Existence and uniqueness of strong solutions of flows of asymmetric fluids in unbounded domain

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Abstract

We consider and initial boundary value problem for a system of equations describing nonstationary flows of incompressible asymmetric fluids in unbounded domains. Under conditions similar to the ones for the usual Navier-Stokes equations, we prove the existence and uniqueness of strong solutions.

Resumo

Consideramos um problema de valor inicial e de contorno para um sistema de equações que descrevem o fluxo dos fluidos assimétricos incompressíveis em domínios não limitados. Sob condições similares às equações de Navier-Stokes usuais, provamos a existência e unicidade de soluções fortes.

1. Introduction

Let Ω be a bounded or unbounded domain in $I\!R^3$, $T > 0$ and $Q_T = \Omega \times [0, T]$. The equations that describe the motion of asymmetric fluids are given by

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$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mu+\mu_r)\Delta\mathbf{u} + \nabla p = 2\mu_r \text{ rot } \mathbf{w} + \mathbf{f}, \\
\text{div } \mathbf{u} = 0\,, \\
\dfrac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla \text{ div } \mathbf{w} \\
\qquad +4\mu_r\mathbf{w} = 2\mu_r \text{ rot } \mathbf{u} + \mathbf{g}\,.
\end{cases} \tag{1.1}
$$

together with the following boundary and initial conditions

$$
\begin{cases}
\mathbf{u} = 0 \quad \text{on} \quad S_T = \partial\Omega \times (0,T)\,, \\
\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in} \quad \Omega\,, \\
\mathbf{w} = 0 \quad \text{on} \quad S_T = \partial\Omega \times (0,T)\,, \\
\mathbf{w}(x,0) = \mathbf{w}_0(x) \quad \text{in} \quad \Omega\,.
\end{cases} \tag{1.2}
$$

The functions $\mathbf{u} = (u_1, u_2, u_3), \mathbf{w} = (w_1, w_2, w_3)$ and p denote the velocity vector, the angular velocity vector of rotation of particles, the pressure of the fluid, respectively. The functions $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ denote external sources of linear and angular momentum, respectively. The positive constants μ, μ_r, c_0, c_a and c_d are viscosities. We consider $c_0 + c_d > c_a$.

For the derivation and physical discussion of equations (1.1)-(1.2) see Petrosyan [12], Condiff and Dalher [1], Eringen [4], [5] and Lukaszewicz [11]. We observe that this model of fluid include as a particular case the classical Navier-Stokes equations, which has been greatly studied (see, for instance, the classical books by Ladyzhenskaya [6], Temam [21] and the references there in). For Newtonian fluids, equations (1.1) and (1.2) decouple since $\mu_r = 0$.

It is appropriate to cite some earlier works on the initial-value problem (1.1)-(1.2) which are related to ours and also located our contribution there in. When Ω is a bounded domain, Lukaszewicz [9], [10] (see, also [11]) established the global existence of weak solutions and local strong solutions for (1.1)-(1.2) under certain assumptions by using linearization and an almost fixed point theorem.

By using the spectral Galerkin method Rojas-Medar and Boldrini [18] proved the global existence of weak solutions and the regularity of solution was studied by Ortega-Torres and Rojas-Medar [13]. More, strong solution was obtained by Rojas-Medar [16] (local), Ortega-Torres and Rojas-Medar [14] (global) by using the spectral Galerkin method. The convergence rates to this method were established in [17]. An interactive method was used in [15] to show the existence and uniqueness of strong solution.

When Ω is a exterior domain, the existence of weak solution for stationary model associated a (1.1)-(1.2) was studied in [2], the evolution case was done in [3].

However, no study of existence and uniqueness has been considered for system (1.1)-(1.2) in unbounded domains.

In work , we use an iterative process to prove the existence and uniqueness of strong solution.

The paper is organized as follows: in Section 2 we state some preliminaries results that will be useful in the rest of the paper; state the results of existence and uniqueness of strong solutions as also some apriori estimates that form the theorical basis for the problem. In Section 3 we study the linear problems associated a (1.1) and (1.2). In Section 4 we prove our result.

Finally, we would like to say that, as it usual in this context, to simplicity the notation in the expressions we will denote by c, C_0, M_0 generics finites positives constants depending only on Ω and the other fixed parameters of the problem (like the initial data) that may have different values in different expressions. In a few points to emphasize the fact that the constants are different we use $C_1, C_2, ..., M_1, M_2, \cdots$ and so on.

2. Preliminaries

We use the classical notations and results of the Sobolev spaces. For $k = 0, 1, 2, ...$ and $1 \leq p \leq \infty$,

$$W_p^k(\Omega) = \{\mathbf{u} \in L_p(\Omega) \ / \ \sum_{|\alpha| \leq k} \|D_x^\alpha \mathbf{u}\| < \infty\}$$

$$W_p^{2,1}(Q_T) = \{\mathbf{u} \in L_p(Q_T) \ / \ \|\mathbf{u}\|_{W_p^{2,1}(Q_T)} = \|\mathbf{u}_t\|_{L_p(Q_T)} + \sum_{|\alpha| \leq 2} \|D_x^\alpha \mathbf{u}\|_{L_p(Q_T)} < \infty\},$$

where $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}$.

It is know that the values of the function from $W_p^{2,1}(Q_T)$ on the hyperplane $t =$ const. belong for $\forall\ t \in [0, T]$ to the Slobodetskii-Besov space $W_p^{2-\frac{2}{p}}(\Omega)$ and depend continuously on t in the norm of $W_p^{2-\frac{2}{p}}(\Omega)$, defined by

$$\|\mathbf{u}\|_{W_p^{2-\frac{2}{p}}(\Omega)} = \left(\sum_{|\alpha| \leq 1} \|D_x^\alpha \mathbf{u}\|_{L_p(\Omega)}^p + \sum_{|\alpha| = 1} \int_\Omega \int_\Omega \frac{|\ D_x^\alpha \mathbf{u}(x) - D_y^\alpha \mathbf{u}(y)\ |^p}{|\ x - y\ |^{1+p}} dx dy \right)^{\frac{1}{p}}.$$

Moreover, we have the inequality

$$\|\mathbf{u}(\cdot,t)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \leq \|\mathbf{u}(\cdot,0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\mathbf{u}\|_{W_p^{2,1}(Q_T)},$$

where the constant $\widehat{c}$ does not depend on t.

For more details of the Slobodetskii-Besov space see [8]

Theorem 2.1. *Let $p > 3$. assume that*

$$\mathbf{u}_0(x) \in W^{2-\frac{2}{p}}(\Omega), \mathbf{u}_0|_{S_T} = 0, \ div\ \mathbf{u}_0 = 0,$$

$$\mathbf{w}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega), \mathbf{w}_0|_{S_T} = 0,$$

$$\mathbf{f}, \mathbf{g} \in L_p(Q_T).$$

Then there exists $T_1 \in (0,T]$ such that problem (1.1)-(1.2) has a unique solution $(\mathbf{u}, \mathbf{w}, p)$ which satisfies

$$\mathbf{u} \in W_p^{2,1}(Q_{T_1}),$$

$$\nabla p \in L_p(Q_{T_1})$$

$$\mathbf{w} \in W_p^{2,1}(Q_{T_1}).$$

3. Linear problems

In this section, we study some linear problems associated with (1.1)-(1.2). The first Lemma is proved in Solonnikov [20]

Lemma 3.1. *Let $F(x,t) \in L_p(Q_T)$ and $\mathbf{u}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$ with $\mathbf{u}_0|_{S_T} = 0$ and $div\ \mathbf{u}_0 = 0$, then the following problem*

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - (\mu + \mu_r)\triangle \mathbf{u} + \nabla p &= F, \\ div\ \mathbf{u} &= 0, \\ \mathbf{u}|_{S_T} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0(x) \end{aligned}$$

has a unique solution $\mathbf{u} \in W_p^{2,1}(Q_T)$, satisfying

$$\|\mathbf{u}\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} \leq K_1(T)(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|F\|_{L_p(Q_T)}),$$

where $K_1(\cdot)$ is an increasing function of T.

The following result is a special case of the result for parabolic system given in [19].

Lemma 3.2. *Let $G(x,t) \in L_p(Q_T)$ and $\mathbf{w}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$ with $\mathbf{w}_0|_{S_T} = 0$, then the following problem*

$$\frac{\partial \mathbf{w}}{\partial t} - (c_a + c_d)\triangle \mathbf{w} - (c_0 + c_d - c_a)\nabla\ div\ \mathbf{w} + 4\mu_r \mathbf{w} = G,$$

$$\begin{aligned} \mathbf{w}|_{S_T} &= 0, \\ \mathbf{w}(0) &= \mathbf{w}_0(x) \end{aligned}$$

has a unique solution $\mathbf{w} \in W_p^{2,1}(Q_T)$, satisfying

$$\|\mathbf{w}\|_{W_p^{2,1}(Q_T)} \leq K_2(T)(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|G\|_{L_p(Q_T)}),$$

where $K_2(\cdot)$ is an increasing function of T.

4. Auxiliar result

We construct approximate solution inductively

$$\mathbf{u}^{(0)} = \mathbf{0},\ \mathbf{w}^{(0)} = \mathbf{0}$$

and for $k = 1, 2, 3, ..., \{\mathbf{u}^{(k)}, p^{(k)}\}$ and $\{\mathbf{w}^{(k)}\}$ are respectively, the solutions of problems

$$\begin{aligned} \frac{\partial \mathbf{u}^{(k)}}{\partial t} - (\mu + \mu_r)\triangle \mathbf{u}^{(k)} + \nabla p^{(k)} &= \mathbf{f} + 2\mu_r \text{ rot } \mathbf{w}^{(k-1)} - (\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}, \\ \text{div } \mathbf{u}^{(k)} &= 0, \\ \mathbf{u}^{(k)}|_{S_T} &= 0, \\ \mathbf{u}^{(k)}(0) &= \mathbf{u}_0(x) \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial \mathbf{w}^{(k)}}{\partial t} - (c_a + c_d)\triangle \mathbf{w}^{(k)} - (c_0 + c_d - c_a)\nabla \text{ div } \mathbf{w}^{(k)} + 4\mu_r \mathbf{w}^{(k)} \\ = \ & \mathbf{g} + 2\mu_r \text{ rot } \mathbf{u}^{(k-1)} - (\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{w}^{(k-1)} \end{aligned}$$

$$\begin{aligned}
\mathbf{w}^{(k)}|_{S_T} &= 0, \\
\mathbf{w}^{(k)}(0) &= \mathbf{w}_0(x).
\end{aligned}$$

Now, we prove the boundeness of above sequence.

Lemma 4.1. *For sufficiently small $T_1 \in (0, T]$, the sequence $\{\mathbf{u}^{(k)}, p^{(k)}, \mathbf{w}^{(k)}\}$ is bounded in $W_p^{2,1}(Q_{T_1}) \times L_p(Q_T) \times W_p^{2,1}(Q_{T_1})$.*

Proof. Let

$$\Phi^{(k)}(T) = \|\mathbf{u}^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\mathbf{w}^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\nabla p^{(k)}\|_{L_p(Q_T)}.$$

From Lemmas (3.1)-(3.2) imply

$$\begin{aligned}
\Phi^{(k)}(T) \leq\ & K_1(T)(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\mathbf{f}\|_{L_p(Q_T)} + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}\|_{L_p(Q_T)} \\
& + \|2\mu_r \text{ rot } \mathbf{w}^{(k-1)}\|_{L_p(Q_T)}) \\
& + K_2(T)(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\mathbf{g}\|_{L_p(Q_T)} + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{w}^{(k-1)}\|_{L_p(Q_T)} \\
& + \|2\mu_r \text{ rot } \mathbf{u}^{(k-1)}\|_{L_p(Q_T)}).
\end{aligned}$$

Now, we estimate the right-hand side of the above inequality.

The following estimate was obtained in

$$\|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}\|_{L_p(Q_T)} \leq C \left[\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^\delta \Phi^{(k-1)}(T)^2 \right]$$

with some positive constant δ and $C \geq 2$.

We will prove

$$\|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{w}^{(k-1)}\|_{L_p(Q_T)} \leq C[\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^\alpha \Phi^{(k-1)}(T)^2],$$

where $\alpha > 0$.

In fact, we have

$$\|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{w}^{(k-1)}\|^p_{L_p(Q_T)} \leq \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(Q_T)} \|\nabla \mathbf{w}^{(k-1)}\|^p_{L_p(Q_T)}.$$

We observe that

$$\begin{aligned}
\|\nabla \mathbf{w}^{(k-1)}(t)\|_{L_p(\Omega)} &\leq \|\mathbf{w}^{(k-1)}(t)\|_{W_p^1(\Omega)} \\
&\leq \|\mathbf{w}^{(k-1)}(t)\|^a_{W_p^2(\Omega)} \|\mathbf{w}^{(k-1)}(t)\|^{(1-a)}_{L_\infty(\Omega)},
\end{aligned}$$

where $a=\frac{p-3}{2p-3}$.

By other hand, see Ladyzhenskaya and Solonnikov [7], we have

$$\|\mathbf{w}^{(k-1)}\|_{L_\infty(Q_T)}\le c(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T)) \tag{4.1}$$

and

$$\|\mathbf{u}^{(k-1)}\|_{L_\infty(Q_T)}\le c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T)). \tag{4.2}$$

Consequently,

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|^p_{L_p(Q_T)} &\le \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(Q_T)}\int_0^T\|\nabla\mathbf{w}^{(k-1)}(t)\|^p_{L_p(\Omega)}dt \qquad (4.3)\\ &\le \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(Q_T)}\|\mathbf{w}^{(k-1)}\|^{(1-a)p}_{L_\infty(Q_T)}\int_0^T\|\mathbf{w}^{(k-1)}(t)\|^{pa}_{W_p^2(\Omega)}dt.\end{aligned}$$

But,

$$\int_0^T\|\mathbf{w}^{(k-1)}(t)\|^{ap}_{W_p^2(\Omega)}dt\le\left(\int_0^T 1^s dt\right)^{\frac{1}{s}}\left(\int_0^T\|\mathbf{w}^{(k-1)}(t)\|^{apr}_{W_p^2(\Omega)}dt\right)^{\frac{1}{r}}$$

since $a<1$, we take $\frac{1}{s}=1-a$, $\frac{1}{r}=a$ then $\frac{1}{r}+\frac{1}{s}=1$ and thus in the last inequality, we have

$$\begin{aligned}\int_0^T\|\mathbf{w}^{(k-1)}(t)\|^{ap}_{W_p^2(\Omega)}dt &\le T^{1-a}\left(\int_0^T\|\mathbf{w}^{(k-1)}(t)\|^{p}_{W_p^2(\Omega)}dt\right)^a\\ &\le T^{1-a}\|\mathbf{w}^{(k-1)}\|^{ap}_{W_p^{2,1}(Q_T)} \qquad (4.4)\\ &\le T^{1-a}\left(\Phi^{(k-1)}\right)^{ap}.\end{aligned}$$

The inequalities (4.1), (4.2), (4.3) and (4.4) imply

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_p(Q_T)} &\le T^{\frac{1-a}{p}}c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))\\ &\quad\times(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))^{1-a}\times\left(\Phi^{(k-1)}\right)^a\end{aligned}$$

We observe that

$$\begin{aligned}&(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))^{1-a}\times T^{\frac{1-a}{p}}\left(\Phi^{(k-1)}\right)^a\\ \le\ & 2^{-a}(\|\mathbf{w}_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})(1-a)}(\Phi^{(k-1)}(T))^{1-a})\times T^{\frac{1-a}{p}}\left(\Phi^{(k-1)}(T)\right)^a\\ =\ & 2^{-a}(\|\mathbf{w}_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a)+T^{\delta_1}\Phi^{(k-1)}(T)),\end{aligned}$$

where $\delta_1 = (1-\frac{1}{p})(1-\frac{3}{p})(1-a) + \frac{1-a}{p}$.

Also,

$$\|\mathbf{w}_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a) \leq \frac{\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}}{\frac{1}{1-a}} + \frac{T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T)}{\frac{1}{r}},$$

where we use the inequality $x^{\frac{1}{r}}y^{\frac{1}{s}} \leq \frac{x}{r} + \frac{y}{s}$, $\frac{1}{s}+\frac{1}{s}=1$, consequently

$$\begin{aligned}\|\mathbf{w}_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a) &\leq (1-a)\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + aT^{\frac{1-a}{ap}}\Phi^{(k-1)}(T)\\ &\leq \|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + aT^{\frac{1-a}{ap}}\Phi^{(k-1)}(T).\end{aligned}$$

Consequently

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_p(Q_T)} &\leq c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))\\ &\quad\times 2^{-a}(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + aT^{\frac{1-a}{ap}}\Phi^{(k-1)}(T)) \times T^{\delta_1}\Phi^{(k-1)}(T)\\ &\leq c2^{-a-1}(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\mathbf{w}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} +\\ &\quad \left(T^{2(1-\frac{1}{p})(1-\frac{3}{p})} + (T^{\frac{1-a}{ap}} + T^{\delta_1})^2\right)\left(\Phi^{(k-1)}(T)\right)^2.\end{aligned}$$

Setting $T^\alpha = T^{2(1-\frac{1}{p})(1-\frac{3}{p})} + (T^{\frac{1-a}{ap}} + T^{\delta_1})^2$, we have

$$\|(\mathbf{u}^{(k-1)}\cdot\nabla)\mathbf{w}^{(k-1)}\|_{L_p(Q_T)} \leq c2^{-a-1}(\left[\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\mathbf{w}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + T^\alpha\left(\Phi^{(k-1)}(T)\right)^2\right]$$

Also, we observe that

$$\begin{aligned}\|\nabla\mathbf{w}^{(k-1)}(t)\|_{L_p(\Omega)} &\leq \|\mathbf{w}^{(k-1)}(t)\|_{W_p^1(\Omega)}\\ &\leq \|\mathbf{w}^{(k-1)}(t)\|_{W_p^{2-\frac{2}{p}}(\Omega)}\end{aligned}$$

follows that

$$\begin{aligned}\sup_{0\leq t\leq T}\|2\mu_r \text{ rot } \mathbf{w}^{(k-1)}(t)\|_{L_p(\Omega)} &\leq c\sup_{0\leq t\leq T}\|\mathbf{w}^{(k-1)}(t)\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\ &\leq c\|\mathbf{w}^{(k-1)}(t)\|_{W_p^{2,1}(Q_t)} + \|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\ &\leq c\Phi^{(k-1)}(T) + \|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}.\end{aligned}$$

Analogously,

$$\begin{aligned}
\sup_{0\le t\le T}\|2\mu_r \text{ rot } \mathbf{u}^{(k-1)}(t)\|_{L_p(\Omega)} &\le c\sup_{0\le t\le T}\|\mathbf{u}^{(k-1)}(t)\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\le c\|\mathbf{u}^{(k-1)}(t)\|_{W_p^{2,1}(Q_t)}+\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\le c\Phi^{(k-1)}(T)+\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}.
\end{aligned}$$

The above estimates imply the following inequality

$$\begin{aligned}
\Phi^{(k)}(T) &\le K_1(T)(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+\|\mathbf{f}\|_{L_p(Q_T)}+(C\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^\delta\Phi^{(k-1)}(T)^2)+\\
&\quad c\Phi^{(k-1)}(T)+\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)})+K_2(T)(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+\|\mathbf{g}\|_{L_p(Q_T)}\\
&\quad +c2^{-a-1}(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)}+\|\mathbf{w}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)}+T^\alpha\left(\Phi^{(k-1)}(T)\right)^2)\\
&\quad +c\Phi^{(k-1)}(T)+\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)})\\
&\le K(T)(C+CT^\gamma\Phi^{(k-1)}(T)^2+C\Phi^{(k-1)}(T),
\end{aligned}$$

where $K(T)=\max(K_1(T),K_2(T))$ and $\gamma=\min(\alpha,\delta)$.

5. Proof of the Theorem

Setting $\mathbf{u}^{(n,s)}(t)=\mathbf{u}^{(n+s)}(t)-\mathbf{u}^{(n)}(t), p^{(n,s)}=p^{(n+s)}-p^{(n)}$ and $\mathbf{w}^{(n,s)}=\mathbf{w}^{(n+s)}-\mathbf{w}^{(n)}$, we have

$$\begin{aligned}
\frac{\partial\mathbf{u}^{(n,s)}}{\partial t}-(\mu+\mu_r)\triangle\mathbf{u}^{(n,s)}+\nabla p^{(n,s)} &= F^{(n,s)},\\
\text{div } \mathbf{u}^{(n,s)} &= 0,\\
\mathbf{u}^{(n,s)}|_{S_T} &= 0,\\
\mathbf{u}^{(n,s)}(0) &= 0,
\end{aligned}\tag{5.1}$$

where $F^{(n,s)}=2\mu_r \text{ rot } \mathbf{w}^{(n-1,s)}-(\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{(n+s-1)}-(\mathbf{u}^{(n-1)}\cdot\nabla)\mathbf{u}^{(n-1,s)}$ and

$$\frac{\partial\mathbf{w}^{(n,s)}}{\partial t}-(c_a+c_d)\triangle\mathbf{w}^{(n,s)}-(c_0+c_d-c_a)\nabla\text{ div }\mathbf{w}^{(n,s)}+4\mu_r\mathbf{w}^{(n,s)}=G^{(n,s)}$$

$$\begin{aligned}
\mathbf{w}^{(n,s)}|_{S_T} &= 0,\\
\mathbf{w}^{(n,s)}(0) &= 0,
\end{aligned}\tag{5.2}$$

where $G^{(n,s)} = 2\mu_r \text{ rot } \mathbf{u}^{(n-1,s)} - (\mathbf{u}^{(n+s-1)} \cdot \nabla)\mathbf{w}^{(n-1,s)} - (\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{w}^{(n-1)}$.

Let

$$\Psi^{(n,s)}(t) = \|\mathbf{u}^{(n,s)}\|_{W_p^{2,1}(Q_t)} + \|\mathbf{w}^{(n,s)}\|_{W_p^{2,1}(Q_t)} + \|\nabla p^{(n,s)}\|_{L_p(Q_t)}.$$

Then, it follows that for $t \in (0, T_1]$,

$$\|F^{(n,s)}\|^p_{L_p(Q_t)} \leq c(\|\nabla \mathbf{w}^{(n-1,s)}\|^p_{L_p(Q_t)} + \|\mathbf{u}^{(n-1,s)} \cdot \nabla \mathbf{u}^{(n+s-1)}\|^p_{L_p(Q_t)} + \|(\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,s)}\|^p_{L_p(Q_t)}.$$

By other hand

$$\begin{aligned}
\|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{(n-1+s)}\|^p_{L_p(Q_t)} &\leq \int_0^t \|\nabla \mathbf{u}^{(n-1+s)}(\tau)\|^p_{L_p(\Omega)} \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\leq \sup_{0 \leq s \leq t} \|\nabla \mathbf{u}^{(n-1+s)}(\tau)\|^p_{L_p(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\leq \sup_{0 \leq s \leq t} \|\mathbf{u}^{(n-1+s)}(\tau)\|^p_{W_p^1(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^1(\Omega)} d\tau \\
&\leq \sup_{0 \leq s \leq t} \|\mathbf{u}^{(n-1+s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\leq (\|\mathbf{u}^{(n-1+s)}(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + \widehat{c}\|\mathbf{u}^{(n-1+s)}(\tau)\|_{W_p^{2,1}(Q_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau
\end{aligned}$$

and

$$\begin{aligned}
\|(\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,s)}\|^p_{L_p(Q_t)} &\leq \int_0^t d\tau \int_\Omega |\mathbf{u}^{(n-1)}|^p |\nabla \mathbf{u}^{(n-1,s)}|^p dx \\
&\leq \|\mathbf{u}^{(n-1)}\|^p_{L_\infty(Q_t)} \int_0^t \|\nabla \mathbf{u}^{(n-1,s)}\|^p_{L_p(\Omega)} d\tau \\
&\leq \sup_{0 \leq \tau \leq t} \|\mathbf{u}^{(n-1)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^1(\Omega)} d\tau \\
&\leq \sup_{0 \leq \tau \leq t} \|\mathbf{u}^{(n-1)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\leq (\|\mathbf{u}^{(n-1)}(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + \widehat{c}\|\mathbf{u}^{(n-1)}\|_{W_p^{2,1}(Q_t)})^p \widehat{c}^p \int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau
\end{aligned}$$

and

$$
\begin{aligned}
\|\nabla \mathbf{w}^{(n-1,s)}\|^p_{L_p(Q_t)} &\le \int_0^t \|\nabla \mathbf{w}^{(n-1,s)}\|^p_{L_p(\Omega)} d\tau \\
&\le \int_0^t \|\mathbf{w}^{(n-1,s)}\|^p_{W^1_p(\Omega)} d\tau \\
&\le \int_0^t \|\mathbf{w}^{(n-1,s)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\le \widehat{c}^p \int_0^t \|\mathbf{w}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau,
\end{aligned}
$$

consequently

$$
\begin{aligned}
\|F^{(n,s)}\|^p_{L_p(Q_t)} \le\; & c\int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau + (\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
& + \widehat{c}\|\mathbf{u}^{(n-1+s)}(\tau)\|_{W_p^{2,1}(Q_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau \\
& + (\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \qquad\qquad\qquad\qquad\qquad\qquad (5.3) \\
& + \widehat{c}\|\mathbf{u}^{(n-1+s)}(\tau)\|_{W_p^{2,1}(Q_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\|(\mathbf{u}^{(n+s-1)}\cdot\nabla)\mathbf{w}^{(n-1,s)}\|^p_{L_p(Q_t)} &\le \int_0^t d\tau \int_\Omega |\mathbf{u}^{(n+s-1)}|^p |\nabla \mathbf{w}^{(n-1,s)}|^p dx \\
&\le \|\mathbf{u}^{(n+s-1)}\|^p_{L_\infty(Q_t)} \int_0^t \|\nabla \mathbf{w}^{(n-1,s)}\|^p_{L_p(\Omega)} d\tau \\
&\le \sup_{0\le\tau\le t} \|\mathbf{u}^{(n+s-1)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\mathbf{w}^{(n-1,s)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\le (\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + \widehat{c}\|\mathbf{u}^{(n+s-1)}\|_{W_p^{2,1}(Q_t)})^p \widehat{c}^p \int_0^t \|\mathbf{w}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)} d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{w}^{(n-1)}\|^p_{L_p(Q_t)} &\le \int_0^t \|\nabla \mathbf{w}^{(n-1)}(\tau)\|^p_{L_p(\Omega)} \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\le \sup_{0\le s\le t} \|\nabla \mathbf{w}^{(n-1)}(\tau)\|^p_{L_p(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{0\le s\le t}\|\mathbf{w}^{(n-1)}(\tau)\|^p_{W^1_p(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W^1_p(\Omega)}d\tau\\
&\le \sup_{0\le s\le t}\|\mathbf{w}^{(n-1)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}\int_0^t\|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}d\tau\\
&\le (\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\quad+\hat{c}\|\mathbf{w}^{(n-1)}(\tau)\|_{W_p^{2,1}(Q_t)})^p\int_0^t\hat{c}^p\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)}d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla\mathbf{u}^{(n-1,s)}\|^p_{L_p(Q_t)} &\le \int_0^t\|\nabla\mathbf{u}^{(n-1,s)}\|^p_{L_p(\Omega)}d\tau\\
&\le \int_0^t\|\mathbf{u}^{(n-1,s)}\|^p_{W^1_p(\Omega)}d\tau\\
&\le \int_0^t\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}d\tau\\
&\le \hat{c}^p\int_0^t\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)}d\tau,
\end{aligned}
$$

Then, from , it follows that for $t\in(0,T_1]$,

$$
\begin{aligned}
\|G^{(n,s)}\|^p_{L_p(Q_t)} &\le c(\|\nabla\mathbf{u}^{(n-1,s)}\|^p_{L_p(Q_t)}+\|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{w}^{(n-1)}\|^p_{L_p(Q_t)}\\
&\quad+\|(\mathbf{u}^{(n+s-1)}\cdot\nabla)\mathbf{w}^{(n-1,s)}\|^p_{L_p(Q_t)}\\
&\le c\int_0^t\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)}d\tau+c(\|\mathbf{w}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\quad+\hat{c}\|\mathbf{w}^{(n-1)}(\tau)\|_{W_p^{2,1}(Q_t)})^p\int_0^t\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)}d\tau\\
&\quad+c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \qquad (5.4)\\
&\quad+\hat{c}\|\mathbf{u}^{(n+s-1)}\|_{W_p^{2,1}(Q_t)})^p\int_0^t\|\mathbf{w}^{(n-1,s)}\|^p_{W_p^{2,1}(Q_\tau)}d\tau.
\end{aligned}
$$

By using the estimates (5.1), (5.4) and together with Lemma 4.1, we have for $t\in[0,T_1]$ and $p>3$

$$
\Psi^{(n,s)}(t)\le c\left(\int_0^t\Psi^{(n-1,s)}(\tau)^p\right)^{\frac{1}{p}} \qquad (5.5)
$$

or

$$
\left[\Psi^{(n,s)}(t)\right]^p\le c^p\int_0^t\left[\Psi^{(n-1,s)}(\tau)\right]^p d\tau,
$$

consequently $\Psi^{(n,s)}(t) \to 0$ as $n \to \infty$, $\forall\ t \in [0, T_1]$. Firstly, we observe that $W^{2,1}_p(Q_t)$ is a Banach space and consequently, we have there exist $\mathbf{u}, \mathbf{w} \in W^{2,1}_p(Q_{T_1})$, such that

$$\begin{aligned} \mathbf{u}^n &\to \mathbf{u} \text{ strongly in } W^{2,1}_p(Q_{T_1}), \\ \mathbf{w}^n &\to \mathbf{w} \text{ strongly in } W^{2,1}_p(Q_{T_1}). \end{aligned}$$

Also, from of the completeness of $L_p(Q_{t_1})$, there exist $p \in L_p(Q_{T_1})$ such that

$$p^n \to p \text{ strongly in } L_p(Q_{T_1}).$$

Now, the next step is to take limit. But, once the above convergences have been established, this is a standard procedure to obtain that $\mathbf{u}, \mathbf{w}$,p is a strong solution of the problem (1.1)-(1.2).

We need only to argument the uniqueness of the solution in order to complete the proof of Theorem . Suppose that there exist another solution $\mathbf{u}_1, \mathbf{w}_1, p_1$ of (1.1) and (1.2) with the same regularity as stated in the Theorem. Define

$$U = \mathbf{u}_1 - \mathbf{u},\ W = \mathbf{w}_1 - \mathbf{w},\ P = p_1 - p.$$

These auxiliar functions verify a set of equations similar to (5.1)-(5.2). Repeat the argument used to obtain (5.5), we get for $\theta(t) = \|U\|^p_{W^{2,1}_p(Q_t)} + \|W\|^p_{W^{2,1}_p(Q_t)} + \|P\|^p_{L_p(Q_t)}$ an inequality of the following type

$$\theta(t) \leq c \int_0^t \theta(\tau) d\tau$$

which, by Gronwall's inequality, is equivalent to assert $U = 0, W = 0, P = 0$.

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