Relative invariance for monoid actions

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Abstract

Let G be a topological group and $S \subset G$ a submonoid of G acting on the topological space M. Let J be a subset of M. Our purpose here is to study the subsets of M which correspond, under the action of S, to the relative (with respect to J) invariant control sets for control systems [4]. The relation $x \sim y$ if $y \in \mathrm{cl}(Sx)$ and $x \in \mathrm{cl}(Sy)$ is an equivalence relation and the classes with respect to this relation with nonempty interior in M are the control sets for the action of S. It is given conditions for the existence and uniqueness of relative invariant classes. As it was done for the control sets, we define an order in the classes and relate it to the relative invariant classes. We also show under certain condition that the relative invariant classes are relatively closed in J.

1 Introduction

One of the principal dynamical concepts in control theory is the study of the controllability of the control systems. Many questions about the control

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system, especially those related to its controllability depend, in fact, only on the action of the semigroup of the system, so that it can be abstracted to arbitrary semigroup actions and solved in a more general setting. The regions of the state space where the controllability occurs are called control sets. The control sets for control systems were studied by Colonius and Kliemann in [1],[2],[3] and [4]. In particular, Colonius and Kliemann introduced the concept of a control set which is relatively invariant with respect to a subset of the phase space of the control system. From a more general point of view, the theory of control sets for semigroup actions was developed by San Martin and Tonelli (see [5],[6] and [7]). Let S be a submonoid of a topological group G and suppose that S acts on a topological space M. Since the control sets are the regions where S is approximate transitive it is natural to define an equivalence relation by saying that two points are equivalent if they are approximate attainable by the action of S. We consider equivalence classes in M with respect to this relation. We show that a class with nonempty interior in M is a control set for S. The purpose of this paper is to study the relative invariant classes in M. We define relatively invariant classes. In case S is the system semigroup of a control system these relatively invariant classes, with nonempty interior in M, are the relative invariant control sets defined by Colonius and Kliemann in [4]. We develop the theory of relative invariant classes. As it was done for the control sets, we define an order in the classes and relate it to the relative invariant classes. We give conditions for the existence and uniqueness of relative invariant classes. Under the hypothesis of acessibility, we show that a relative invariant class is relatively closed.

2 Relative invariance

In this paper we assume that G is a topological group and S is a submonoid of G, i.e., $S \subset G$ is a subsemigroup with $1 \in S$. We also suppose that S acts continuously on a topological space M. We define a control set.

Definition 1 *A control set for S on M is a subset $D \subset M$ which satisfies*

1. $\mathrm{int}(D) \neq \emptyset$,
2. $\forall x \in D$, $D \subset \mathrm{cl}(Sx)$ *and*

3. D is maximal with these properties.

The action on M induces the pre-order relation defined by

$$x \preceq y \text{ if } y \in \mathrm{cl}(Sx),\ \ x, y \in M.$$

We define $x \sim y$ if $x \preceq y$ and $y \preceq x$. Therefore $[x] = [y]$ if and only if $y \in \mathrm{cl}(Sx)$ and $x \in \mathrm{cl}(Sy)$. Thus $\sim$ is the equivalence relation associated with $\preceq$. The pre-order in M induces a partial order in the quotient space $M/\sim$. This order in the quotient space is also denoted by $\preceq$. We denote by $[x] \in M/\sim$ the equivalence class of $x \in M$. From the control theoretic point of view a class $[x]$ with $\mathrm{int}_M([x]) \neq \emptyset$ is a control set for the action of the submonoid S.

Proposition 1 *Let $D = [x]$ be a class with respect to the equivalence relation $\sim$. Suppose that $\mathrm{int}_M(D) \neq \emptyset$. Then D is a control set for S on M.*

Proof: We first show that $D = [x] \subset \mathrm{cl}(Sy)$, for every $y \in [x]$. Take $y \in [x]$ then $x \sim y$. For $z \in [x]$, we have $z \sim x$. By the transitivity of the relation $\sim$ we have $z \sim y$ and $y \preceq z$. This implies that $z \in \mathrm{cl}(Sy)$. Now, suppose $D \subseteq D'$ with D' satisfying the condition $D' \subset \mathrm{cl}(Sz)$ for every $z \in D'$. Take $y \in D'$. Then $y \in \mathrm{cl}(Sz)$, for every $z \in D'$, in particular for $x \in D$. Therefore $y \in \mathrm{cl}(Sx)$ and $x \preceq y$. On the other hand, since $x \in D \subseteq D'$, we have $x \in \mathrm{cl}(Sy)$ once $y \in D'$ and $y \preceq x$. Hence $x \sim y$ and $y \in [x] = D$, showing the maximality of D. □

A class $[x]$ is said *maximal* if every class $[y]$ with $[x] \preceq [y]$ satisfies $[x] = [y]$. We will show later that a maximal class with $\mathrm{int}_M([x]) \neq \emptyset$ is an invariant control set for the submonoid S.

We observe that $[x] \subset \mathrm{cl}(Sx)$ for every $x \in M$.

We say that a subset $A \subset M$ is S-*invariant,* or invariant for the monoid S, if for every $y \in A$ we have $\mathrm{cl}(Sy) \subset A$.

A S-invariant class is always maximal. In fact, suppose $[x]$ is S-invariant. Then $\mathrm{cl}(Sx) \subset [x]$. If we take $[x] \preceq [y]$ we have $x \preceq y$ and $y \in \mathrm{cl}(Sx) \subset [x]$. Therefore $x \sim y$ and $[x] = [y]$, showing the maximality.

We define $\mathcal{O} = \{\mathrm{cl}(Sx) \subset M : x \in M\}$. Now, we relate maximality and S-invariance.

Lemma 1 *For $x \in M$ the following statements are equivalent:*

1. *$[x]$ is maximal*
2. *$\mathrm{cl}(Sx)$ is minimal in $\mathcal{O}$ with respect to the inclusion of sets*
3. *$[x] = \mathrm{cl}(Sx)$*
4. *$[x]$ is closed and S-invariant*

Proof: Let's assume that $\mathrm{cl}(Sy) \subset \mathrm{cl}(Sx)$ for some $y \in M$. Take $z \in \mathrm{cl}(Sy)$. By the continuity of the action of S on M we have $\mathrm{cl}(Sz) \subset \mathrm{cl}(Sy)$. Since $z \in \mathrm{cl}(Sx)$ the maximality of $[x]$ implies that $\mathrm{cl}(Sx) \subset \mathrm{cl}(Sz)$. Therefore $\mathrm{cl}(Sx) \subset \mathrm{cl}(Sy)$ showing that $\mathrm{cl}(Sx)$ is minimal. Suppose that $\mathrm{cl}(Sx)$ is minimal in $\mathcal{O}$. Then $\mathrm{cl}(Sy) \subset \mathrm{cl}(Sx)$ for all $y \in \mathrm{cl}(Sx)$. By minimality $\mathrm{cl}(Sy) = \mathrm{cl}(Sx)$ for all $y \in \mathrm{cl}(Sx)$. This implies that $\mathrm{cl}(Sx)$ is entirely contained in an equivalence class so that $\mathrm{cl}(Sx) = [x]$. If $[x] = \mathrm{cl}(Sx)$ it is immediate that $[x]$ is closed and S-invariant. Finally, a S-invariant class is maximal. □

The maximal classes with nonempty interior in M are the invariant control sets, more specifically, we have.

Corollary 1 *Let $D = [x]$ be a maximal class with respect to the relation $\sim$. Assume $\mathrm{int}_M(D) \neq \emptyset$. Then D is an invariant control set for S on M.*

Proof: First we show that $\mathrm{cl}(D) = \mathrm{cl}(Sy)$ for every $y \in D$. Since $[x]$ is maximal we have by the Lemma 1 that $[x] = \mathrm{cl}(Sx)$. Hence $D = \mathrm{cl}(D)$. For $y, z \in [x]$ we have $z \in \mathrm{cl}(Sy)$. Conversely, take $z \in \mathrm{cl}(Sy)$, since $x \sim y$ we have $y \in \mathrm{cl}(Sx)$. Therefore $z \in \mathrm{cl}(Sx) = [x]$. Now we will show the maximality of D. Suppose $[x] = D \subseteq D'$ and D' satisfies the equality $\mathrm{cl}(D') = \mathrm{cl}(Sy)$ for every $y \in D'$. Take $z \in D'$. Then $\mathrm{cl}(D') = \mathrm{cl}(Sz)$. Since $x \in D \subseteq D' \subseteq \mathrm{cl}(D')$ we have $x \in \mathrm{cl}(Sz)$ and $z \preceq x$. On the other hand, since $x \in D \subseteq D'$ we have $\mathrm{cl}(D') = \mathrm{cl}(Sx)$. But, $z \in D' \subseteq \mathrm{cl}(D') = \mathrm{cl}(Sx)$ and therefore $x \preceq z$. It follows that $x \sim z$ and $z \in [x] = D$, showing the maximality of D. □

On the existence of maximal classes we have the following proposition.

Proposition 2 *Let J be a S-invariant compact subset of M. Then for every $x \in J$ there exists a maximal equivalence class $[w] \subset \mathrm{cl}(Sx)$.*

Proof: Fix $x \in J$ and consider the family of subsets

$$\mathcal{O}_x = \{\mathrm{cl}(Sy) : \mathrm{cl}(Sy) \subset \mathrm{cl}(Sx)\}.$$

This family is not empty because it contains $\mathrm{cl}(Sx)$. Let us order $\mathcal{O}_x$ by inclusion and show with the aid of Hausdorff 's maximality principle that it contains minimal elements: Take a chain $\{\mathrm{cl}(Sy)\}_{y \in \mathcal{I}}$ of subsets in $\mathcal{O}_x$, where $\mathcal{I}$ is an index set. Since J is S-invariant $\mathrm{cl}(Sx) \subset J$. Therefore we have a chain of closed subsets of J. Hence they are compact which implies that the intersection $\bigcap_{y \in \mathcal{I}} \mathrm{cl}(Sy)$ is not empty. Take $z \in \bigcap_{y \in \mathcal{I}} \mathrm{cl}(Sy)$. Then $\mathrm{cl}(Sz)$ belongs to $\mathcal{O}_x$ and is contained in $\mathrm{cl}(Sy)$ for all $y \in \mathcal{I}$. These means that $\mathrm{cl}(Sz)$ is a lower bound of the chain. Applying the maximality principle we conclude that $\mathcal{O}_x$ contains a minimal element, say $\mathrm{cl}(Sw)$. Any element of $\mathcal{O} = \{\mathrm{cl}(Sx) \subset M : x \in M\}$ contained in $\mathrm{cl}(Sw)$ is an element of $\mathcal{O}_x$ because $\mathrm{cl}(Sw) \subset \mathrm{cl}(Sx)$. Hence $\mathrm{cl}(Sw)$ is also minimal in $\mathcal{O}$ so the proof follows from Lemma 1. □

We define a maximal class relatively to a subset J contained in the manifold M.

Definition 2 *Given a subset $J \subset M$ a class $[x] \subset J$ is said to be J-maximal, if every class $[y] \subset J$ with $[x] \preceq [y]$ satisfies $[x] = [y]$.*

For subsets J of the manifold M which are compact and S-invariant we have.

Corollary 2 *Suppose that $J \subset M$ is compact and S-invariant. Then there are J-maximal classes and they are the maximal classes contained in J.*

Proof: Take $x \in J$. Since J is compact and invariant under the action of S we have by the Lemma 2 that there exist a maximal class, say, $[w] \subset \mathrm{cl}(Sx) \subset J$. We also have that $[w]$ is J-maximal. Conversely, a maximal class contained in J is J-maximal. □

Now, we define a class which is invariant with respect to a subset J in M.

Definition 3 *For a subset $J \subset M$ a class $[x] \subset J$ is called SJ-invariant, if $z \in \mathrm{cl}(Sx)$ with $z \notin [x]$ implies $z \notin J$.*

Therefore, if a class $[x] \subset J$ is SJ-invariant it cannot leave by the closure of an orbit without leaving J.

If a class $[x]$ has nonempty interior in M then a SJ-invariant class is called an *SJ-invariant control set*. The relative invariant control sets for control systems were studied in [4] by Colonius and Kliemann.

It follows immediately that a SJ-invariant class is J-maximal. As a consequence a SM-invariant class is M-maximal and therefore it is a maximal class. Conversely, by the Lemma 1, a maximal class is SM-invariant. Therefore, if $\mathrm{int}_M([x]) \neq \emptyset$ a class $[x]$ is an invariant control set if and only if $[x]$ is SM-invariant.

On the existence of SJ-invariant classes we have.

Corollary 3 *Suppose that $J \subset M$ is compact and S-invariant. Then $[x]$ is SJ-invariant class if and only if $[x]$ is J-maximal. In this case there exist SJ-invariant classes.*

Proof: Suppose $[x]$ is J-maximal. By the Corollary 2 $[x]$ is an S-invariant class contained in J. Therefore $[x]$ is a SJ-invariant class. □

The no-return condition defined bellow was introduced, in the context of control systems, by Colonius and Kliemann [4] in the study of relatively invariant control sets.

We say that a subset $J \subset M$ satisfy the *no-return* condition if $z \in \mathrm{cl}(Sx)$ for some $x \in J$ and $\mathrm{cl}(Sz) \cap J \neq \emptyset$, then $z \in J$. This condition says that if we leave J we cannot go back to J again thorough the closure of an orbit of S.

Now, we translate the no-return condition in terms of an union of equivalence classes.

Proposition 3 *Suppose that $J \subset M$ satisfy the no-return condition. Then J is exhaustive for the equivalence relation $\sim$, i.e., any class $[x]$ is entirely contained in J or in J^c.*

Proof: Let $[x]$ be a class such that $[x] \cap J \neq \emptyset$ and $[x] \cap J^c \neq \emptyset$ and we will obtain a contradiction. Take $z \in [x] \cap J$ and $y \in [x] \cap J^c$. Then $z \sim y$, i.e.,

$z \in \mathrm{cl}(Sy)$ and $y \in \mathrm{cl}(Sz)$. By the no-return condition we have that $y \in J$ which is a contradiction. □

Corollary 4 *Suppose J satisfies the no-return condition. Then*

$$J = \bigcup_{x \in J} [x]$$

Proof: Take $x \in J$. Then by the Proposition 3 we have $x \in [x] \subset J$. Thus $\bigcup_{x \in J}[x] \subset J$. Conversely, since $x \in [x]$ we have $J \subset \bigcup_{x \in J}[x]$. □

Let $\mathcal{F} = \{[x] \subset J : [y] \subset J \text{ if } y \in M \text{ and } [y] \preceq [x]\}$. As a converse of the last corollary we have.

Proposition 4 *Assume $\mathcal{F} \neq \emptyset$ and suppose that $J = \bigcup_{[x] \in \mathcal{F}} [x]$. Then J satisfies the no-return condition.*

Proof: Take $x \in J$ and $z \in \mathrm{cl}(Sx)$ with $\mathrm{cl}(Sz) \cap J \neq \emptyset$. Pick $w \in \mathrm{cl}(Sz) \cap J$. Then $x \preceq z \preceq w$. It follows that $[x] \preceq [z] \preceq [w]$. Since $w \in J$ and $w \in [w] \in \mathcal{F}$ we have $[z] \subset J$ and $z \in J$. □

The next theorem gives conditions for the existence of SJ-invariant classes. It also generalizes Proposition 3.3.3 of [4].

Theorem 1 *Let J be a subset of M satisfying the no-return condition. Take $x \in J$ and assume that there exists a compact set $K \subset J$ such that for all $y \in \mathrm{cl}(Sx) \cap J$*

$$\mathrm{cl}(Sy) \cap K \neq \emptyset$$

Then there exists a SJ-invariant class $[w] \subset \mathrm{cl}(Sx)$.

Proof: For $y \in \mathrm{cl}(Sx) \cap J$ we define the compact $K_y = \mathrm{cl}(Sy) \cap K$. Since $1 \in S$ we have that K_x is defined. Now, consider the family

$$\mathcal{F} = \{K_y : y \in K_x\}$$

define the following order on $\mathcal{F}$

$$K_y \preceq K_z \text{ if } y \preceq z$$

thus if $K_y \preceq K_z$ then $z \in \mathrm{cl}(Sy)$ and $K_z = \mathrm{cl}(Sz) \cap K \subset \mathrm{cl}(Sy) \cap K = K_y$. Therefore every linearly ordered set $\{K_{y_i} : i \in I\}$ has an upper bound $K_y = \bigcap_{i\in I} K_{y_i}$ for some $y \in \bigcap_{i\in I} K_{y_i}$. The Zorn's lemma implies that the family $\mathcal{F}$ has a maximal element K_y. Since $y \in Sy \cap K$ we have $y \in K_y \subset J$. Let's define

$$D = \mathrm{cl}(Sy) \cap J.$$

We will show that $D \subset \mathrm{cl}(Sx)$ is a SJ-invariant class. We know that $y \in Sy \cap J \subset D$. By its only definition, every $z \in D$ is approximately reachable from y. Conversely, $y \in \mathrm{cl}(Sz)$, since otherwise $y \notin \mathrm{cl}(Sz) \cap K = K_z$, hence this is a proper subset of K_y contradicting the maximality of K_y. Therefore we have approximate transitivity in D. We also have that D is a class. Otherwise, there exists a class $[w] \supset D$ containing a point $\omega \notin D = \mathrm{cl}(Sy) \cap J$. The no-return condition and Proposition 3 implies that $D \subset [w] \subset J$. It follows that $w \sim y$ and $w \in \mathrm{cl}(Sy) \cap J$ contradicting the choice of w. It remains to show the SJ-invariance of D. Take $z \in D = \mathrm{cl}(Sy) \cap J$ and suppose that there exists $k \in \mathrm{cl}(Sz) \cap J$. Then $k \in \mathrm{cl}(Sz) \subset \mathrm{cl}(Sy)$ and $k \in \mathrm{cl}(Sy) \cap J$, showing the SJ-invariance of D. □

The theorem above allows us to show that the J-maximal and JS-invariant classes coincide.

Corollary 5 *Let $J \subset M$ be a subset satisfying the no-return condition. Take $x \in J$ and assume that there exists a compact set $K \subset J$ such that for all $y \in \mathrm{cl}(Sx) \cap J$*

$$\mathrm{cl}(Sy) \cap K \neq \emptyset$$

Then for a class $[w] \subset \mathrm{cl}(Sx) \cap J$ we have that $[w]$ is J-maximal if and only if $[w]$ is SJ-invariant.

Proof: Suppose $[w]$ is J-maximal. Assume that there exists $y \in [w] \subset \mathrm{cl}(Sx)$ and $z \in \mathrm{cl}(Sy) \cap J$ with $z \notin [w]$. By the last proposition there exist a SJ-invariant class $[w_1] \subset \mathrm{cl}(Sy) \subset \mathrm{cl}(Sx)$. Thus $[w] \preceq [w_1]$ and $[w] \neq [w_1]$. This contradicts the J-maximality of $[w]$. Hence $[w]$ is SJ-invariant. □

We will see next that a closed subset is a maximal class if and only if it is the closure of the orbit of its elements.

Proposition 5 *Let C be a closed subset of M. Then C is maximal class if and only if $C = \mathrm{cl}(Sx)$ for every $x \in C$.*

Proof: Suppose $C = \mathrm{cl}(Sx)$ for every $x \in C$. Then for $x, y \in C$ we have $y \in \mathrm{cl}(Sx)$ and $x \in \mathrm{cl}(Sy)$, i.e., $x \sim y$ and therefore C is contained in a equivalence class, say, $[x] \supset C$. Furthermore, for $y \in [x]$ we have $y \in \mathrm{cl}(Sx)$, and therefore $[x] \subset \mathrm{cl}(Sx) = C$, showing that $[x] = C$. Now, suppose that $C = [x] \preceq [y]$. Then $x \preceq y$ and $y \in \mathrm{cl}(Sx) = [x]$. Therefore $C = [x] = [y]$ is a maximal class. Conversely, assume that $C = [x]$ is a maximal class. Thus by the Lemma 1 we have $C = [x] = \mathrm{cl}(Sx)$ for every $x \in C$. □

Now, we give conditions for the existence of a unique SJ-invariant class.

Proposition 6 *Let $J \subset M$ be a subset satisfying the no-return condition. Take $x \in J$ and assume that there exists a compact set $K_x \subset J$ which is S-invariant and such that for all $y \in \mathrm{cl}(Sx) \cap J$*

$$\mathrm{cl}(Sy) \cap K_x \neq \emptyset$$

Suppose that

$$C = \bigcap_{x \in J} \bigcap_{y \in \mathrm{cl}(Sx) \cap J} (\mathrm{cl}(Sy) \cap K_x) \neq \emptyset$$

Then C is a unique SJ-invariant class contained in J.

Proof: First, we show that C is a SJ-invariant class. It is easy to see that C is closed and it is contained in J. By the Proposition 5 and Corollary 5 it is enough to show that $C = \mathrm{cl}(Sx)$ for every $x \in C$. Take $x \in J$ and $y \in \mathrm{cl}(Sx) \cap J$. Then $\mathrm{cl}(Sy) \subset \mathrm{cl}(Sx)$ and $\mathrm{cl}(Sx) \cap K_x \subset \mathrm{cl}(Sx)$. Therefore $C \subset \bigcap_{x \in J} (\mathrm{cl}(Sx)) \subset \mathrm{cl}(Sx)$ for every $x \in C$. Now take $w \in \mathrm{cl}(Sz)$ with $z \in C$. Then for every $x \in J$ and every $y \in \mathrm{cl}(Sx) \cap J$ we have $z \in \mathrm{cl}(Sy)$ $\cap K_x$. Since K_x is S-invariant we have $w \in \mathrm{cl}(Sz) \subset \mathrm{cl}(Sy) \cap K_x$. Thus $w \in C$. It remains to show the uniqueness of C. Suppose that $C_1 \subset J$ is a maximal class. By the Proposition 5, $\mathrm{cl}(Sx) = C_1$ for every $x \in C_1$.Thus $C_1 = \bigcap_{x \in C_1} \mathrm{cl}(Sx)$. Therefore

$$C = \bigcap_{x \in J} \bigcap_{y \in \mathrm{cl}(Sx) \cap J} (\mathrm{cl}(Sy) \cap K_x) \subset \bigcap_{x \in J} (\mathrm{cl}(Sx)) \subset \bigcap_{x \in C_1} \mathrm{cl}(Sx) = C_1.$$

By the maximality of C it follows that $C = C_1$. □

As the invariant control sets are closed in M under the hypothesis of accessibility of S, we show that the SJ-invariant control sets are relatively closed control sets in J.

We say that a subset $J \subset M$ satisfies the J-*accessibility* condition for a submonoid S if for all $y \in J$, $\mathrm{int}_M(Sy \cap J) \neq \emptyset$.

Theorem 2 *Suppose that $J \subset M$ satisfy the J-accessibility condition for S. Then any SJ-invariant class $[w] \subset J$ is relatively closed in J, i.e., $(\partial([w]) - [w]) \cap J = \emptyset$.*

Proof: Assume that $[w]$ is a SJ-invariant class and $(\partial([w]) - [w]) \cap J \neq \emptyset$. Pick $y \in (\partial([w]) - [w]) \cap J$. We have that $Sy \cap [w] = \emptyset$. Otherwise, there exist $g \in S$ such that $gy \in [w]$. Thus $[w] \subset \mathrm{cl}(Sgy) \subset \mathrm{cl}(Sy)$ and we would have $y \in [w]$, cause if $y \in \partial([w])$ is not in $[w]$ then $\mathrm{cl}(Sy)$ do not contains $[w]$. Since $y \in J$ the J-accessibility condition guarantees the existence of $z \in (\mathrm{int}(Sy \cap J)) - [w]$. There exists $g \in S$ such that $gy = z$. Let V be a neighborhood of z contained in $Sy \cap J$ and outside $[w]$. Now, $g^{-1}V$ is an open neighborhood of $y \in \partial([w])$. Therefore there exists $x \in [w]$ such that $gx \in V$, which is a contradiction with the SJ-invariance of $[w]$. □

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