Solving Generalized Nonlinear Complementarity Problems: Numerical Experiments on Polyhedral Cones

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Abstract

In a previous work, the minimization of a differentiable function subject to box constraints was proposed as a strategy to solve the generalized nonlinear complementarity problem (GNCP) defined on a polyhedral cone. Theoretical results that relate stationary points of the function that is minimized to the solutions of the GNCP were presented. These theoretical results show that local methods for box constrained optimization applied to the associated problem are efficient tools for solving the GNCP. In this work, numerical experiments are presented that encourage the use of this approach.

Keywords. Box constrained optimization, complementarity.

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1 Introduction

The Generalized Nonlinear Complementarity Problem (GNCP) is to find $x \in I\!R^m$ such that

$$F(x) \in \mathcal{K}, \;\; G(x) \in \mathcal{K}^\circ, \;\; F(x)^T G(x) = 0, \tag{1}$$

where F and G are continuous functions from $I\!R^m$ to $I\!R^n$, $\mathcal{K}$ is a nonempty closed convex cone in $I\!R^n$, and $\mathcal{K}^\circ$ denotes the polar cone of $\mathcal{K}$.

We consider the case $n = m$, $F, G \in C^1$ and $\mathcal{K}$ a polyhedral cone in R^n that is, given $A \in I\!R^{p \times n}$ and $B \in I\!R^{s \times n}$, we have

$$\mathcal{K} = \{v \in I\!R^n \mid Av \geq 0, \;\; Bv = 0\}$$

and

$$\mathcal{K}^\circ = \{u \in I\!R^n \mid u = A^T \lambda_1 + B^T \lambda_2, \;\; \lambda_1 \geq 0\}.$$

This problem has many interesting applications and its solution using special techniques has been considered extensively in the literature. See [8, 9, 14] among others. If $\mathcal{K} = I\!R^m_+ \equiv \{x \in I\!R^m \mid x \geq 0\}$, $G(x) = x - F(x)$ and $F : I\!R^m \to I\!R^m$, the GNCP$(F, G, \mathcal{K})$ reduces to the so-called *implicit complementarity problem* [11, 12]. In particular, if $G(x) = x$, the GNCP reduces to the *nonlinear complementarity problem*, denoted by NCP.

Our approach in this paper is to solve the GNCP by means of an equivalent box-constrained smooth optimization problem. Differentiable bound-constrained minimization is a well developed area of practical optimization and many methods and reliable software are available for large-scale problems. See, for example, [3, 4, 6, 16].

Any efficient algorithm for smooth box-constrained minimization can be used, in particular, algorithms that do not rest upon matrix factorizations at all, allowing us to deal with large-scale problems. Unlike the formulations in [13, 15], the computation of the objective function of the equivalent minimization problem is straightforward and projections on convex sets are not necessary to compute neither the objective function nor the derivatives.

Our set of experiments contains four families: randomly generated problems in the positive orthant; implicit complementarity problems from Outrata & Zowe [10]; problems with general polyhedral cones in $I\!R^n$ and problems in 3D-cones with control of generated faces.

For the first family of problems, functions F and G are affine and both cones are the positive orthant. Although quite simple, these problems contain essential elements to start the investigation. By varying dimensions and features of the matrices that define F and G, we have produced an extensive set of tests for which the theoretical hypothesis of equivalence might hold or not.

In the second family our main objective was to solve problems already addressed in the literature. We also extended the family of implicit complementarity problems proposed in [10] to variable dimension, producing large-scale tests. For such problems, however, the cones are the positive orthant as well.

General polyhedral cones were treated in the third and fourth families of problems. In the third one, functions F and G are affine and matrices A and B that define the cones are generated to accomplish well defined problems, but whithout any specific control. In the fourth family, we produced three dimensional tests, so that geometrical features of the cone, like control of edges and number of faces, were exploited in great extent.

The paper is organized as follows: the equivalent formulation which allows turning the GNCP into a nonlinear programming problem is given in Section 2. Numerical experiments are presented in Section 3. Conclusions and lines for future research are discussed in Section 4.

Notation. We denote by $\langle\cdot,\cdot\rangle$ the Euclidean inner product in $I\!R^n$ and by $\|\cdot\|$ the norm induced by this inner product and its corresponding matricial norm. If B is a real $n \times n$ matrix, $B \geq 0$ ($B > 0$) means that B is positive semidefinite (positive definite).

2 Equivalent formulation

The following minimization problem with simple bounds is associated to the GNCP$(F, G, \mathcal{K})$ defined in (1):

$$\begin{array}{rl} \min & f(x,z,\lambda) \\ \text{subject to} & \left\{ \begin{array}{l} z^1 \geq 0, \\ \lambda^1 \geq 0, \end{array} \right. \end{array} \tag{2}$$

where

$$f(x,z,\lambda) = \|RF(x) - z\|^2 + \|G(x) - R^T\lambda\|^2 + \rho\langle z^1, \lambda^1\rangle^2$$

and

$$R = \begin{pmatrix} A \\ B \end{pmatrix}, \quad z = \begin{pmatrix} z^1 \\ 0 \end{pmatrix} \in I\!R^p \times I\!R^s, \quad \lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \in I\!R^p \times I\!R^s.$$

The next theorem, proved in [1], states that solving problem GNCP$(F, G, \mathcal{K})$ is equivalent to finding the global minimizer of the optimization problem (2).

Theorem 1. *If* (x_*, z_*, λ_*) *is a global minimizer of problem (2) with* $f(x_*, z_*, \lambda_*) = 0$*, then* x_* *is a solution of the* GNCP$(F, G, \mathcal{K})$. *Conversely, if* x_* *is a solution of the* GNCP$(F, G, \mathcal{K})$*, then there exist* z_*, λ_* *such that* (x_*, z_*, λ_*) *is a global minimizer of (2) with* $f(x_*, z_*, \lambda_*) = 0$.

For completeness, we include in the following a result (proved in [1]) concerning solution of problem GNCP$(F, G, \mathcal{K})$ and stationary points of (2) whenever $F(x), G(x)$ are affine functions. The theoretical hypothesis about matrix $G'F'^{-1}$ will be exploited in the numerical experiments.

Theorem 2. *Let* $F(x), G(x)$ *be affine,* $G'F'^{-1}$ *positive semidefinite in the null space of* B *and* GNCP$(F, G, \mathcal{K})$ *feasible. If* (x_*, z_*, λ_*) *is a stationary point of (2) then,* x_* *is a solution of* GNCP$(F, G, \mathcal{K})$.

3 Computational experiments

The equivalent minimization problems (2), with simple bounded variables, were solved using `BOX-QUACAN`, a software developed by our research group at the State University of Campinas. It is based on the trust-region approach for solving large-scale bound constrained minimization, and uses the infinity norm to define the trust-region, so that the quadratic subproblems have also simple bounded variables. The subproblems are solved by combining conjugate gradients with projected gradients and a mild active set strategy (see [2, 6] or [5, p.459]).

The code was developed in Fortran 77 double precision (Microsoft PowerStation) and run on a Pentium 64MB RAM. The stopping criteria used is tolerance for the objective function value $\varepsilon_f = 10^{-10}$ and tolerance for the norm of the continuous projected gradient $\varepsilon_g = 10^{-6}$. We set $\rho = 1$ for all the tests.

3.1 Randomly generated problems in the positive orthant

In our first set of experiments we considered the problem of finding $x \in I\!R^n$ such that $Mx + c \geq 0$, $Px + d \geq 0$ and $(Mx + c)^T(Px + d) = 0$, where matrices $M, P \in I\!R^{n \times n}$ and vectors $c, d \in I\!R^n$ are given.

The problems were randomly generated as follows: first we obtained two vectors $y, z \in I\!R^n$, $y, z \geq 0$ such that $y^T z = 0$. Using a function $rnd(a, b)$ that randomly generates a real value between a and b, for $i = 1, 2, \ldots, n$ we computed $\alpha = rnd(0, 1)$; if $\alpha < 0.5$, we set $y_i = rnd(1, 10)$ and $z_i = 0$, otherwise, $y_i = 0$ and $z_i = rnd(1, 10)$. We also generated a vector $x_* \in I\!R^n$ with components $[x_*]_i = rnd(-10, 10), i = 1, 2, \ldots, n$. Matrices M and P

have the pattern $M = Q_{ML}D_M Q_{MR}$, $P = Q_{PL}D_P Q_{PR}$, where D_M and D_P are diagonal matrices and $Q_{(\cdot)}$ are orthogonal Householder matrices defined by $Q_{(\cdot)} = I - 2\frac{u_{(\cdot)}u_{(\cdot)}^T}{u_{(\cdot)}^T u_{(\cdot)}}$, with vector $u_{(\cdot)} \in I\!R^n$ with components generated by $rnd(-1, 1)$. Hence, only three vectors are used to define and are stored for each matrix $M(u_{ML}, d_M, u_{MR})$ and $P(u_{PL}, d_P, u_{PR})$, since we just need to compute their products by a vector. To generate positive definite matrices, the elements of D_M and/or D_P were generated by $rnd(1, 10)$. To generate singular matrices we forced 20% of the elements of the diagonal to be identically zero. To generate indefinite matrices P, each element of D_P was multiplied by the signal of $rnd(-1, 1)$. Finally, vectors c and d were computed by $c = z - Mx_*$ and $d = y - Px_*$.

We started with a randomly generated initial approximation as follows: $[x_0]_i = rnd(-10, 10)$, $[z_0]_i = rnd(1, 10)$ and $[\lambda_0]_i = rnd(1, 10), i = 1, \ldots, n$.

According to the features of matrices M and P, we divided the set of tests in fourteen families: M and P may be identical or not, M and P may be symmetric or not and matrices M and P may be regular or singular. M is either positive definite or positive semidefinite, whereas P might be positive definite, positive semidefinite or indefinite (but regular). Whenever M or P is invertible, the theoretical hypotheses of the equivalence results of Section 2 can be checked by analysing properties of matrices PM^{-1} or MP^{-1}.

For each family, four values for the dimension n were used (5, 50, 500 and 5000). For each dimension, three problems were solved, with different seeds. The arithmetic means of the results are reported in Tables 1-14, where we inform the number of iterations (`INNER`) and matrix-vector products (`MVP`) performed by the inner (quadratic) solver, and the number of iterations (`OUTER`) and functional evaluations (`FE`) performed by the outer (trust-region) algorithm.

n	INNER	MVP	OUTER	FE
5	90.0	148.0	8.0	9.0
50	211.7	409.7	10.0	11.0
500	336.0	962.0	12.7	13.7
5000	484.3	2038.0	16.0	17.3

Table 1: Average results: $M = P$, $M = M^T$, M positive definite ($PM^{-1} = MP^{-1} = I$).

n	INNER	MVP	OUTER	FE
5	114.7	199.0	9.0	10.0
50	247.3	501.3	10.3	11.3
500	344.7	894.3	12.3	11.3
5000	355.3	1312.0	13.7	14.7

Table 2: Average results: $M = P$, $M = M^T$, M regular indefinite ($PM^{-1} = MP^{-1} = I$).

n	INNER	MVP	OUTER	FE
5	106.3	169.7	8.3	9.3
50	220.3	419.7	10.7	11.7
500	340.7	849.0	13.0	14.0
5000	379.3	1258.7	14.3	15.3

Table 3: Average results: $M = P$, $M = M^T$, M singular (positive semidefinite).

n	INNER	MVP	OUTER	FE
5	102.3	161.0	8.7	9.7
50	224.7	385.3	10.3	11.3
500	337.7	965.0	12.7	13.7
5000	564.7	2771.0	15.7	17.0

Table 4: Average results: $M = P$, $M \neq M^T$, M positive definite ($PM^{-1} = MP^{-1} = I$).

n	INNER	MVP	OUTER	FE
5	100.0	153.0	8.3	9.3
50	236.0	433.3	9.7	10.7
500	291.7	723.3	12.0	13.0
5000	453.3	1846.3	16.0	18.0

Table 5: Average results: $M = P$, $M \neq M^T$, M regular indefinite ($PM^{-1} = MP^{-1} = I$).

n	INNER	MVP	OUTER	FE
5	121.7	207.7	9.3	10.3
50	224.7	397.0	10.0	11.0
500	336.3	854.7	13.3	14.3
5000	392.7	1256.0	14.7	16.0

Table 6: Average results: $M = P$, $M \neq M^T$, M singular (positive semidefinite).

n	INNER	MVP	OUTER	FE
5	165.7	232.7	10.0	11.0
50	660.7	1042.7	15.7	16.7
500	1207.0	2533.3	18.7	20.0
5000	1480.7	3928.3	26.7	28.7

Table 7: Average results: $M \neq P$, $M = M^T$, $P = P^T$, M and P positive definite.

n	INNER	MVP	OUTER	FE
5	172.3	235.7	9.7	11.0
50	local	local	local	local
500	local	local	local	local
5000	local	local	local	local

Table 8: Average results: $M \neq P$, $M = M^T$, $P = P^T$, M positive definite, P regular indefinite.

n	INNER	MVP	OUTER	FE
5	134.7	188.3	9.7	10.7
50	1026.7	1308.3	17.7	18.7
500	2596.7	3520.7	37.3	38.3
5000	13555.7	20585.7	214.0	215.0

Table 9: Average results: $M \neq P$, $M = M^T$, $P = P^T$, M positive definite, P singular (positive semidefinite).

n	INNER	MVP	OUTER	FE
5	203.3	268.0	10.7	11.7
50	825.3	1096.3	15.3	16.7
500	1178.0	2130.0	18.7	19.7
5000	1473.7	4151.3	25.7	26.7

Table 10: Average results:$M \neq P$, $M = M^T$, $P = P^T$, M and P singular (positive semidefinite).

n	INNER	MVP	OUTER	FE
5	209.7	294.0	11.3	13.0
50	2493.7	2977.3	24.3	25.3
500	6774.0	7625.0	29.0	30.7
5000	17119.7	19131.3	50.0	51.7

Table 11: Average results: $M \neq P$, $M \neq M^T$, $P \neq P^T$, M and P positive definite.

n	INNER	MVP	OUTER	FE
5	120.0	286.7	8.7	9.7
50	local	local	local	local
500	local	local	local	local
5000	local	local	local	local

Table 12: Average results: $M \neq P$, $M \neq M^T$, $P \neq P^T$, M positive definite, P regular indefinite.

n	INNER	MVP	OUTER	FE
5	local	local	local	local
50	1143.7	1398.3	15.7	16.7
500	3919.3	4762.0	39.0	40.0
5000	23520.7	31786.3	233.3	234.3

Table 13: Average results: $M \neq P$, $M \neq M^T$, $P \neq P^T$, M positive definite, P singular (positive semidefinite).

n	INNER	MVP	OUTER	FE
5	163.3	240.7	10.0	11.0
50	1057.0	1359.3	15.7	16.7
500	4094.7	5037.7	26.7	27.7
5000	8300.3	10061.3	29.7	30.7

Table 14: Average results: $M \neq P$, $M \neq M^T$, $P \neq P^T$, M and P singular (positive semidefinite).

There were some problems, the results of which are reported in Tables 8, 12 and 13, that converged to local non-global minimizers of (2), with merit function value greater than 10^{-1}. For problems reported in Tables 1, 2, 4 and 5 the theoretical hypotheses hold, representing 28.5% of the total number of tests. For Tables 1, 2, 4, 5 and 7, the algorithm computed the same solution that was generated for assembling the problem data. In Tables 3, 6, 10 and 14, since both matrices M and P are singular, the theoretical hypotheses fail, representing 28.5% of tests. For these tests, however, the global solution of (2) was always obtained. There is no guarantee that the theoretical hypotheses are valid for the test problems of Tables 7, 8, 9, 11, 12 and 13, which represent 43% of tests. In fact, in 18 out of the 60 problems of these last six tables, at least one of the values $u^T PM^{-1}u$ or $v^T MP^{-1}v$, where $u = Mx + c - z$ and $v = Px + d - \lambda$, was negative. In the total of 168 problems solved, the hypotheses fail for 66 (39%), but only 16 converged to local solutions of (2), which correspond to 24% of the candidates for failure, and to 9.5% of the total of tests.

Denoting figures of Tables 1-14 by T^k_{ij}, $k = 1, 2, \ldots, 14$, $i = 1, 2, 3, 4$ (rows $n = 5, 50, 500$ and 5000, respec.), $j = 1, 2, 3, 4$ (columns INNER, MVP, OUTER and FE, respec.), we define average values to guide our analysis. Concerning the effort spent by the algorithm, there are two aspects we would like to address: how is such effort related to the problem dimension and how is it related to the problem features? In order to do so, considering each dimension separately, we start by defining two cost measures: per inner iteration (MVP/INNER) and global (INNER/OUTER), as follows:

$$me_1(i) = \frac{1}{K}\sum_k \frac{T_{i2}^k}{T_{i1}^k} \quad \text{and} \quad me_2(i) = \frac{1}{K}\sum_k \frac{T_{i1}^k}{T_{i3}^k},$$

for $i = 1, 2, 3, 4$, where $K = 13$ if $i = 1$ and $K = 12$ if $i = 2, 3, 4$, to exclude the local solutions.

To allow a better understanding of the average values represented by these two measures, we also computed the minimum and maximum values:

$$m_1(i) = \min_k \frac{T_{i2}^k}{T_{i1}^k},\ M_1(i) = \max_k \frac{T_{i2}^k}{T_{i1}^k},\ m_2(i) = \min_k \frac{T_{i1}^k}{T_{i3}^k},\ \text{and}\ M_2(i) = \max_k \frac{T_{i1}^k}{T_{i3}^k}.$$

Results are reported in Table 15, where the triples contain

$$(m_1(i), me_1(i), M_1(i)) \quad \text{and} \quad (m_2(i), me_2(i), M_2(i)),$$

for $i = 1, 2, 3, 4$. We observe that, in the average, less than three matrix-vector products are required per inner iteration, and this inner effort grows quite slowly as n increases. The global effort, however, increases with n, as well as the dispersion between the average, minimum and maximum values.

Dimension (n)	(m_1, me_1, M_1)	(m_2, me_2, M_2)
5	(1.32, 1.58, 2.39)	(11.25, 14.58, 19.00)
50	(1.19, 1.59, 2.03)	(20.59, 44.27, 102.62)
500	(1.13, 2.06, 2.86)	(24.31, 70.12, 233.59)
5000	(1.12, 2.84, 4.91)	(25.93, 89.38, 342.39)

Table 15: Measures of effort per problem dimension.

With the aim of analyzing results according to the family of generated problems, we define two additional measures for each one of Tables 1 to 14. The weights $\ln(n)$ and $\sqrt{\ln(n)}$ were introduced to filter dependence of dimension and somehow uniformize the computed values:

$$me_3(k) = \frac{1}{T}\sum_i \frac{T_{i2}^k}{\sqrt{\ln\left(\frac{10^i}{2}\right) T_{i1}^k}} \quad \text{and} \quad me_4(k) = \frac{1}{T}\sum_i \frac{T_{i1}^k}{\ln\left(\frac{10^i}{2}\right) T_{i3}^k},$$

for $k = 1, 2, \ldots, 14$, $k \neq 8$, $k \neq 12$, where $T = 3$ if $k = 13$ and $T = 4$ otherwise, to exclude results corresponding to local solutions. We stress that for $i = 1, 2, 3, 4$, the values $\frac{10^i}{2}$ are the dimensions 5, 50, 500 and 5000 used in the tests.

Table	Problem features	me_3	me_4
1	$M = P, M = M^T, M > 0$	1.85	11.64
2	$M = P, M = M^T, M$ indef.	1.78	12.44
3	$M = P, M = M^T, M \geq 0$	1.65	11.83
4	$M = P, M \neq M^T, M > 0$	1.87	12.30
5	$M = P, M \neq M^T, M$ indef.	1.72	12.06
6	$M = P, M \neq M^T, M \geq 0$	1.65	12.13
7	$M \neq P, \begin{array}{c} M = M^T \\ P = P^T \end{array}, \begin{array}{c} M > 0 \\ P > 0 \end{array}$	1.39	21.85
8	$M \neq P, \begin{array}{c} M = M^T \\ P = P^T \end{array}, \begin{array}{c} M > 0 \\ P \text{ indef.} \end{array}$	local	local
9	$M \neq P, \begin{array}{c} M = M^T \\ P = P^T \end{array}, \begin{array}{c} M > 0 \\ P \geq 0 \end{array}$	1.07	24.23
10	$M \neq P, \begin{array}{c} M = M^T \\ P = P^T \end{array}, \begin{array}{c} M \geq 0 \\ P \geq 0 \end{array}$	1.29	24.44
11	$M \neq P, \begin{array}{c} M \neq M^T \\ P \neq P^T \end{array}, \begin{array}{c} M > 0 \\ P > 0 \end{array}$	0.96	66.52
12	$M \neq P, \begin{array}{c} M \neq M^T \\ P \neq P^T \end{array}, \begin{array}{c} M > 0 \\ P \text{ indef.} \end{array}$	local	local
13	$M \neq P, \begin{array}{c} M \neq M^T \\ P \neq P^T \end{array}, \begin{array}{c} M > 0 \\ P \geq 0 \end{array}$	0.79	35.79
14	$M \neq P, \begin{array}{c} M \neq M^T \\ P \neq P^T \end{array}, \begin{array}{c} M \geq 0 \\ P \geq 0 \end{array}$	1.03	48.84

Table 16: Measures of effort per problem features.

Results are presented in Table 16, for which some observations are pertinent. First, whenever matrices M and P are equal (k from 1 to 6) results are pretty much similar. In fact, taking minimum, average and maximum values for both columns of Table 16, for $k = 1$ to 6 we obtain (1.65, 1.75, 1.87) and (11.64, 12.07, 12.44), respectively. Although for problems the results of which are given in Tables 3 and 6 do not satisfy the theoretical hypotheses, since M and P are both singular, this aspect does not seem to interfere in the results.

Now, if $M \neq P$ (k from 7 to 14), symmetric and non symmetric problems behave slightly different. When symmetry takes place, a little more matrix-vector products are performed, whereas the global effort is significantly reduced. Comparing results for $k = 7$ against $k = 11$, 8 against 12, 9 against 13 and 10 against 14, we can see that the effort for symmetric problems is about one third of the effort for non-symmetric ones, for positive definite matrices M and P, and about one half when the matrices are positive semidefinite. Taking minimum, average and maximum values for both columns of Table 16 for $k = 7$ to 14 we obtain (0.79, 1.09, 1.39) and (21.85, 36.94, 65.52). Comparatively with the figures computed for $k = 1$ to 6, we observe a much larger dispersion of the values. Moreover, the inner effort is smaller, whereas the global one increases.

3.2 Implicit complementarity problems from Outrata and Zowe

In the second set of experiments we solved implicit complementarity problems (see [10]) of the form:

Find $y \in \mathbb{R}^n$ such that

$$y - m(y) \geq 0, \quad F(y) \geq 0 \quad \text{and} \quad \langle F(y), y - m(y) \rangle = 0,$$

where $m_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n,$

$$F(y) = Ay + b = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

and $m(y) = \varphi(Ay+b)$, with $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ twice continuously differentiable.

As in examples 4.3 and 4.4 of [10], the following choices for function φ defined our test problems:

POZ1: $\varphi_i(\lambda) = -0.5 - \lambda_i, i = 1, 2, 3, 4$ and

POZ2: $\varphi_i(\lambda) = -0.5 - 1.5\lambda_i + 0.25\lambda_i^2, i = 1, 2, 3, 4.$

For each problem, three starting vectors were used, namely,

(a) $(0.0, 0.0, 0.0, 0.0)^T$
(b) $(-0.5, -0.5, -0.5, -0.5)^T$
(c) $(-1.0, -1.0, -1.0, -1.0)^T$.

In [10] Newtonian strategies were adopted to solve problems POZ1 and POZ2. In the *first approach*, the iterative scheme to compute fixed points of an operator S was

$$y_{k+1} = y_k - (E - V^k)^{-1}(y_k - S(y_k)),$$

where $V^k \in \partial S(y_k)$. In the *second approach*, a Newton variant scheme was applied to the semismooth operator

$$H(y) := \min\{y - m(y), F(y)\} = 0,$$

where min denotes the componentwise minimum of the two vectors in brackets.

Problems POZ1 and POZ2 were also solved in [8], with a trust-region approach for solving the GNCP$(F, G, I\!R^n_+)$ using the merit function $\Phi : I\!R^n \to R$ defined by

$$\Phi(x) := \frac{1}{2}\sum_{i=1}^{n} \phi(F_i(x), G_i(x))^2.$$

The function $\phi(a,b) = \sqrt{a^2+b^2} - a - b$ is the Fischer-Burmeister one, with the property $\phi(a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$.

In Tables 17 and 18 we present, for comparative purposes, numerical results of [10] and [8] for problems POZ1 and POZ2, respectively. Our results are reported in Table 19, where the notation of Tables 1-14 is used. We also included the final value of our merit function $f(x, z, \lambda)$, together with the norm of the projected gradient $\|g_p\|$ at the final approximation.

The results of our approach compared quite well with [8] and were by far superior than the results of [10]. For problem POZ1, starting points (a) and (b) provide similar results in terms of effort spent, although point (b) generates a solution with slightly better quality. For this problem, starting with point (c), on the other hand, requires twice as much inner iterations and matrix-vector products than starting with (a) or (b). For problem POZ2, the starting point that generated higher cost was (b).

OZ95			JFQS98		
start	first approach ITER	second approach ITER	ITER	FE	Φ
(a)	2	14	5	17	7.65D−18
(b)	2	41	4	16	9.71D−15
(c)	V^2 singular	56	5	11	3.43D−24

Table 17: Previous results - Problem 1 (POZ1 - $n = 4$).

OZ95			JFQS98		
start	first approach ITER	second approach ITER	ITER	FE	Φ
(a)	3	15	5	17	1.05D−18
(b)	V^2 singular	15	4	16	4.89D−15
(c)	V^2 singular	no convergence	5	11	7.05D−22

Table 18: Previous results - Problem 2 (POZ2 - $n = 4$).

Problem	start	OUTER	FE	INNER	MVP	$f(x,z,\lambda)$	$\|g_p\|$
POZ1	(a)	4	5	24	30	2.31D−10	8.61D−06
	(b)	4	5	22	39	1.55D−14	7.03D−08
	(c)	4	5	45	68	6.63D−11	7.77D−06
POZ2	(a)	5	6	48	74	4.25D−12	2.33D−06
	(b)	6	8	104	171	1.15D−14	8.25D−08
	(c)	3	4	31	60	9.43D−11	2.25D−05

Table 19: Results using our approach ($n = 4$).

To assess the performance of our approach, we enlarged the dimension n of problems POZ1 and POZ2, allowing $n = 40$, $n = 400$ and $n = 4000$. Matrix $A \in I\!R^{n\times n}$ and vector $b \in I\!R^n$ are the natural extensions of (3), as are the starting vectors (a), (b) and (c). Results are presented in Table 20, where one can see that the computational effort grows very slowly as n increases. The greatest difference happens between $n = 4$ and $n = 40$, but from 40 to 400 and from 400 to 4000 the cost does not grow as much as in the first case. Such differences in the increasing factors can be better appreciated by the average values shown in Table 21.

Problem	start	OUTER	FE	INNER	MVP	$f(x,z,\lambda)$	$\|g_p\|$
POZ1 $n = 40$	(a)	7	10	125	236	1.26D−11	5.48D−06
	(b)	6	8	102	313	3.16D−13	4.52D−07
	(c)	5	7	84	176	1.11D−10	6.62D−06
POZ1 $n = 400$	(a)	8	12	146	205	2.42D−12	8.69D−07
	(b)	7	10	126	201	5.66D−12	1.59D−06
	(c)	6	8	94	206	1.44D−11	2.32D−06
POZ1 $n = 4000$	(a)	9	14	143	311	1.59D−12	7.79D−07
	(b)	8	12	123	377	7.43D−12	2.26D−06
	(c)	7	9	99	289	1.96D−11	2.91D−06
POZ2 $n = 40$	(a)	7	11	127	248	4.36D−12	2.45D−06
	(b)	6	9	116	201	1.89D−11	2.56D−06
	(c)	6	8	104	176	6.90D−13	7.44D−07
POZ2 $n = 400$	(a)	9	14	143	227	6.64D−13	5.35D−07
	(b)	7	11	135	367	1.75D−11	2.74D−06
	(c)	7	10	120	203	6.30D−13	4.93D−07
POZ2 $n = 4000$	(a)	10	15	157	394	2.98D−12	9.12D−07
	(b)	9	14	161	385	7.84D−11	5.18D−06
	(c)	8	12	161	309	1.21D−12	4.94D−07

Table 20: Additional tests with larger dimensions.

Problem	n	OUTER	FE	INNER	MVP
POZ1	4	4.0	5.0	30.3	45.7
	40	6.0	8.3	103.7	241.7
	400	7.0	10.0	122.0	204.0
	4000	8.0	11.7	121.7	325.7
POZ2	4	4.7	6.0	61.0	101.7
	40	6.3	9.3	115.7	208.3
	400	7.7	11.7	132.7	265.7
	4000	9.0	13.7	159.7	362.7

Table 21: Average results of our approach.

3.3 Problems with general polyhedral cones in $\mathbb{R}^n$

In this third set of experiments we address the problem of finding $x \in \mathbb{R}^n$ such that $Mx + c \in \mathcal{K}$, $Px + d \in \mathcal{K}^\circ$ and $(Mx + c)^T(Px + d) = 0$, where the sets $\mathcal{K}$, $\mathcal{K}^\circ$ are defined by

$$\mathcal{K} = \{v \in \mathbb{R}^n \mid Av \geq 0, \quad Bv = 0\},$$

$$\mathcal{K}^\circ = \{u \in \mathbb{R}^n \mid u = A^T\lambda_1 + B^T\lambda_2, \quad \lambda_1 \geq 0\},$$

with $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{s\times n}$ given. Matrices $M, P \in \mathbb{R}^{n\times n}$ and vectors $c, d \in \mathbb{R}^n$ are also given.

The problems were randomly generated quite similarly to our first set of experiments. We started by generating matrices M and P with the pattern $M = Q_{ML}D_MQ_{MR}$, $P = Q_{PL}D_PQ_{PR}$, where D_M and D_P are diagonal matrices and $Q_{(\cdot)}$ are orthogonal Householder matrices defined by $Q_{(\cdot)} = I - 2\frac{u_{(\cdot)}u_{(\cdot)}^T}{u_{(\cdot)}^Tu_{(\cdot)}}$, with vector $u_{(\cdot)} \in \mathbb{R}^n$ with components generated by $rnd(-1, 1)$. To generate singular matrices we forced 20% of the elements of the diagonal to be identically zero. To generate indefinite matrices, each element of diagonal $D_{(\cdot)}$ was multiplied by the signal of $rnd(-1, 1)$. Next, vectors x_* and c were computed with components generated by $rnd(-10, 10)$. Then, we calculated $y_* = Mx_* + c$ and generated matrix B such that $By_* = 0$. This was accomplished by applying the modified Gram-Schmidt algorithm (see, e.g. [7]) to obtain the QR factorization of matrix

$$\left(y_* \,\middle|\, \begin{matrix} I_s \\ 0 \end{matrix}\right) = QR = \left(\frac{y_*}{\|y_*\|_2} \,\middle|\, B^T\right) R.$$

and the rows of matrix B are defined by the s last columns of the Q factor. Matrix A was defined by the product $A = S_AQ_{AL}D_AQ_{AR}$, with $S_A \in \mathbb{R}^p$ diagonal with 1 or -1 so that $Ay_* \geq 0$. Orthogonal matrices $Q_{AL} \in \mathbb{R}^{p\times p}$ and $Q_{AR} \in \mathbb{R}^{n\times n}$ are as matrices $Q_{(\cdot)}$ above. Matrix $D_A \in \mathbb{R}^{p\times n}$ is diagonal, without any prior assumption about p and n. We generated its elements

$(D_A)_i \in rnd[-0.1, 0.1], i = 1, \min\{p, n\}$. To complete the problem data, we generated λ_1^* such that $\lambda_1^* \perp z_* = Ay_*$, $\lambda_2^* \in rnd[-10, 10]$ and computed $d = A^T\lambda_1^* + B^T\lambda_2^* - Px_*$.

The initial approximation was randomly generated as follows: $[x_0]_i = rnd(-10, 10)$, $i = 1, \ldots, n$, $[z_0]_i = rnd(1, 10)$ and $[\lambda_1^0]_i = rnd(1, 10), i = 1, \ldots, p$ and $[\lambda_2^0]_i = 0, i = 1, \ldots, s$.

According to the features of matrices M and P, we divided the set of tests in three families: 1) $M = P$, indefinite and non symmetric; 2) $M = P$, indefinite and symmetric; 3) $M \neq P$, indefinite, non symmetric and singular. For families 1) and 2) the theoretical hypotheses of the equivalence results hold since $PM^{-1} = I$.

For each family, six sets for the dimensions (n, p, s) were considered: (10, 5, 1); (10, 10, 1); (10, 15, 1); (100, 50, 5); (100, 100, 5) and (100, 150, 5). For each set of dimensions, three problems were generated, with different seeds. The arithmetic means of the results are reported in Tables 22-23, where we inform the number of iterations (`INNER`) and matrix-vector products (`MVP`) performed by the inner (quadratic) solver, and the number of iterations (`OUTER`) and functional evaluations (`FE`) performed by the outer (trust-region) algorithm.

p	family	`INNER`	`MVP`	`OUTER`	`FE`
5	1	136.7	170.3	9.0	10.0
10		184.0	257.0	11.3	12.3
15		309.0	436.7	18.0	19.0
5	2	168.0	213.0	11.3	12.3
10		168.7	232.3	11.8	12.8
15		208.3	282.3	12.7	13.7
5	3	208.7	253.3	10.0	11.0
10		278.7	371.7	13.0	14.0
15		485.7	640.7	19.7	20.7

Table 22: Average results - problems with $n = 10, s = 1$.

p	family	`INNER`	`MVP`	`OUTER`	`FE`
50	1	1021.0	1373.0	35.7	36.7
100		2199.3	2971.3	72.0	73.0
150		3946.3	5103.0	113.3	114.0
50	2	1064.7	1421.0	37.0	38.0
100		2167.7	2833.0	67.3	68.3
150		4291.3	5720.3	124.3	125.3
50	3	7397.7	7922.0	101.0	102.0
100		160724.0	166259.0	1856.0	1857.0
150		102189.0	112886.0	957.3	963.0

Table 23: Average results - problems with $n = 100, s = 5$.

We denote the figures of Tables 22 and 23 by T_{ij}^k, where $k \in \{1,2,3\}$ represents each family, $i \in \{1,2,3\}$ corresponds to rows with $p = 5, 10, 15$ (Table 22), $i \in \{4,5,6\}$ corresponds to rows with $p = 50, 100, 150$ (Table 23), and $j \in \{1,2,3,4\}$ is the corresponding column with the values `INNER`, `MVP`, `OUTER` and `FE`. Based on these values, and similarly to the first set of tests, we define two cost measures to guide our analysis, per inner iteration (`MVP/INNER`) and global (`INNER/OUTER`), as follows:

$$me_1(i) = \frac{1}{3}\sum_k \frac{T_{i2}^k}{T_{i1}^k} \quad \text{and} \quad me_2(i) = \frac{1}{3}\sum_k \frac{T_{i1}^k}{T_{i3}^k},$$

for $i = 1,2,3,4,5,6$.

For a better understanding of the average values represented by these two measures, we also computed the minimum and maximum values:

$$m_1(i) = \min_k \frac{T_{i2}^k}{T_{i1}^k},\ M_1(i) = \max_k \frac{T_{i2}^k}{T_{i1}^k},\ m_2(i) = \min_k \frac{T_{i1}^k}{T_{i3}^k},\ \text{and } M_2(i) = \max_k \frac{T_{i1}^k}{T_{i3}^k}.$$

Results are reported in Table 24, where the triples contain

$$(m_1(i), me_1(i), M_1(i)) \quad \text{and} \quad (m_2(i), me_2(i), M_2(i)),$$

for $i = 1, \ldots, 6$.

With the aim of analyzing results according to the family of generated problems, we define two additional measures for each one of sets 1 to 3. The weights $\ln(n+2p+s)$ and $\sqrt{\ln(n+2p+s)}$ were introduced to filter dependence of dimension and somehow uniformize the computed values:

$$me_3(k) = \frac{1}{6}\left(\sum_{i=1}^{3} \frac{T_{i2}^k}{\ln(11+10i)T_{i1}^k} + \sum_{i=4}^{6} \frac{T_{i2}^k}{\ln(100i-195)T_{i1}^k}\right)$$

and

$$me_4(k) = \frac{1}{6}\left(\sum_{i=1}^{3} \frac{T_{i2}^k}{\sqrt{\ln(11+10i)}T_{i1}^k} + \sum_{i=4}^{6} \frac{T_{i2}^k}{\sqrt{\ln(100i-195)}T_{i1}^k}\right)$$

for $k = 1,2,3$. We stress that the values $11+10i, i = 1,2,3$ and $100i-195, i = 4,5,6$ are, respectively, the dimensions 21, 31, 41 and 205, 305, 405 used in the tests. Results are shown in Table 25, where we also include minimum (m_3, m_4) and maximum values (M_3, M_4).

Dimension (p)	(m_1, me_1, M_1)	(m_2, me_2, M_2)
5	(1.22, 1.24, 1.26)	(14.69, 16.54, 20.03)
10	(1.34, 1.36, 1.38)	(14.79, 17.50, 21.34)
15	(1.33, 1.37, 1.42)	(16.49, 19.28, 24.11)
50	(1.09, 1.25, 1.34)	(28.24, 41.04, 66.12)
100	(1.06, 1.24, 1.35)	(30.49, 48.74, 83.55)
150	(1.09, 1.24, 1.34)	(34.24, 54.99, 95.91)

Table 24: Measures of effort per problem dimension.

Family	(m_3, me_3, M_3)	(m_4, me_4, M_4)
1	(0.50, 0.54, 0.58)	(23.08, 37.16, 54.70)
2	(0.51, 0.54, 0.56)	(23.19, 37.02, 53.86)
3	(0.43, 0.48, 0.54)	(31.34, 81.58, 151.38)

Table 25: Measures of effort per problem family.

Observing Table 24 one can see that the effort of the inner solver is always inferior to 1.5 matrix-vector products per iteration. Moreover, it is slightly larger for smaller problems (dimensions $n + 2p + s \in \{21, 31, 41\}$ than for larger ones ($n + 2p + s \in \{205, 305, 405\}$), although the dispersion between minimum and maximum values grows with increasing p. This last comment also applies to the global effort measure me_2, that grows as p increases, together with the length of intervals $[m_2, M_2]$. Although dimension differs from a factor of ten for the two sets of problems, figures of (m_2, me_2, M_2) are about twice as large when the two sets are compared.

Concerning Table 25, the main conclusions are that symmetry of matrices M and P does not seem to interfere in the performance of our approach, since families 1 and 2 produced quite similar results for both triples (m_3, me_3, M_3) and (m_4, me_4, M_4). Singularity of matrices M and P, on the other hand, showed significative effects, especially as far as the global performance is concerned.

This set of experiments contains a total of 54 tests. For the 27 problems of smaller dimension, the final objective function value was always inferior to 10^{-5}. Considering the 27 larges ones, for 8 problems of the third family the final objective function values were greater than 10^{-2}, indicating convergence to a local non-global solution. This amounts to 55.6% of success among problems for which the theoretical condition of equivalence does not hold. We stress, however, that whenever such hypothesis is valid, a global solution was reached.

3.4 Problems in 3D-cones with control of generated faces

In the fourth set of experiments we addressed the problem of finding $x \in \mathcal{K} = \{v \in I\!R^n \mid Av \geq 0\}$ such that $Tx + q \in \mathcal{K}^\circ = \{v \in I\!R^n \mid A^T\lambda = v, \lambda \geq 0\}$. We generated the polyhedral cones $\mathcal{K}$ with p faces, such that their edges were the lines

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\left(\frac{2\pi}{p}k\right) \\ r\sin\left(\frac{2\pi}{p}k\right) \\ 1 \end{pmatrix} t, \quad t \in I\!R,\ k = 1, \ldots, p.$$

Therefore, $\mathcal{K}$ was defined by computing the rows of matrix A as the normal vectors to the support planes to the faces of the cone. In other words, the vector that defines the $i-$th row of matrix A $(i = 1, \ldots, p)$ is given by the cross product:

$$\begin{pmatrix} \cos\left(\frac{2\pi}{p}(i-1)\right) \\ \sin\left(\frac{2\pi}{p}(i-1)\right) \\ \frac{1}{r} \end{pmatrix} \times \begin{pmatrix} r\cos\left(\frac{2\pi}{p}i\right) \\ r\sin\left(\frac{2\pi}{p}i\right) \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\left(\frac{2\pi}{p}i\right)\left(\cos\frac{2\pi}{p} - 1\right) - \cos\left(\frac{2\pi}{p}i\right)\sin\frac{2\pi}{p} \\ \cos\left(\frac{2\pi}{p}i\right)\left(1 - \cos\frac{2\pi}{p}\right) - \sin\left(\frac{2\pi}{p}i\right)\sin\frac{2\pi}{p} \\ r\sin\frac{2\pi}{p} \end{pmatrix}$$

The problems were generated as follows. Once defined the values of the radius r and of the dimension p (number of faces of cone $\mathcal{K}$), we built matrix A and created two types of x_*, at the border and in the interior of $\mathcal{K}$, respectively. Next we generated matrix T using the pattern $T = Q_{TL}D_TQ_{TR}$, where D_T is diagonal and Q_{TL}, Q_{TR} are orthogonal Householder matrices defined as in the first and third set of tests. We kept T symmetric, and produced four families of problems, namely 1) T indefinite; 2) T positive definite; 3) T positive semidefinite and 4) T negative semidefinite. We built $0 \leq \lambda_* \perp Ax_* \equiv z_*$. Finally we computed $q = A^T\lambda_* - Tx_*$.

The tests were produced by varying $r \in \{0.1, 1, 10\}$, $p \in \{3, 4, 5, 6, 9, 12\}$, the four families of matrices T and the two kinds of generated solution x_*, which amounted to 144 problems. Three distinct seeds were chosen to generate problems for each selection of r, p, T and x_*. Tables 26-33 contain average values of the results obtained with the three seeds. To present the results we have separated information concerning number of inner iterations, of matrix-vector products, of outer iterations and of functional evaluations in distinct tables, in order to keep together variation of dimension p, radius r and features of matrix T.

To analyse the robustness of the proposed approach, since half of the generated problems do not satisfy the hypothesis of the equivalence result (families 1 and 4, with matrices T indefinite and negative semidefinite, respectively), we observed that for the 72 problems with x_* generated at the boundary of the cone, 29 out of the 72×3 tests stopped at local non global solutions. This corresponds to success for 86.6% of the total and 73.2% of the candidates for failure. For problems with x_* generated in the interior of the cone, there were six problems that converged to local non-global solutions, in a total of 72×3 problems. In this case, the measures of success are 97.2% of the total and 94.4% of the problems without theoretical guarantee of convergence. Summing up the two blocks of tests, there were 35 failures, representing success in 91.9% of total and 83.8% of the universe of problems that do not satisfy the hypothesis of equivalence result.

There are some salient features that emerge from Tables 26-33. First, the computational cost of the inner solver grows with problem dimension, reaching its maximum for $p = 9$ and $p = 5$ if x_* is generated at the boundary and in the interior of $\mathcal{K}$, respectively.

It is also evident that the degree of difficulty of the generated problems grows as the radius r decreases: $r = 10$ produces the easiest problems whereas $r = 0.1$ generates the most difficult ones. Recall that in this set of experiments our problem is to find $x \in \mathcal{K} = \{v \in I\!R^n \mid Av \geq 0\}$ such that $Tx + q \in \mathcal{K}^\circ = \{v \in I\!R^n \mid A^T\lambda = v, \lambda \geq 0\}$, so the requirements for $\mathcal{K}$ and $\mathcal{K}^\circ$ are different.

family	r	INNER					
		$p=3$	$p=4$	$p=5$	$p=6$	$p=9$	$p=12$
1	1	62.0	75.3	120.3	120.0	261.7	216.7
	10	118.0	139.3	155.0	340.7	363.3	377.3
	0.1	74.3	107.0	255.3	210.3	768.0	593.3
2	1	70.3	110.7	160.3	134.0	228.7	252.7
	10	88.0	110.3	161.7	191.3	357.0	300.7
	0.1	60.3	101.3	158.7	188.0	502.0	585.0
3	1	70.0	118.7	190.0	220.0	207.3	220.7
	10	75.0	61.3	223.7	140.3	307.0	391.0
	0.1	88.0	149.7	190.3	182.3	402.3	722.0
4	1	66.7	114.7	116.7	158.7	380.7	339.0
	10	58.0	81.0	120.7	139.0	264.7	300.0
	0.1	78.7	167.3	240.0	172.7	970.3	163.7

Table 26: Average number of inner iterations (INNER) for problems with x_* generated at the boundary of cone $\mathcal{K}$.

family	r	MVP					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	84.7	105.0	172.7	171.7	364.0	344.3
	10	176.3	197.3	227.0	430.7	542.0	558.3
	0.1	99.3	141.7	343.7	307.0	1010.3	811.0
2	1	98.7	158.0	223.0	187.0	301.0	345.3
	10	119.0	158.0	256.3	297.7	612.0	505.7
	0.1	76.3	138.7	206.7	250.0	647.0	770.7
3	1	97.0	166.3	272.7	313.0	254.0	303.7
	10	100.3	86.0	332.0	209.3	468.3	654.7
	0.1	120.7	192.7	252.7	253.3	544.7	950.0
4	1	90.0	155.7	182.7	215.7	521.3	477.3
	10	75.3	108.3	160.0	194.7	371.3	482.0
	0.1	111.3	224.7	294.3	257.7	1281.3	263.0

Table 27: Average number of matrix vector products (MVP) for problems with x_* generated at the boundary of cone $\mathcal{K}$.

family	r	OUTER					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	5.3	6.0	7.3	7.7	10.3	10.7
	10	6.3	7.7	8.0	10.0	11.7	12.7
	0.1	5.7	7.3	9.3	9.7	21.3	20.0
2	1	6.3	7.3	8.0	7.0	10.0	12.0
	10	5.7	6.7	8.0	7.3	10.7	11.7
	0.1	6.0	7.0	9.0	11.0	16.7	23.0
3	1	6.0	6.7	8.3	8.3	10.0	10.3
	10	5.0	3.7	7.0	6.7	10.7	12.7
	0.1	6.0	7.0	8.7	8.3	15.3	24.3
4	1	5.7	7.3	6.7	7.3	10.3	10.3
	10	6.0	6.0	9.3	8.7	10.7	9.7
	0.1	6.7	8.3	9.3	8.7	18.3	7.3

Table 28: Average number of outer iterations (OUTER) for problems with x_* generated at the boundary of cone $\mathcal{K}$.

family	r	FE					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	6.3	7.0	8.3	8.7	11.3	11.7
	10	7.3	8.7	9.0	11.0	13.3	15.7
	0.1	6.7	8.3	10.3	10.7	23.0	21.3
2	1	7.3	8.3	9.0	8.0	11.0	13.0
	10	7.0	7.7	9.0	8.3	12.3	12.7
	0.1	7.0	8.0	10.0	12.0	17.7	24.0
3	1	7.0	7.7	9.7	9.3	11.7	11.3
	10	6.0	4.7	8.0	7.7	11.7	14.3
	0.1	7.0	8.0	9.7	9.3	16.3	25.3
4	1	6.7	8.3	7.7	8.3	11.3	11.3
	10	7.0	7.0	10.3	9.7	12.3	11.7
	0.1	7.7	9.3	10.3	9.7	19.3	8.3

Table 29: Average number of functional evaluations (FE) for problems with x_* generated at the boundary of cone $\mathcal{K}$.

family	r	INNER					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	107.0	137.7	99.0	109.7	217.7	257.3
	10	77.3	89.7	139.0	157.0	133.3	227.3
	0.1	144.3	485.7	211.3	371.3	947.3	1380.3
2	1	84.7	92.0	95.0	104.7	113.0	113.3
	10	52.7	63.3	78.3	66.7	33.7	134.0
	0.1	146.7	299.7	434.0	479.3	471.3	492.7
3	1	81.0	90.3	142.0	107.3	106.3	129.3
	10	43.0	100.3	77.0	59.0	154.7	233.0
	0.1	214.0	393.3	368.0	409.0	574.7	814.0
4	1	73.3	57.3	122.7	87.3	150.7	197.0
	10	96.3	189.0	177.0	132.0	71.3	45.3
	0.1	197.7	391.0	1993.3	274.7	483.7	228.3

Table 30: Average number of inner iterations (INNER) for problems with x_* generated in the interior of cone $\mathcal{K}$.

family	r	MVP					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	138.3	175.7	140.7	158.3	319.0	344.7
	10	107.3	129.7	203.7	255.7	239.7	374.3
	0.1	174.7	654.3	283.7	476.7	1248.3	1778.3
2	1	108.0	126.0	139.7	156.0	179.0	178.7
	10	73.3	88.3	128.3	104.7	58.0	241.3
	0.1	188.3	378.7	560.3	623.0	637.0	664.0
3	1	115.0	128.3	225.0	166.7	176.3	229.0
	10	61.0	141.7	116.3	92.0	285.7	433.0
	0.1	298.3	505.7	474.0	552.0	775.7	1126.3
4	1	99.7	79.0	175.0	137.3	248.3	309.7
	10	135.0	283.0	278.3	211.7	136.0	86.7
	0.1	270.0	548.7	2666.7	356.0	657.0	339.7

Table 31: Average number of matrix vector products (MVP) for problems with x_* generated in the interior of cone $\mathcal{K}$.

family	r	OUTER					
		$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 9$	$p = 12$
1	1	7.3	8.0	7.3	7.7	9.7	10.0
	10	4.7	4.7	6.3	6.3	5.3	8.0
	0.1	7.3	15.3	10.0	13.0	27.0	28.0
2	1	6.7	7.3	7.3	7.3	6.7	7.0
	10	4.0	4.7	4.3	4.7	3.3	6.3
	0.1	10.3	13.3	13.7	14.0	16.3	19.7
3	1	7.0	6.3	8.0	7.0	7.0	7.7
	10	3.0	4.7	4.3	4.0	5.7	7.7
	0.1	9.7	11.7	12.7	11.7	15.3	18.3
4	1	6.0	4.3	7.3	6.3	7.3	9.3
	10	5.7	6.7	8.0	6.7	4.0	4.0
	0.1	8.0	13.3	27.2	10.0	10.0	11.7

Table 32: Average number of outer iterations (OUTER) for problems with x_* generated in the interior of cone $\mathcal{K}$.

family	r	FE					
		$p=3$	$p=4$	$p=5$	$p=6$	$p=9$	$p=12$
1	1	8.3	9.0	8.3	8.7	11.3	11.0
	10	5.7	5.7	7.3	7.3	6.3	9.0
	0.1	8.3	16.3	11.0	15.3	28.0	29.0
2	1	7.7	8.3	8.3	8.3	7.7	8.0
	10	5.0	5.7	5.3	5.7	4.7	7.3
	0.1	11.3	14.3	14.7	16.0	19.0	21.0
3	1	8.0	7.3	9.0	8.0	8.0	8.7
	10	4.0	5.7	5.3	5.0	6.7	8.7
	0.1	10.7	12.7	13.7	12.7	16.3	19.3
4	1	7.0	5.3	8.3	7.3	8.3	10.3
	10	6.7	7.7	9.0	7.7	5.0	5.0
	0.1	9.0	14.3	28.7	11.0	11.0	12.7

Table 33: Average number of functional evaluations (FE) for problems with x_* generated in the interior of cone $\mathcal{K}$.

Grouping problems according to the features of matrix T, there are 36 problems for each family (6 dimensions p, 3 values for r and 2 types of generated x_*). We have computed the ratios INNER$/n_t$ and OUTER$/n_t$, where $n_t = n + 2p$ is the dimension of problem (1) and calculated average values, presented in Table 34, together with minimum and maximum values.

family	INNER$/n_t$			OUTER$/n_t$		
	minimum	average	maximum	minimum	average	maximum
1	6.3	15.6	51.1	0.3	0.6	1.4
2	1.6	12.2	33.4	0.2	0.6	1.2
3	3.9	13.6	35.8	0.3	0.6	1.1
4	1.7	16.4	153.3	0.1	0.6	2.1

Table 34: Measures of effort per problem features.

Observing the figures of Table 34, one can see that families 1 and 4 (T indefinite and negative semidefinite, respectively) demand more effort to be solved than those from families 2 and 3 (T positive definite and positive semidefinite, respectively). The largest dispersion, that is the largest interval [minimum, maximum] occurs for the fourth family, because of an outlier. Removing this discrepant value, the triples become (1.7, 14.5, 46.2) and (0.1, 0.7, 1.2), with dispersions similar to the ones of the first family.

4 Conclusions

We have used a smooth box constrained minimization reformulation of the GNCP$(F, G, \mathcal{K})$, assuming that $\mathcal{K}$ is a polyhedral cone. Any efficient minimization algorithm for solving this kind of problems may be used. Computational experiments are presented which encourage the use of our approach.

Four groups of problems were addressed: randomly generated problems in the positive orthant; implicit complementarity problems from Outrata & Zowe; problems with general cones in $\mathbb{R}^n$ and problems in 3D-cones with control of generated faces.

The numerical results showed that the solution of the GNCP using (2) was found in the majority of the tests, even without accomplishment of theoretical hypothesis, meaning that the behavior of the method does not depend strongly on the sufficient conditions that guarantee the equivalence.

Quantifying this robustness, considering only the universe of problems without theoretical support for convergence, for the first set of experiments the amount of failure was 24%. In the third and fourth sets, local non-global solutions were reached in 44% and 16% of the tests, respectively. No doubt, in the absence of theoretical support, the convergence to global solutions is more frequent for problems of smaller dimensions. The second set of problems, included for comparative purposes, formed by implicit complementarity problems, contained large-scale experiments (dimension up to $3 \times 4000 = 12000$) for which our approach had a very good performance. The third set of experiments revealed that general polyhedral cones might produce quite difficult problems, especially as the dimension increases. The fourth group of tests was created to investigate geometrical features of the cone $\mathcal{K}$. Besides noticing that, for the generated 3D-problems, thinner cones need more effort than wider ones, we observed that the increasing number of edges and faces did not substantially augment the amount of effort needed to solve the problems. As a natural extension of this work we would like to investigate the possibility of approximating a general cone by a polyhedral one. This leads us to look for further connections between theory and practice concerning geometrical and algebraic properties of general cones and their relationship with GNCP defined in such sets. We are also interested in studying the behavior of our approach applied to problems with nonlinear functions F and G and polyhedral cones.

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