micropolar fluid equations has not been studied, thus it is the aim of this paper to construct strong solutions for the Cauchy problem for the micropolar fluids.

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The governing system of Cauchy problem for equations of micropolar fluids is the following

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\nu+\nu_r)\Delta\mathbf{u} + \nabla p = 2\nu_r \text{ rot } \mathbf{w} + \mathbf{f}, \text{ in } I\!\!R^3 \times (0,T), \\ \text{div } \mathbf{u} = 0 \text{ in } I\!\!R^3 \times (0,T), \\ \dfrac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w} \\ \qquad\qquad +4\nu_r\mathbf{w} = 2\nu_r \text{ rot } \mathbf{u} + \mathbf{g} \text{ in } I\!\!R^3 \times (0,T), \\ \mathbf{u} \to 0 \text{ and } \mathbf{w} \to 0 \text{ as } |\mathbf{x}| \to \infty, \\ \mathbf{u}(\mathbf{x},0) = a(\mathbf{x}) \text{ and } \mathbf{w}(\mathbf{x},0) = b(\mathbf{x}) \text{ in } I\!\!R^3 \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1,\, u_2,\, u_3)$ is the velocity field, p is the pressure, and $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is the microrotational interpreted as the angular velocity field of rotational of particles. The fields $\mathbf{f} = (f_1,\, f_2,\, f_3)$ and $\mathbf{g} = (g_1,\, g_2,\, g_3)$ are external forces and moments respectively. Positive constants $\nu,\, \nu_r,\, c_0,\, c_a,\, c_d$ represent viscosity coefficients, ν is the usual Newtonian viscosity and ν_r is called the microrotational viscosity.

We observe that the classical Navier-Stokes equations is a special case of the system (1.1). To the authors knowledge the first work the existence for the Cauchy problem for the Navier-Stokes equations is due to Leray (see the Ladyzhenskaya' book [10])

Recently, He [9] prove the local and global existence and the asymptotic behaviours of strong solutions to the Cauchy problem for the Navier-Stokes equations.

Our purpose, roughly speaking, is to extend the previous work to the micropolar fluid equations. In fact, we observe that, with the results of this paper, our knowledge about strong solutions of the micropolar fluid equations approaches a level similar to that of the classical Navier-Stokes equations.

The structure of this paper is as follows. In Section 2, we give the notations, results that we will used in this article and we state the results. In Section 3, we discuss the linearized problem for (1.1) and finally in Section 4 we give the proofs of theorems.

2. Statements and notations

Let $W^{l,p}(\mathbb{R}^3)$, $l \in \mathbb{N}, 1 \leq p \leq +\infty$, be the usual Sobolev space over $\mathbb{R}^3$ such that $W^{0,p}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$. We denote by $\|\cdot\|_p$ the norm of $L^p(\mathbb{R}^3)$ and that of $(L^p(\mathbb{R}^3))^3$, $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\mathbb{R}^3)$. Let $C^\infty_{0,\sigma}(\mathbb{R}^3)$ denote the set of all C^∞ real vector fields $\mathbf{v}$ with compact support in $\mathbb{R}^3$ such that div $\mathbf{v}$ =0. By H we denote the completion of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$ and by V the completion $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in $W^{1,2}(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

If X is a Banach space, with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1 \leq p \leq +\infty$, for the space of functions v on $(0,T)$ with values in X such that the real valued function $t \to \|v(t)\|_X$ belongs to $L^p(0,T)$. The space of continuous functions from $[0,T]$ into X is denoted by $C([0,T)],X)$.

We will consider the following space $W^{1,1}_2(Q_T) = \left\{\mathbf{u} \ / \ \mathbf{u} \in L^2(0,T;H^1(\mathbb{R}^3)), \frac{d\mathbf{u}}{dt} \in L^2(Q_T)\right\}$ and $Q_T = \mathbb{R}^3 \times [0,T)$.

We adopted of definition in, given for Navier-Stokes equation, then we give the respective for micropolar fluids equations.

Definition 2.1. *The pair* $(\mathbf{u}(x,t), \mathbf{w}(x,t))$ *is called a strong solution of micropolar fluids equations with initial values* $a, b,$ *if* $\mathbf{u} \in L^\infty(0,T;H \cap L^6(\mathbb{R}^3)) \cap L^2(0,T;V)$ *and* $\mathbf{w} \in L^\infty(0,T;L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)) \cap L^2(0,T;H^1_0(\mathbb{R}^3))$, *for* $T > 0$ *and it satisfy the identities*

$$\int_0^{T_0} \{(\mathbf{u}, \varphi_t) - (\nu + \nu_r)(\nabla \mathbf{u}, \nabla \varphi) + ((\mathbf{u} \cdot \nabla)\varphi, \mathbf{u})\} dt - (a, \varphi(0)) =$$

$$\int_0^{T_0} (2\nu_r \ rot \ \mathbf{w} + \mathbf{f}, \varphi), \quad \forall \ \varphi \in W^{1,1}_2(Q_T)$$

$$\int_0^{T_0} \{(\mathbf{w}, \phi_t) - (L^{\frac{1}{2}}\mathbf{w}, L^{\frac{1}{2}}\phi) + ((\mathbf{u} \cdot \nabla)\phi, \mathbf{w}) - 4\nu_r(\mathbf{w}, \phi)\} dt - (b, \phi(0)) =$$

$$\int_0^{T_0} (2\nu_r \ rot \ \mathbf{u} + \mathbf{g}, \phi) \quad \forall \ \phi \in W^{1,1}_2(Q_T)$$

with $\varphi(T) = \phi(T) = 0$.

The main Theorems of this paper are the following.

Theorem 2.2. *Let the initial values* $a \in H \cap L^6(\mathbb{R}^3)$, $b \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, *and the external forces* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $L^2(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^{4/3}(\mathbb{R}^+; L^6(\mathbb{R}^3))$. *Then on a* *time interval* $[0, T_0]$, $T_0 < T$ *exist a unique solution* $(\mathbf{u}, \mathbf{w})$ *of Cauchy problem for micropolar equations. (1.1), such that*

$$\mathbf{u} \in L^\infty(0, T_0; H \cap L^6(I\!R^3)) \text{ and } \mathbf{w} \in L^\infty(0, T_0; L^2(I\!R^3) \cap L^6(I\!R^3))$$

$$\mathbf{u} \in L^2(0, T_0; V) \text{ and } \mathbf{w} \in L^2(0, T_0; H_0^1(I\!R^3)),$$

and

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T_0; V') \text{ and } \frac{\partial \mathbf{w}}{\partial t} \in L^2(0, T_0; H^{-1}(I\!R^3)).$$

and

$$\mathbf{u}(t) \to a \text{ and } \mathbf{w}(t) \to b \text{ as } t \to 0.$$

And

Theorem 2.3. *Let the initial values* $a \in H \cap L^6(I\!R^3)$, $b \in L^2(I\!R^3) \cap L^6(I\!R^3)$, *and the external forces* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $L^2(I\!R^+; L^2(I\!R^3)) \cap L^{4/3}(I\!R^+; L^6(I\!R^3))$. *Then exist positives constants*

$$\lambda_1 = \lambda_1(\|a\|_2, \|b\|_2, \|a\|_2, \|\mathbf{f}\|_{L^2(I\!R^+; L^2(I\!R^3))}, \|\mathbf{f}\|_{L^{\frac{4}{3}}(I\!R^+; L^6(I\!R^3))}, \|\mathbf{g}\|_{L^2(I\!R^+; L^2(I\!R^3))}),$$
$$\lambda_2 = \lambda_2(\|a\|_2, \|b\|_2, \|\mathbf{f}\|_{L^2(I\!R^+; L^2(I\!R^3))}, \|\mathbf{g}\|_{L^2(I\!R^+; L^2(I\!R^3))}),$$

such that if

$$\|a\|_2^2 + \|b\|_2^2 + \int_0^\infty (\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2) d\tau < \lambda_1,$$

or

$$\|a\|_2^2 + \|b\|_2^2 + \|\mathbf{u}_0\|_6 + \|\mathbf{g}\|^2_{L^2(I\!R^+; L^2(I\!R^3))} + 2\|\mathbf{f}\|^2_{L^2(I\!R^+; L^2(I\!R^3))} + \|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+; L^6(I\!R^3))} < \lambda_2,$$

then the solution of Theorem 1 is global, i.e., $T_0 = +\infty$. *Moreover*

$$\mathbf{u} \in L^\infty(0, +\infty; H \cap L^6(I\!R^3)) \text{ and } \mathbf{w} \in L^\infty(0, +\infty; L^2(I\!R^3) \cap L^6(I\!R^3)).$$

Furthermore, if

$$\|\mathbf{f}(t)\|_2^2 + \|\mathbf{f}(t)\|_6 \leq C(1+t)^{-\alpha}, \quad \alpha > 0, \tag{2.1}$$
$$\|\mathbf{g}(t)\|_6 \leq C(1+t)^{-\alpha}, \quad \alpha > 0, \tag{2.2}$$

then

$$\|\mathbf{u}(t)\|_6^6 \leq C(1+t)^{-\beta} \text{ for } t \geq 1, \tag{2.3}$$
$$\|\mathbf{w}(t)\|_6^6 \leq C(1+t)^{-\beta} \text{ for } t \geq 1, \tag{2.4}$$

with $\beta = \min\{3, \frac{3}{4}\alpha\}$. *Especially, if* $\mathbf{f} = \mathbf{g} = 0$, *then*

$$\|\mathbf{u}(t)\|_6^6 \leq C(1+t)^{-3} \textit{ for } t \geq 1,$$
$$\|\mathbf{w}(t)\|_6^6 \leq C(1+t)^{-3} \textit{ for } t \geq 1.$$

Finally, we would like to say that, as it usual in this context, to simplify the notation we will denote by C generic finite positive constants depending only on Sobolev embedding, other fixed parameters of the problem, etc., that may have different values in different expressions. Sometimes, to emphasize the fact that the constants are different, we use $C_1, C_2, ...$, and so on.

3. Linearized Problems

We define the approximate solution, in order to establish some basic estimates. Let $a \in H \cap L^6(I\!R^3)$, $b \in L^2(I\!R^3) \cap L^6(I\!R^3)$. We choose $a^k \in V$, and $b^k \in H_0^1(I\!R^3)$ such that $a^k \to a$ in $H \cap L^6(I\!R^3)$ and $b^k \to b$ in $L^2(I\!R^3) \cap L^6(I\!R^3)$, and

$$\|a^k\|_2 \leq C\|a\|_2, \qquad \|a^k\|_6 \leq C\|a\|_6,$$
$$\|b^k\|_2 \leq C\|b\|_2, \qquad \|b^k\|_6 \leq C\|b\|_6,$$

We now consider the Cauchy problem for linearized of the micropolar fluid equations in $I\!R^3$, inspired in the process iterative given in [17].

$$\begin{cases} \mathbf{u}_t^0 - (\nu+\nu_r)\Delta\mathbf{u}^0 + \nabla p^0 = \mathbf{f} \text{ in } I\!R^3 \times (0,T), \\ \text{div } \mathbf{u}^0 = 0 \text{ in } I\!R^3 \times (0,T), \\ \mathbf{w}_t^0 - (c_a+c_d)\Delta\mathbf{w}^0 - (c_0+c_d-c_a)\nabla(\text{ div } \mathbf{w}^0) + 4\nu_r\mathbf{w}^0 = \mathbf{g} \text{ in } I\!R^3 \times (0,T), \\ \mathbf{u}^0 \to 0 \text{ and } \mathbf{w}^0 \to 0 \text{ as } |\mathbf{x}| \to \infty, \\ \mathbf{u}^0(\mathbf{x},0) = a^0(\mathbf{x}) \text{ and } \mathbf{w}^0(\mathbf{x},0) = b^0(\mathbf{x}) \text{ in } I\!R^3 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \mathbf{u}_t^k - (\nu+\nu_r)\Delta \mathbf{u}^k + (\mathbf{u}^{k-1}\cdot\nabla)\mathbf{u}^k + \nabla p^k = 2\nu_r \text{ rot } \mathbf{w}^k + \mathbf{f} \text{ in } I\!R^3 \times (0,T), \\ \\ \text{div } \mathbf{u}^k = 0 \text{ in } I\!R^3 \times (0,T), \\ \\ \mathbf{w}_t^k - (c_a+c_d)\Delta \mathbf{w}^k - (c_0+c_d-c_a)\nabla(\text{ div } \mathbf{w}^k) + (\mathbf{u}^{k-1}\cdot\nabla)\mathbf{w}^k \\ \qquad +4\nu_r \mathbf{w}^k = 2\nu_r \text{ rot } \mathbf{u}^k + \mathbf{g} \text{ in } I\!R^3 \times (0,T), \\ \\ \mathbf{u}^k \to 0 \text{ and } \mathbf{w}^k \to 0 \text{ as } |\mathbf{x}| \to \infty, \\ \\ \mathbf{u}^k(\mathbf{x},0) = a^k(\mathbf{x}) \text{ and } \mathbf{w}^k(\mathbf{x},0) = b^k(\mathbf{x}) \text{ in } I\!R^3 \end{cases} \tag{3.2}$$

for $k \geq 1$.

If $\mathbf{f},\mathbf{g} \in L^2(I\!R^+; L^2(I\!R^3))$ is easily to prove (see [17] or [10]) that there exist a unique solution $(\mathbf{u}^k, \mathbf{w}^k)$, $(k \geq 0)$ to (3.1) and (3.2) satisfying

$$\frac{\partial \mathbf{u}^k}{\partial x_i}, \frac{\partial^2 \mathbf{u}^k}{\partial x_i \partial x_j}, \frac{\partial \mathbf{u}^k}{\partial t}, \frac{\partial p^k}{\partial x_i}, \frac{\partial \mathbf{w}^k}{\partial x_i}, \frac{\partial^2 \mathbf{w}^k}{\partial x_i \partial x_j}, \frac{\partial \mathbf{w}^k}{\partial t} \in L^2(I\!R^3 \times (0,T)), \tag{3.3}$$

and

$$\mathbf{u}^k, \mathbf{w}^k \in L^2(0,T; L^2(I\!R^3)) \tag{3.4}$$

for $i,j = 1,2,3,\ k = 0,1,\dots$.

Utilizing the Sobolev embedding theorem (see [11])

$$H^1(I\!R^3) \subset L^6(I\!R^3), \tag{3.5}$$

and

$$H^2(I\!R^3) \subset C_B(I\!R^3),$$

here B means bounded.

We define the operator $L : D(L) \to L^2(I\!R^3)$ with the Dirichlet boundary conditions with domain $D(L) \equiv H^2(I\!R^3) \cap H_0^1(I\!R^3)$ by

$$L\mathbf{w} = -(c_a+c_d)\Delta \mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w}.$$

We give the next Lemma, analogously to Lemma 3.1 of [9].

Lemma 3.1. *Let the initial values $a^k \in H$, $b^k \in L^2(I\!R^3)$, and the external forces* $\mathbf{f}$ *and* $\mathbf{g}$ *in $L^1(I\!R^+; L^2(I\!R^3)) \cap L^2(I\!R^+; L^2(I\!R^3))$. Then the solution $(\mathbf{u}^k, \mathbf{w}^k)$ of (3.1)-(3.2) satisfy*

$$\|\mathbf{u}^k(t)\|_2^2 + \|\mathbf{w}^k(t)\|_2^2 + (\nu+\nu_r)\int_0^t \|\nabla \mathbf{u}^k(\tau)\|_2^2 d\tau + \int_0^t \|L^{\frac{1}{2}}\mathbf{w}^k(\tau)\|_2^2 d\tau$$
$$\leq K(\|a\|_2^2 + \|b\|_2^2 + \int_0^\infty (\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2) d\tau)$$
$$= A^2 = A^2(\|a\|_2, \|b\|_2, \|\mathbf{f}\|_{L^2(I\!R^+;L^2(I\!R^3))}, \|\mathbf{g}\|_{L^2(I\!R^+;L^2(I\!R^3))}) \qquad (3.6)$$

for every $t > 0$ and $k \geq 0$.

Proof. Multiplying the equations (3.2) by $\mathbf{u}^k$ and $\mathbf{w}^k$ respectively, and using equivalence of norm between rot and ∇, we obtain,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^k(t)\|_2^2 + (\nu+\nu_r)\|\nabla \mathbf{u}^k(t)\|_2^2 \leq \|\mathbf{f}(t)\|_{H^{-1}}\|\mathbf{u}^k(t)\|_{H_0^1} + 2\nu_r(\text{ rot } \mathbf{u}^k(t), \mathbf{w}^k(t)),$$
$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}^k(t)\|_2^2 + \|L^{1/2}\mathbf{w}^k(t)\|_2^2 + 4\nu_r\|\mathbf{w}^k(t)\|_2^2 \leq \|\mathbf{g}(t)\|_{H^{-1}}\|\mathbf{w}^k(t)\|_{H_0^1} + 2\nu_r(\text{ rot } \mathbf{u}^k(t), \mathbf{w}^k(t)).$$

Adding the above equalities and applying Young inequality, we have

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}^k(t)\|_2^2 + \|\mathbf{w}^k(t)\|_2^2) + (\nu+\nu_r)\|\nabla \mathbf{u}^k(t)\|_2^2 + \|L^{1/2}\mathbf{w}^k(t)\|_2^2 + 4\nu_r\|\mathbf{w}^k(t)\|_2^2 \leq$$
$$C_\varepsilon\|\mathbf{f}(t)\|_{H^{-1}}^2 + \varepsilon\|\nabla \mathbf{u}^k(t)\|_2^2 + \frac{1}{2}\|\mathbf{g}(t)\|_{H^{-1}}^2 + \frac{1}{2}\|L^{\frac{1}{2}}\mathbf{w}^k(t)\|_2 + 4\nu_r\|\mathbf{w}^k(t)\|_2^2 + \nu_r\|\nabla \mathbf{u}^k(t)\|_2^2$$

using the injection $L^2 \hookrightarrow H^{-1}$ after taking $K = \max\{2C_\varepsilon, 1, C\}$,we get

$$\frac{d}{dt}\|\mathbf{u}^k(t)\|_2^2 + \frac{d}{dt}\|\mathbf{w}^k(t)\|_2^2 + (\nu+\nu_r)\|\nabla \mathbf{u}^k(\tau)\|_2^2 d\tau + \|L^{\frac{1}{2}}\mathbf{w}^k(\tau)\|_2^2 d\tau$$
$$\leq K(\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2)d\tau. \qquad (3.7)$$

Now integrating in $[0, T)$, we obtain

$$\|\mathbf{u}^k(t)\|_2^2 + \|\mathbf{w}^k(t)\|_2^2 + (\nu+\nu_r)\int_0^t \|\nabla \mathbf{u}^k(\tau)\|_2^2 d\tau + \int_0^t \|L^{\frac{1}{2}}\mathbf{w}^k(\tau)\|_2^2 d\tau$$
$$\leq \|a^k\|_2^2 + \|b^k\|_2^2 + K\int_0^t (\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2) d\tau) \qquad (3.8)$$

finally we observe that

$$\int_0^t (\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2) \leq \int_0^\infty (\|\mathbf{f}(\tau)\|_2^2 + \|\mathbf{g}(\tau)\|_2^2),$$

and

$$\begin{aligned} \|a^k\|_2 &\leq C\|a\|_2, \\ \|b^k\|_2 &\leq C\|b\|_2. \end{aligned}$$

Because of (3.3) and (3.4) from (3.2), it follows that

$$-\Delta p^k = \text{ div } ((\mathbf{u}^{k-1} \cdot \nabla)\mathbf{u}^k - \mathbf{f})$$

for $k \geq 1$. Since p^k is unique up to the addition of a constant, without loss of generality, analogously to [9], we assume the pressure $p^k(x,t)$ is determined by the supplementary condition

$$\lim_{|x| \to +\infty} p^k(x,t) = 0.$$

Let $p^k = p_1^k + p_2^k$, with p_i^k satisfying

$$\begin{aligned} -\Delta p_1^k &= \text{div } ((\mathbf{u}^{k-1} \cdot \nabla)\mathbf{u}^k) \\ -\Delta p_2^k &= \text{div } \mathbf{f}. \end{aligned}$$

Thus if $p \in L^2(I\!R^3)$, has Calderon-Zygmund theory (see [22]) on singular integrals yields the estimates of the following lemma.

Lemma 3.2. *Let $p^k \in L^2(I\!R^3)$, and $\mathbf{u}^k$ satisfy (3.3) then for $k \geq 1$ the estimates hold*

$$\begin{aligned} \left\|p_1^k\right\|_3 &\leq C\left\|\mathbf{u}^{k-1}\right\|_3 \leq C\left\|\mathbf{u}^{k-1}\right\|_6 \left\|\mathbf{u}^k\right\|_6, && (3.9) \\ \left\|p_2^k\right\|_6 &\leq C\left\|\mathbf{f}\right\|_2, && (3.10) \end{aligned}$$

Analogously to [9], we establish the key estimate for $\mathbf{u}^k$ and $\mathbf{w}^k$.

Lemma 3.3. *Let the initial values $a \in H \cap L^6(I\!R^3)$, $b \in L^2(I\!R^3) \cap L^6(I\!R^3)$, and the external forces $\mathbf{f}$ and $\mathbf{g}$ in $L^2(I\!R^+; L^2(I\!R^3)) \cap L^{4/3}(I\!R^+; L^6(I\!R^3))$. Then the differential inequalities*

$$\begin{aligned} \frac{1}{6}\frac{d}{dt}\|\mathbf{u}^k\|_6^6 + CA^{-2}\|\mathbf{u}^k\|_6^8 &\leq C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^6 + C\|\mathbf{f}\|_2^2\|\mathbf{u}^k\|_6^4 && (3.11) \\ &\quad + \|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^4. \end{aligned}$$

and

$$\frac{1}{6}\frac{d}{dt}\|\mathbf{w}^k\|_6^6 + CA^{-2}\|\mathbf{w}^k\|_6^8 + 4\nu_r\|\mathbf{w}^k\|_6^6 \leq \|\mathbf{g}\|_6\|\mathbf{w}^k\|_6^5 + C\|\nabla\mathbf{u}^k\|_2^2\|\mathbf{u}^k\|_6^4. \quad (3.12)$$

hold uniformly for $k \geq 1$.

Proof. The i^{th} equation of (3.2) is:

$$\frac{d}{dt}u_i^k - (\nu+\nu_r)\Delta u_i^k + (\mathbf{u}^{k-1}\cdot\nabla)u_i^k = -\frac{\partial p^k}{\partial x_i} + f_i + 2\nu_r\xi_{ijl}\frac{\partial w_l^k}{\partial x_j}$$

$$\frac{d}{dt}w_i^k + Lw_i^k + (\mathbf{u}^{k-1}\cdot\nabla)w_i^k + 4\nu_r w_i^k = g_i + 2\nu_r\xi_{ijl}\frac{\partial u_l^k}{\partial x_j}$$

We now multiply both sides of the above equations by $\left|u_i^k\right|^4 u_i^k$ and by $\left|w_i^k\right|^4 w_i^k$, respectively, and integrate in $I\!R^3$. Because of (3.3) and (3.4), it follows that

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\|u_i^k\|_6^6 + 5(\nu+\nu_r)\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx \\
= \;& 5\int_{I\!R^3} p^k\left|u_i^k\right|^4\frac{\partial u_i^k}{\partial x_i}dx + \int_{I\!R^3} f_i\left|u_i^k\right|^4 u_i^k dx + 2\nu_r\int_{I\!R^3}\xi_{ijl}\frac{\partial w_l^k}{\partial x_j}\left|u_i^k\right|^4 u_i^k dx \\
\leq \;& 5\int_{I\!R^3}\left|p_1^k\right|\left|u_i^k\right|^4\frac{\partial u_i^k}{\partial x_i}dx + 5\int_{I\!R^3}\left|p_2^k\right|\left|u_i^k\right|^4\frac{\partial u_i^k}{\partial x_i}dx + \|\mathbf{f}\|_6\|u_i^k\|_6^5 \\
& +2\nu_r\|\xi_{ijl}\frac{\partial w_l^k}{\partial x_j}\|_2\|\left|u_i^k\right|^2\|_6\|\left|u_i^k\right|^2 u_i^k\|_6 \\
\leq \;& \frac{5}{2}\nu\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx + 5\nu^{-1}\int_{I\!R^3}\left|p_1^k\right|^2\left|u_i^k\right|^4 dx + 5\nu^{-1}\int_{I\!R^3}\left|p_2^k\right|^2\left|u_i^k\right|^4 dx \\
& +\|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C2\nu_r\|L^{\frac{1}{2}}\mathbf{w}^k\|_2\|\mathbf{u}^k\|_6^2\|\nabla(\left|u_i^k\right|^2 u_i^k)\|_2 \\
\leq \;& \frac{5}{2}\nu\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx + 5\nu^{-1}\int_{I\!R^3}\left|p_1^k\right|^2\left|u_i^k\right|^4 dx + 5\nu^{-1}\|p_2^k\|_6^2\|\mathbf{u}^k\|_6^4 \quad (3.13) \\
& +\|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C_\varepsilon\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^6 + \varepsilon\|\left|u_i^k\right|^2\nabla u_i^k\|_2^2.
\end{aligned}$$

here was used in the last term Sobolev embedding $H^1 \hookrightarrow L^6$, and Young inequality.

Taking $\varepsilon = \frac{5}{2}(\nu+\nu_r)$ in the above estimate, we have

$$\begin{aligned}
\frac{1}{6}\frac{d}{dt}\|u_i^k\|_6^6 + \frac{5}{2}(\nu+\nu_r)\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx \;=\;& 5\nu^{-1}\int_{I\!R^3}\left|p_1^k\right|^2\left|u_i^k\right|^4 dx + 5\nu^{-1}\|p_2^k\|_6^2\|\mathbf{u}^k\|_6^4 \\
& +\|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C_\varepsilon\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^6 + C_\delta\|\mathbf{u}_6^{k6}\|_6^6.
\end{aligned}$$

Now by (3.6), we obtain

$$\frac{1}{6}\frac{d}{dt}\left\|u_i^k\right\|_6^6 + \frac{5}{2}(\nu+\nu_r)\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx = 5\nu^{-1}\int_{I\!R^3}\left|p_1^k\right|^2\left|u_i^k\right|^4 dx \qquad (3.14)$$
$$+5\nu^{-1}C\left\|\mathbf{f}\right\|_2^2\left\|\mathbf{u}^k\right\|_6^4 + \left\|\mathbf{f}\right\|_6\left\|\mathbf{u}^k\right\|_6^5$$
$$+C\left\|L^{\frac{1}{2}}\mathbf{w}^k\right\|_2^2\left\|\mathbf{u}^k\right\|_6^4$$

Using the Ladyzhenskaya inequality (see [10]) and estimate (3.6), the first term at the right hand side can be estimated by

$$\begin{aligned}\int_{I\!R^3}\left|p_1^k\right|^2\left|u_i^k\right|^4 dx &\leq \|p_1^k\|_3^2\|u_i^k\|_{12}^4\\ &= \|p_1^k\|_3^2\|\left|u_i^k\right|^2 u_i^k\|_4^{\frac{4}{3}}\\ &\leq C\|\mathbf{u}^{k-1}\|_6^2\|\mathbf{u}^k\|_6^3\left(\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx\right)^{\frac{1}{2}}\\ &\leq \frac{5}{4}\nu\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx + C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^6.\end{aligned}$$

Substituting the above estimate it follows that

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|u_i^k\|_6^6 + \left(\frac{5}{4}\nu+\frac{5}{2}\nu_r\right)\int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx\\ \leq\ & C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^6 + C\|\mathbf{f}\|_2^2\|\mathbf{u}^k\|_6^4 \qquad (3.15)\\ & \|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^4.\end{aligned}$$

Newly by the Sobolev inequality (see [1])

$$C\|u_i^k\|_{18}^6 \leq \int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx$$

Using the interpolation inequality and estimate (3.6) we see that

$$\|u_i^k\|_6 \leq \|u_i^k\|_2^{\frac{1}{4}}\|u_i^k\|_{18}^{\frac{3}{4}}.$$

So

$$CA^{-2}\|u_i^k\|_6^8 \leq \int_{I\!R^3}\left|u_i^k\right|^4\left|\nabla u_i^k\right|^2 dx$$

substituting the above inequality into (3.16), adding for i from 1 to 3, it follows that

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|\mathbf{u}^k\|_6^6 + CA^{-2}\sum_{i=1}^{3}\|u_i^k\|_6^8\\ \leq\ & C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^6 + C\|\mathbf{f}\|_2^2\|\mathbf{u}^k\|_6^4\\ & \|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^4.\end{aligned}$$

By the Hölder inequality

$$\|\mathbf{u}^k\|_6^6 = \sum_{i=1}^{3}\|u_i^k\|_6^6 \leq 3^{1/4}(\sum_{i=1}^{3}\|u_i^k\|_6^8)^{\frac{3}{4}}.$$

Thus

$$\sum_{i=1}^{3}\|u_i^k\|_6^8 \geq C\|\mathbf{u}^k\|_6^8.$$

So

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\|\mathbf{u}^k\|_6^6 + CA^{-2}\|\mathbf{u}^k\|_6^8 \quad\leq\quad & C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^6 + C\|\mathbf{f}\|_2^2\|\mathbf{u}^k\|_6^4\\ & +\|\mathbf{f}\|_6\|\mathbf{u}^k\|_6^5 + C\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2\|\mathbf{u}^k\|_6^4.\end{aligned}$$

of analogous form we obtain

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|\mathbf{w}^k\|_6^6 + CA^{-2}\|\mathbf{w}^k\|_6^8 + 4\nu_r\|\mathbf{w}^k\|_6^6\\ \leq\ & \|\mathbf{g}\|_6\|\mathbf{w}^k\|_6^5 + C\|\nabla\mathbf{u}^k\|_2^2\|\mathbf{w}^k\|_6^4,\end{aligned}$$

these last inequalities correspond to (3.12) and (3.13) respectively.

Lemma 3.4. *Suppose the conditions of Lemma 3.3 hold. Then there exists* $T_0 > 0$, *such that*

$$\|\mathbf{u}^k\|_6 \leq C, \quad \forall\, t \in [0, T_0], \tag{3.16}$$

and

$$\|\mathbf{w}^k\|_6 \leq C, \quad \forall\, t \in [0, T_0], \tag{3.17}$$

holds uniformly for $k \geq 1$.

Proof. Here we follow the paper [9], then we work with (3.12), but the procedure is analogous for equation (3.13).

Using the Young inequality, from (3.12) and using (3.7), it follows that

$$\frac{d}{dt}\|\mathbf{u}^k\|^2 \leq C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^2 + C(\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}}).$$

Integrating the above inequality from 0 to s, and using (3.7), we get

$$\begin{aligned}\|\mathbf{u}^k\|_6^2 &\leq \|a^k\|_6^2 + C\int_0^s(\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^2 + C\int_0^s(\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}})d\tau,\\ &\leq \|a\|_6^2 + C\int_0^s(\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^2 + CA^2 + C\int_0^s(\|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}})d\tau,\end{aligned}$$

applying Gronwall's inequality, we conclude that

$$\|\mathbf{u}^k\|_6^2 \leq Ce^{C(\int_0^t\|\mathbf{u}^{k-1}\|_6^4 d\tau}\left[\|a\|_6^2 + A^2 + \int_0^s(\|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}})d\tau\right].$$

Analogously for equation (3.13) we have,

$$\|\mathbf{w}^k\|_6^2 \leq C\left[\|b\|_6^2 + A^2 + \int_0^s A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}}d\tau\right].$$

Let $Y_k(t) = \int_0^t\|\mathbf{u}^{k-1}\|_6^4 d\tau$ then

$$Y_k \leq Ce^{2C(Y_{k-1})}\left[\|a\|_6^2 + A^2 + \int_0^s(\|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}})d\tau\right]^2 t.$$

By a similar calculation, we deduce from (3.1)

$$\begin{aligned}Y_0(\infty) &= \int_0^\infty\|\mathbf{u}^0\|_6^4 dt \leq C\left[\|a\|_6^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_{L^{4/3}(I\!R^+;L^6(I\!R^3))}^{\frac{4}{3}}\right]\\ &= CM.\end{aligned}$$

Thus, we take T_0 such that

$$Ce^{2CM}\left[\|a\|_6^2 + A^2 + \int_0^{T_0}(\|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}})d\tau\right]^2 T_0 \leq CM$$

Then by induction we obtain estimates (3.17).
Analogously

$$\|\mathbf{w}^k\|_6^2 \leq C\left[\|b\|_6^2 + A^2 + \int_0^{T_0} A^{\frac{2}{3}}\|\mathbf{g}\|_6^{\frac{4}{3}}d\tau\right] \leq CM.$$

Then we obtain estimative (3.18).

Lemma 3.5. *Let the assumption of Lemma 3.3 hold. If there exist constants* C_1 *and* C_2 *independent of* a *and* $\mathbf{f}$*, which satisfy*

$$C_1 A^2 e^{C_2 M}\left[\|a\|_6^2 + A^2 + \|\mathbf{f}\|^2_{L^2(I\!R^+;L^2(I\!R^3))} + A^{\frac{2}{3}}\|\mathbf{f}\|^{\frac{4}{3}}_{L^{4/3}(I\!R^+;L^6(I\!R^3))}\right] \le M,$$

then the following estimates are true

$$\begin{aligned} \|\mathbf{u}^k\|_6 &\le C, \quad \forall\, t \ge 0, &(3.18)\\ \|\mathbf{w}^k\|_6 &\le C, \quad \forall\, t \ge 0, &(3.19)\end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \|\mathbf{u}^k\|_6^4 dt &\le C, &(3.20)\\ \int_0^\infty \|\mathbf{w}^k\|_6^4 dt &\le C, &(3.21)\end{aligned}$$

uniformly for $k \ge 1$.

Proof. From (3.12), (resp. (3.13)),

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}^k\|_6^2 + CA^{-2}\|\mathbf{u}^k\|_6^4 &\le C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^2\\ &\quad + C(\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + \|\mathbf{u}^k\|_6\|\mathbf{f}\|_6).\end{aligned}$$

By using the Young inequality, apply to last term, we have

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}^k\|_6^2 + CA^{-2}\|\mathbf{u}^k\|_6^4 &\le C\|\mathbf{u}^{k-1}\|_6^4\|\mathbf{u}^k\|_6^2\\ &\quad + C(\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + \|A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}}).\end{aligned}$$

Let $Y(s) = e^{-C\int_0^s \|\mathbf{u}^{k-1}\|_6^4 d\tau}\|\mathbf{u}^k\|_6^2$. Then

$$\begin{aligned}&\frac{d}{dt}Y + CA^{-2}e^{-C\int_0^s \|\mathbf{u}^{k-1}\|_6^4 d\tau}Y^2\\ \le\ & Ce^{-C\int_0^s \|\mathbf{u}^{k-1}(\tau)\|_6^4 d\tau}\left[\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + A^{\frac{2}{3}}\|\mathbf{f}\|_6^{\frac{4}{3}}\right].\end{aligned}$$

Integrating the last inequality over $[0,t]$, $t > 0$, we get

$$\begin{aligned}&Y + CA^{-2}\int_0^t e^{C\int_0^s \|\mathbf{u}^{k-1}\|_6^4 d\tau}Y^2 ds &(3.22)\\ \le\ & C\left[\|a\|_6^2 + A^2 + \|\mathbf{f}\|^2_{L^2(I\!R^+;L^2(I\!R^3))} + A^{\frac{2}{3}}\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}\right].\end{aligned}$$

So

$$\int_0^t \|\mathbf{u}^k\|_6^4 dt \le CA^2 e^{C\int_0^t (\|\mathbf{u}^{k-1}(\tau)\|_6^4 d\tau} \quad \times \tag{3.23}$$

$$\left[\|a\|_6^2 + A^2 + \|\mathbf{f}\|^2_{L^2(I\!R^+;L^2(I\!R^3))} + A^{\frac{2}{3}}\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}\right]$$

Analogously to lemma before

$$Y_0(\infty) = \int_0^\infty \|\mathbf{u}^0\|_6^4 dt \le M$$

utilizing assumption (3.19), by induction, from (3.26), we obtain

$$\int_0^\infty \|\mathbf{u}^k\|_6^4 dt \le C.$$

Analogously for $\mathbf{w}$, form (3.13) we obtain

$$\|\mathbf{w}^k\|_6^2 + CA^{-2}\int_0^t \|\mathbf{w}^k\|_6^4 \le C\left\{\|b\|_6^2 + A^2 + \int_0^s A^{\frac{2}{3}}\|\mathbf{g}\|_6^{\frac{4}{3}} d\tau\right\}$$

Then

$$CA^{-2}\int_0^t \|\mathbf{w}^k\|_6^4 \le C\left\{\|b\|_6^2 + A^2 + \int_0^\infty A^{\frac{2}{3}}\|\mathbf{g}\|_6^{\frac{4}{3}} d\tau\right\} \le C$$

taking $\liminf$ as $t \to \infty$ we conclude that

$$\int_0^\infty \|\mathbf{w}^k\|_6^4 dt \le C.$$

Then, combining (3.23) and (3.25), we obtain estimates (3.21). Analogously from (3.13), we obtain the estimates (3.25) and (3.26), and using the assumption (3.24).

Lemma 3.6. *Suppose the conditions of Lemma 3.3 hold. If* $\|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6 \le C(1+t)^{-\alpha}$ *(resp.* $\|\mathbf{g}\|_6 \le C(1+t)^{-\alpha}$*)* $(\alpha > 0)$, *then*

$$\begin{aligned} \|\mathbf{u}^k\|_6^6 &\le C(1+t)^{-\beta}, \quad \forall\ t \ge 1, \\ \|\mathbf{w}^k\|_6^6 &\le C(1+t)^{-\beta}, \quad \forall\ t \ge 1, \end{aligned}$$

with $\beta = \min\{3, \frac{3}{4}\alpha\}$. *Especially, if* $\mathbf{f} = 0$ *(respect. if* $\mathbf{g} = 0$*), then*

$$\begin{aligned} \|\mathbf{u}^k\|_6^6 &\le C(1+t)^{-3}, \quad \forall\ t \ge 1, \\ \|\mathbf{w}^k\|_6^6 &\le C(1+t)^{-3}, \quad \forall\ t \ge 1, \end{aligned}$$

Proof. Let now $Y(t) = e^{-C\int_0^t \|\mathbf{u}^k\|_6^4 d\tau}\|\mathbf{u}^k\|_6^6$. From (3.12) and (3.21) we deduce that

$$\begin{aligned}\frac{d}{dt}Y + CY^{1+\frac{1}{3}} &\leq C((\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2)Y^{\frac{2}{3}} + \|\mathbf{f}\|_6 Y^{\frac{5}{6}} \\ &\leq C(\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6)Y^{\frac{2}{3}},\end{aligned}$$

since $\|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6 \leq C(1+t)^{-\alpha}$and , also $\int_0^t(\|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6)dt \leq C(1+t)^{-\alpha'}$ then we can suppose that $\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 + \|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6 \leq C(1+t)^{-\alpha}$ from Lemma 3.1 follows

$$\int_0^t (\|L^{\frac{1}{2}}\mathbf{w}^k\|_2^2 dt + \int_0^t \|\mathbf{f}\|_2^2 + \|\mathbf{f}\|_6 dt \leq A^2 + C(1+t)^{-\alpha'}$$

Then

$$\frac{d}{dt}Y + CY^{1+\frac{1}{3}} \leq C(1+t)^{-\alpha}Y^{\frac{2}{3}}.$$

Because of (3.21) we now suppose $Y(t) \leq C_m(1+t)^{-\beta_m}$ with $C_m > 0$, $\beta_m > 0$, $m \in I\!N$. By Lemma 3.6 of [9], we conclude that

$$Y(t) \leq C_{m+1}(1+t)^{-\beta_{m+1}}$$

with $\beta_{m+1} = \min\{3, \frac{2\beta_m+3\alpha}{4}\}$, and $C_{m+1} = \gamma C_m^{1/2}$

Let $m \to \infty$, $\beta_m \to \min\{3, \frac{3}{4}\alpha\}$, and $\limsup C_m \leq \gamma^2$. So

$$\begin{aligned}\|\mathbf{u}^k\|_6^6 &\leq e^{C\int_0^t \|\mathbf{u}^{k-1}\|_6^4 d\tau}Y(t) \\ &\leq \gamma^2 e^{C\int_0^t \|\mathbf{u}^{k-1}\|_6^4 d\tau}(1+t)^{-\beta} \\ &\leq C(1+t)^{-\beta} \quad \forall\ t \geq 1,\end{aligned}$$

with $\beta = \min\{3, \frac{3}{4}\alpha\}$.

Analogously setting $Z(t) = \|\mathbf{w}^k\|_6^6$, from (3.13) we deduce that

$$\begin{aligned}\frac{d}{dt}Z + CZ^{1+\frac{1}{3}} &\leq C(\|\nabla\mathbf{u}^k\|_2^2 Y^{\frac{2}{3}} + \|\mathbf{g}\|_6 Y^{\frac{5}{6}}) \\ &\leq C(\|\nabla\mathbf{u}^k\|_2^2 + \|\mathbf{g}\|_6)Y^{\frac{2}{3}}.\end{aligned}$$

Since $\|\mathbf{g}\|_6 \leq C(1+t)^{-\alpha}$, then we can suppose that $\|\nabla\mathbf{u}^k\|_2^2 + \|\mathbf{g}\|_6 \leq C(1+t)^{-\alpha}$, thus

$$\frac{d}{dt}Z + CZ^{1+\frac{1}{3}} \leq C(1+t)^{-\alpha}Z^{\frac{2}{3}}.$$

Because of (3.22) we now suppose $Z(t) \leq C_m(1+t)^{-\beta_m}$ with $C_m > 0$, $\beta_m > 0$, $m \in I\!N$. By Lemma 3.6 of [9], we conclude that

$$Z(t) \leq C_{m+1}(1+t)^{-\beta_{m+1}}$$

with $\beta_{m+1} = \min\{3, \frac{2\beta_m+3\alpha}{4}\}$, and $C_{m+1} = \gamma C_m^{1/2}$.

Let $m \to \infty$, $\beta_m \to \min\{3, \frac{3}{4}\alpha\}$, and $\limsup C_m \leq \gamma^2$. So

$$\begin{aligned} \|\mathbf{w}^k\|_6^6 &\leq Z(t) \\ &\leq \gamma^2(1+t)^{-\beta} \\ &\leq C(1+t)^{-\beta} \quad \forall\, t \geq 1, \end{aligned}$$

with $\beta = \min\{3, \frac{3}{4}\alpha\}$.

If $\mathbf{f}$, $\mathbf{g} = 0$ we conclude easily by using of Lemma 3.6 of [9], with $\beta = 3$.

4. PROOFS OF THEOREMS

Analogously to [9], we proof the main Theorems.

4.1. Proof of Theorem 1

Proof. The uniqueness of the strong solutions follows from Theorem 2 of [19] or [18]. In the following, we prove the existence of the solution. Since the approximation of $\mathbf{u}^k$ and $\mathbf{w}^k$ of micropolar equations (1.1) constructed in Section 3 satisfies (3.22) , then we obtain estimates (3.6) and (3.12). Applying these estimates, we easily deduce that

$$\|\frac{\partial \mathbf{u}^k}{\partial t}\|_{L^2(0,T_0;V')} \leq C, \tag{4.1}$$

and

$$\|\frac{\partial \mathbf{w}^k}{\partial t}\|_{L^2(0,T_0;H^{-1}(I\!R^3))} \leq C \tag{4.2}$$

holds uniformly for $k \geq 1$.

The estimates (3.6), (3.7), (3.12), (4.1), (3.8), (3.9), (3.13) and (4.2) enable us to assert the existence of an element

$$\mathbf{u} \in L^\infty(0,T_0; H \cap L^6(I\!R^3)) \text{ and } \mathbf{w} \in L^\infty(0,T_0; L^2(I\!R^3) \cap L^6(I\!R^3)),$$

$$\nabla\mathbf{u}, \nabla\mathbf{w} \in L^\infty(0,T_0; L^2(I\!R^3)$$

and

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T_1;V') \text{ and } \frac{\partial \mathbf{w}}{\partial t} \in L^2(0,T_1;H^{-1}(I\!R^3)),$$

and a subsequences of $\mathbf{u}^k$ and $\mathbf{w}^k$ that we denote of the same form (in fact, the sequence itself converges because of the uniqueness) such that

$$\begin{array}{rcl} \mathbf{u}^k & \rightharpoonup & \mathbf{u}, \text{ in } L^\infty(0,T_0;H\cap L^6(I\!R^3)) \text{ weak-start }, \\ \mathbf{w}^k & \rightharpoonup & \mathbf{w}, \text{ in } L^\infty(0,T_0;L^2(I\!R^3)\cap L^6(I\!R^3)) \text{ weak }, \\ \frac{\partial \mathbf{u}^k}{\partial t} & \rightharpoonup & \frac{\partial \mathbf{u}}{\partial t}, \text{ in } L^2(0,T_1;V') \text{ weak }, \\ \frac{\partial \mathbf{w}^k}{\partial t} & \rightharpoonup & \frac{\partial \mathbf{w}}{\partial t}, \text{ in } L^2(0,T_1;H^{-1}(I\!R^3)) \text{ weak }, \\ \nabla\mathbf{u}^k & \rightharpoonup & \nabla\mathbf{u}, \text{ in } L^\infty(0,T_0;L^2(I\!R^3)) \text{ weak }, \\ \nabla\mathbf{w}^k & \rightharpoonup & \nabla\mathbf{w}, \text{ in } L^\infty(0,T_0;L^2(I\!R^3)) \text{ weak }. \end{array} \tag{4.3}$$

For arbitrary $\phi \in C^\infty([0,T_0];C^\infty_{0,\sigma})$ with $\phi(T_0)=0$, let supp $\phi(T_0)=0$, let supp $\phi\subset\Omega\times[0,T_0]$ for some bounded set Ω. By the Sobolev compact Theorem

$$H^1(\Omega)\subset L^2(\Omega).$$

Applying a compact result ([11] or [23]) and (4.1) (respect. (4.2)), we have

$$\begin{array}{rcl} \mathbf{u}^k & \rightarrow & \mathbf{u}, \text{ in } L^\infty(0,T_0;H\cap L^6(I\!R^3)), \\ \mathbf{w}^k & \rightarrow & \mathbf{w}, \text{ in } L^\infty(0,T_0;L^2(I\!R^3)\cap L^6(I\!R^3)) \end{array} \tag{4.4}$$

We now multiply both sides of (3.2) by ϕ, (resp. φ) and integrate over $I\!R^3\times(0,T_0)$ to get

$$\begin{array}{rl} & \int_0^{T_0}\{(\mathbf{u}^k,\varphi_t)-(\nu+\nu_r)(\nabla\mathbf{u}^k,\nabla\varphi)+((\mathbf{u}^{k-1}\cdot\nabla)\varphi,\mathbf{u}^k)\}dt-(a^k,\varphi(0)) \\ = & \int_0^{T_0}(2\nu_r \text{ rot } \mathbf{w}^k+\mathbf{f}^k,\varphi), \end{array}$$

$$\begin{array}{rl} & \int_0^{T_0}\{(\mathbf{w}^k,\phi_t)-(L^{\frac12}\mathbf{w}^k,L^{\frac12}\phi)+((\mathbf{u}^k\cdot\nabla)\phi,\mathbf{w}^k)-4\nu_r(\mathbf{w}^k,\phi)\}dt+(\mathbf{w}^k_0,\phi(0)) \\ = & -\int_0^{T_0}(2\nu_r \text{ rot } \mathbf{u}^k+\mathbf{g}^k,\phi). \end{array}$$

Because $C^\infty([0,T_0);H)$ is dense in $W_2^{1,1}(Q_{T_0})$, using (4.1) and (4.4) and procedures standard we conclude that the limit $(\mathbf{u},\mathbf{w})$ of $(\mathbf{u}^k,\mathbf{w}^k)$ is the unique generalized solution of (3.2) and, $\mathbf{u}(\cdot,t)\to a$ and $\mathbf{w}(\cdot,t)\to b$ in $L^2(I\!R^3)$ as $t\to 0^+$.

4.2. Proof of Theorem 2.

Proof. If $\|a\|_6^2+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}+\|\mathbf{f}\|^2_{L^2(I\!R^+;L^2(I\!R^3))}$, and $\|\mathbf{g}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}$, are given, we let $A<1$. In order to obtain (3.19) (respect. (3.20)), we only need

$$A^2\le A_0^2=C_1^{-1}e^{-C_2(\|a\|_6^2+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}+C_\varepsilon B)}\times$$
$$\frac{\|a\|_6^2+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}}{\|a\|_6^2+A^2+\|\mathbf{f}\|^2_{L^2(I\!R^+;L^2(I\!R^3))}+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}}$$

so we take $\lambda_1=\min\{1,A_0\}$

If $\|a\|_6^2+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}$ (respect. $\|\mathbf{g}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))}$) are given, we take

$$\lambda_2=\min\{1,C_1^{-1}(\|a\|_6^2+\|\mathbf{f}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(I\!R^+;L^6(I\!R^3))})^{-2}e^{-C_2}\},$$

where C_1, C_2 are constants independent of $\mathbf{u}_0$ and $\mathbf{f}$, in (3.19). Then the assumptions of Theorem 2 guarantee that (3.19), is valid. Thus, we hold global estimate (3.21)-(3.24) instead of local estimates (3.17),(3.18). Then proof of Theorem 2 is similar to that of Theorem 1, considering $t\in I\!R^+$, so we omit the details. Since norm is weak lower-semicontinuos, by Lemma 3.6, we obtain decay estimates (2.1) and (2.4).

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