Strong periodic solutions for a class of abstract evolution equations

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Abstract. We consider a class of abstract evolution equations in a Hilbert space for which we prove existence and uniqueness of strong time periodic solutions. The result covers many models in hydrodynamics.

Key words: periodic solution, existence, uniqueness, hydrodynamics, Galerkin approximation

AMS subject classification: 35Q35

*Supported by the Polish Government grant KBN 2 P301 003 14, and FAPESP/Brazil, 2000/01569-7

†PhD. Student, IMECC-UNICAMP, Supported by CNPq-Brazil.

‡Supported by research grants: 300116/93, CNPq-BRAZIL and FAPESP-BRAZIL, 1997/3711-0.

1. Introduction.

We prove existence and uniqueness of strong periodic solutions for a class of abstract evolution equations of the form

$$\frac{d}{dt}U + \mathcal{L}U + B(U,U) + RU = F \tag{1.1}$$

in a Hilbert space $\mathcal{H}$. As an application we consider the system of equations of magneto-micropolar fluids. Our result covers several other models in hydrodynamics, including, e.g., Navier-Stokes equations with nonhomogenous boundary conditions, magnetohydrodynamics equations, and micropolar fluid equations.

Our assumptions are as follows.

$\mathcal{L}$ is a self-adjoint, strictly positive operator in $\mathcal{H}$ with domain $\mathcal{D}(\mathcal{L})$ and compact inverse.

B is a bilinear form in $\mathcal{H} \times \mathcal{H}$ such that

$$|\mathcal{B}(U,V,W)| \le c_1 |\mathcal{L}^\theta U| \cdot |\mathcal{L}^\rho U| \cdot |W| \tag{1.2}$$

for $U \in D(\mathcal{L}^\theta)$, $V \in D(\mathcal{L}^\rho)$, $W \in \mathcal{H}$, where $\mathcal{B}(U,V,W) = (B(U,V),W)$, $(\cdot,\cdot)$ and $|\cdot|$ denote scalar product and norm in $\mathcal{H}$, respectively,

$$\theta + \rho \ge \frac{5}{4}, \quad \rho > \frac{1}{2}, \quad \theta > 0, \tag{1.3}$$

and

$$(B(U,V),V) = 0, \quad U,V \in D(\mathcal{L}^{1/2}) \tag{1.4}$$

R is a linear operator in $\mathcal{H}$ such that

$$|\mathcal{R}(U,V)| \le c_2 |\mathcal{L}^{1/2} U| \cdot |V| \tag{1.5}$$

where $\mathcal{R}(U,V) = (RU,V)$, and

$$\mathcal{A}(U,U) + \lambda \mathcal{R}(U,U) \ge k_1 |\mathcal{L}^{1/2}U|^2, \quad \lambda \in [0,1] \tag{1.6}$$

where $\mathcal{A}(U,V) = (\mathcal{L}^{1/2}U, \mathcal{L}^{1/2}V), \quad U,V \in D(\mathcal{L}^{1/2}),\ k_1 > 0$. We prove the following

Theorem 1.1 *Let the assumptions (1.2)-(1.6) hold, and let $F \in C^1(\tau, \mathcal{H})$ for some $\tau > 0$. Then there exists a constant M such that if*

$$\sup_t |F(t)| \le M \tag{1.7}$$

then (1.1) has a unique strong τ-periodic solution

$$U \in H^2(\tau;\mathcal{H}) \cap H^1(\tau;D(\mathcal{L})) \cap L^\infty(\tau;D(\mathcal{L})) \cap W^{1,\infty}(\tau;D(\mathcal{L}^{1/2})) \quad \blacksquare \tag{1.8}$$

By $C^k(\tau,\mathcal{H})$, $k=0,1,2,...$, we denote the Banach space of $\mathcal{H}$-valued τ-periodic functions on R^1 with continuous derivatives up to order k, with the usual norm

$$||f||_{C^k(\tau,\mathcal{H})} = \sup\{\sum_{i=0}^{k} |D_t^i f(t)| : 0 \le t \le \tau\}$$

For a Hilbert space X, by $H^k(\tau,X)$, $k=0,1,2,...$, we denote the Hilbert space of X valued functions on R^1 that belong, together with their derivatives up to order k, to the space of τ-periodic measurable functions $L^2(\tau;X)$ with the norm $||f||_{L^2(\tau;X)} = (\int_0^\tau ||f(t)||_X^2 dt)^{1/2}$. Similarly, we define Banach spaces $L^\infty(\tau;X)$ and $W^{1,\infty}(\tau;X)$ with norms

$$||f||_{L^\infty(\tau;X)} = \sup \text{ess}\{||f(t)||_X : 0 \le t \le \tau\}$$

and

$$||f||_{W^{1,\infty}(\tau;X)} = (||f||^2_{L^\infty(\tau;X)} + ||D_t f||^2_{L^\infty(\tau;X)})^{1/2}$$

By C we denote various numeric constants.

The proof of theorem 1.1 is based on a set of estimates applied to the family $U_n \in C^1(\tau;\mathcal{H}_n)$, $n = 1,2,3,...$ of approximate solutions, where $\mathcal{H}_n = lin\{\omega_1,\omega_2,...,\omega_n\}$ and (ω_i) is a complete orthonormal system in $\mathcal{H}$ consisting of eigenfunctions of the operator $\mathcal{L}$, cf. [5].

In section 2 we prove the existence of solutions U_n. In section 3 we obtain a set of estimates of U_n that allows us to pass to the limit with n. In section 4 we prove the existence of a solution of equation (1.1) satisfying (1.8) by letting n to infinity as well as its uniqueness.

Section 5 presents an application of theorem 1.1 to the system of equations of magneto-micropolar fluids.

2. Approximate solutions and first order estimates.

Let

$$(U_t,\omega_i) + \mathcal{A}(U,\omega_i) + \mathcal{B}(U,U,\omega_i) + \mathcal{R}(U,\omega_i) = (F,\omega_i), \quad i=1,2,...,n \tag{2.1}$$

where $U(t) = U_n(t) = \sum_{i=1}^{n} c_{in}(t)\omega_i$, $U(t) = U(t+\tau)$, $n = 1, 2, 3, ...$ be the system of ODE defining approximate solutions of equation (1.1).

Lemma 2.1 *For each n there exists a solution $U \in C^1(\tau, \mathcal{H}_n)$ of the nonlinear problem (2.1).*

Proof. The linear problem

$$(U_t, \phi) + \mathcal{A}(U, \phi) = (F, \phi) - \mathcal{R}(V, \phi) - \mathcal{B}(V, V, \phi), \quad \phi \in \mathcal{H}_n, \tag{2.2}$$

$$U(t) = U(t+\tau)$$

has a unique solution (cf. [1], [2]) $U \in C^1(\tau, \mathcal{H}_n)$ for each $V \in C^0(\tau, \mathcal{H}_n)$. Consider the map $\Phi : V \to U$ in the space $C^0(\tau, \mathcal{H}_n)$. We shall show that Φ has a fixed point by using the Leray-Schauder theorem.

We prove that for every U and $\lambda \in [0, 1]$ satisfying $\lambda\Phi(U) = U$,

$$\sup_{0 \le t \le \tau} |U(t)| \le CM \quad \text{for some } C > 0 \tag{2.3}$$

where M is a constant in (1.7).

For $\lambda = 0$, $U = 0$. Let $\lambda > 0$ and assume $\lambda\Phi(U) = U$. Then, from (2.2)

$$(U_t, \phi) + \mathcal{A}(U, \phi) = \lambda(F, \phi) - \lambda\mathcal{R}(U, \phi) - \lambda\mathcal{B}(U, U, \phi)$$

With $\phi = U$, in view of (1.4), we obtain

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \mathcal{A}(U, U) + \lambda\mathcal{R}(U, U) = \lambda(F, U)$$

Denoting by μ the smallest eigenvalue of $\mathcal{L}$ we have

$$|\mathcal{L}^{\alpha}U| \le \mu^{\alpha-\beta}|\mathcal{L}^{\beta}U|, \quad 0 \le \alpha \le \beta, \tag{2.4}$$

whence $|(F, U)| \le C|F||\mathcal{L}^{1/2}U|$, and by (1.6) we obtain

$$\frac{d}{dt}|U|^2 + k_1|\mathcal{L}^{1/2}U|^2 \le CM^2 \tag{2.5}$$

Integrating in t and using the periodicity of U we have

$$\int_o^{\tau} |\mathcal{L}^{1/2}U(t)|^2 dt \le CM^2\tau,$$

whence, by the mean value theorem for integrals and (2.4), there exists $t^* \in [0, \tau]$ such that

$$|U(t^*)| \leq C|\mathcal{L}^{1/2}U(t^*)| \leq CM \tag{2.6}$$

Integrating again (2.5) from t^* to $t+\tau$, $t \in [0,\tau]$ we obtain (2.3). As the map Φ is continuous and compact in $C^0(\tau; \mathcal{H}_n)$ we conclude the existence of a fixed point U for Φ. Observe that (2.3) holds for this U. ■

Now we shall prove a fundamental estimate.

Lemma 2.2 *Let $U = U_n$ be the solution from (2.1). Then there exists a constant C independent of n and such that*

$$\sup_{t \in R^1} |\mathcal{L}^{1/2}U(t)| \leq CM^{1/2} \tag{2.7}$$

Proof. From (2.6) with $M < 1$ we have

$$|\mathcal{L}^{1/2}U(t^*)| < CM^{1/2}$$

Let $T^* = \sup\{T : |\mathcal{L}^{1/2}U(t)| \leq CM^{1/2} \text{ for } t \in [t^*, T)\}$. We shall show that $T^* = \infty$, whence (2.7) by periodicity. Let us assume to the contrary that $T^* < \infty$. Then

$$|\mathcal{L}^{1/2}U(T^*)| = CM^{1/2} \tag{2.8}$$

Our aim is to derive, using (2.8), the differential inequality

$$\frac{d}{dt}|\mathcal{L}^{1/2}U(T^*)|^2 \leq 0 \tag{2.9}$$

that shows the contradiction in view of the definition of T^*.

For all $\phi \in \mathcal{H}_n$ we have

$$(U_t, \phi) + \mathcal{A}(U, \phi) + \mathcal{B}(U, U, \phi) + \mathcal{R}(U, \phi) = (F, \phi) \tag{2.10}$$

Setting $\phi = \mathcal{L}U$ we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathcal{L}^{1/2}U|^2 + |\mathcal{L}U|^2 \leq |F||\mathcal{L}U| + |\mathcal{R}(U, \mathcal{L}U)| + |\mathcal{B}(U, U, \mathcal{L}U)|$$

where we have used $(\mathcal{L}U, \phi) = \mathcal{A}(U, \phi)$, $U \in D(\mathcal{L})$, $\phi \in \mathcal{H}$. Now by (1.2) and (1.5),

$$\frac{1}{2}\frac{d}{dt}|\mathcal{L}^{1/2}U|^2 + |\mathcal{L}U|^2 \leq |F||\mathcal{L}U| + c_2|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| + c_1|\mathcal{L}^{1/4}U| \cdot |\mathcal{L}U|^2 \tag{2.11}$$

Let us consider this inequality at point $t = T^*$. Then, by (2.3), (2.8), and the interpolation inequality

(2.12)
$$|\mathcal{L}^\sigma U| \le c_2 |\mathcal{L}^\alpha U|^\gamma \cdot |\mathcal{L}^\beta U|^{1-\gamma}, \qquad U \in D(\mathcal{L}^\beta)$$

where $\sigma = \alpha\lambda + \beta(1-\lambda)$, $0 \le \alpha \le \sigma \le \beta$, $\lambda \ge 0$, we have

(2.13)
$$|\mathcal{L}^{1/4}U| \cdot |\mathcal{L}U|^2 \le C|U|^{1/2} \cdot |\mathcal{L}^{1/2}U|^{1/2} \cdot |\mathcal{L}U|^2 \le CM^{1/2}M^{1/4}|\mathcal{L}U|^2$$

(2.14)
$$\begin{aligned} |\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| &\le \varepsilon|\mathcal{L}U|^2 + c_\varepsilon|\mathcal{L}^{1/2}U|^2 \\ &\le \varepsilon|\mathcal{L}U|^2 + c_\varepsilon|U| \cdot |\mathcal{L}U| \\ &\le \varepsilon|\mathcal{L}U|^2 + c_\varepsilon M \cdot |\mathcal{L}U| \\ &\le \varepsilon|\mathcal{L}U|^2 + c_\varepsilon M^{1/2} \cdot |\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| \\ &\le \varepsilon|\mathcal{L}U|^2 + c_\varepsilon M^{1/2}|\mathcal{L}U|^2 \end{aligned}$$

and, by (1.7),

(2.15)
$$|F||U| \le CM|\mathcal{L}U| = CM^{1/2}|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| \le CM^{1/2}|\mathcal{L}U|^2$$

Thus, from (2.11), together with (2.13)-(2.15), we get

$$\frac{1}{2}\frac{d}{dt}|\mathcal{L}^{1/2}U|^2 + (1 - C(M))|\mathcal{L}U|^2 \le 0$$

where $C(M) \searrow 0$ as $M \searrow 0$, whence (2.9), provided M is small enough. Q.E.D. ■

Now using (2.7) and (2.11) we have, for small M,

$$\frac{d}{dt}|\mathcal{L}^{1/2}U|^2 + |\mathcal{L}U|^2 \le |F|^2 + CM$$

In view of the periodicity of U, by integration we obtain

(2.16)
$$\int_0^\tau |\mathcal{L}U(t)|^2 dt \le C(M, M_0), \qquad M_0 \equiv \left(\int_0^\tau |F(t)|^2 dt\right)^{1/2}$$

Lemma 2.3 *Let U be the solution from (2.1). Then*

(2.17)
$$\sup_{t \in R^1} |U_t(t)|^2 \le C(M, M_0, M_1), \quad C(M, M_0, M_1) \text{ independent of } n,$$

where M_0 is as in (2.16) and $M_1 = (\int_0^\tau |F_t(t)|^2 dt)^{1/2}$.

Proof. Set $\phi = U_t$ in (2.10). Then we have, by (1.5),

$$|U_t|^2 + \frac{1}{2}\frac{d}{dt}|\mathcal{L}^{1/2}U|^2 \le |F||U_t| + c_2|\mathcal{L}^{1/2}U|\cdot|U_t| + |\mathcal{B}(U,U,U_t)| \tag{2.18}$$

In view of (1.2) and (2.7),

$$\begin{aligned} |\mathcal{B}(U,U,U_t)| &\le c_1|\mathcal{L}^{1/4}U|\cdot|\mathcal{L}U|\cdot|U_t|. \\ &\le CM^{1/2}|\mathcal{L}U|^2 + CM^{1/2}|U_t|^2 \end{aligned} \tag{2.19}$$

and from (2.18), (2.19), for small M,

$$|U_t|^2 + \frac{d}{dt}|\mathcal{L}^{1/2}U|^2 \le |F|^2 + C|\mathcal{L}^{1/2}U|^2 + CM^{1/2}|\mathcal{L}U|^2$$

After integration in t we obtain, in view of (2.16), (2.7),

$$\int_0^\tau |U_t(t)|^2 dt \le C(M, M_0). \tag{2.20}$$

Now, let us differentiate (2.10) with respect to t and set $\phi = U_t$. Then we get, by (1.4),

$$(U_{tt}, U_t) + \mathcal{A}(U_t, U_t) + \mathcal{B}(U_t, U, U_t) + \mathcal{R}(U_t, U_t) = (F_t, U_t)$$

whence, from (1.6),

$$\frac{d}{dt}|U_t|^2 + 2k_1|\mathcal{L}^{1/2}U_t|^2 \le C|F_t|^2 + \frac{k_1}{2}|\mathcal{L}^{1/2}U_t|^2 + 2|\mathcal{B}(U_t, U, U_t)|$$

By (1.2),

$$\begin{aligned} 2|\mathcal{B}(U_t, U, U_t)| &\le 2c_1|\mathcal{L}^{1/2}U_t|\cdot|\mathcal{L}U|\cdot|U_t| \\ &\le \frac{k_1}{2}|\mathcal{L}^{1/2}U_t|^2 + C|\mathcal{L}U|^2\cdot|U_t|^2 \end{aligned}$$

From the two last inequalities we conclude

$$\frac{d}{dt}|U_t|^2 + k_1|\mathcal{L}^{1/2}U_t|^2 \le C|F_t|^2 + C|\mathcal{L}U|^2\cdot|U_t|^2 \tag{2.21}$$

and, in particular,

$$\frac{d}{dt}|U_t|^2 \le C|F_t|^2 + C|\mathcal{L}U|^2\cdot|U_t|^2 \tag{2.22}$$

We shall use now the uniform Gronwall lemma [8]. We have

$$\int_t^{t+\tau} |U_t(s)|^2 ds = \int_0^\tau |U_t(s)|^2 ds \le C(M, M_0)$$

by (2.20), and by (2.16)

$$\int_t^{t+\tau} |\mathcal{L}U(s)|^2 ds = \int_0^\tau |\mathcal{L}U(s)|^2 ds \le C(M, M_0)$$

By our assumptions,

$$\int_t^{t+\tau} |F_t(s)|^2 ds = \int_0^\tau |F_t(s)|^2 ds \leq M_1^2$$

¿From the uniform Gronwall lemma applied to inequality (2.22) we obtain

$$|U_t(t+\tau)|^2 \leq \left\{ \frac{C(M, M_0)}{\tau} + M_1^2 \right\} \exp C(M, M_0) \qquad \text{for all } t \geq 0$$

Since U is τ-periodic, we obtain (2.17). Q.E.D. ■

3. Higher order estimates.

Lemma 3.1 *Let U be the approximate solution from (2.1). Then*

(3.1) $$\sup_{t \in R^1} |\mathcal{L}U(t)| \leq C(M, M_0, M_1)$$

and

(3.2) $$\sup_{t \in R^1} |\mathcal{L}^{1/2}U_t(t)| \leq C(M, M_0, M_1)$$

Proof. Set $\phi = \mathcal{L}U$ in (2.10), use (1.2) and (1.5), and then (2.17), (2.7), and (1.7) to get

$$|\mathcal{L}U|^2 \leq |U_t| \cdot |\mathcal{L}U| + c_1|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| \cdot |\mathcal{L}U| + c_2|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U| + |F| \cdot |\mathcal{L}U|$$

$$\leq C(M, M_0, M_1)|\mathcal{L}U| + CM^{1/2}|\mathcal{L}U|^2$$

For M such that $1 - CM^{1/2} > 0$ we obtain (3.1).

In order to obtain the second estimate of the lemma we differentiate identity (2.10) with respect to t and then set $\phi = \mathcal{L}U_t$. Using (1.2) and (1.5) we get

$$\frac{1}{2}\frac{d}{dt}|\mathcal{L}^{1/2}U_t|^2 + |\mathcal{L}U_t|^2 \leq c_1|\mathcal{L}^{1/2}U_t| \cdot |\mathcal{L}^{3/4}U| \cdot |\mathcal{L}U_t| + c_1|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}^{3/4}U_t| \cdot |\mathcal{L}U_t|$$

$$+C|\mathcal{L}^{1/2}U_t|^2 + \frac{1}{4}|\mathcal{L}U_t|^2 + |F_t|^2 + \frac{1}{4}|\mathcal{L}U_t|^2$$

Using (2.4) and (3.1) we get

(3.3) $$\frac{d}{dt}|\mathcal{L}^{1/2}U_t|^2 + |\mathcal{L}U_t|^2 \leq C(M, M_0, M_1)|\mathcal{L}^{1/2}U_t| \cdot |\mathcal{L}U_t|$$

$$+CM|\mathcal{L}U_t|^2 + C|\mathcal{L}^{1/2}U_t|^2 + |F_t|^2$$

From (2.21), (2.20),

$$\int_0^\tau |\mathcal{L}^{1/2}U_t|^2 dt \leq C(M, M_0, M_1) \tag{3.4}$$

and from (3.3), (3.4) and for M small enough,

$$\int_0^\tau |\mathcal{L}U_t|^2 dt \leq C(M, M_0, M_1)^{3/2} \cdot (\int_0^\tau |\mathcal{L}U_t|^2 dt)^{1/2} + C(M, M_0, M_1)$$

whence

$$\int_0^\tau |\mathcal{L}U_t(t)|^2 dt \leq C(M, M_0, M_1) \tag{3.5}$$

Thus, for some $t^* \in [0, \tau]$,

$$|\mathcal{L}^{1/2}U_t(t^*)|^2 \leq C(M, M_0, M_1)$$

and, after integration of (3.3) in t from t^* to $t + \tau$, $t \in [0, \tau]$ we obtain (3.2). Q.E.D. ∎

Lemma 3.2 *Let U be the approximate solution from (2.1) Then*

$$\int_0^\tau |U_{tt}(t)|^2 dt \leq C(M, M_0, M_1) \tag{3.6}$$

Proof. Differentiate (2.10) with respect to t and set $\phi = U_{tt}$. We get

$$|U_{tt}|^2 \leq |\mathcal{A}(U_t, U_{tt})| + |\mathcal{R}(U_t, U_{tt})| + |\mathcal{B}(U_t, U, U_{tt})| + |\mathcal{B}(U, U_t, U_{tt})| + |(F_t, U_{tt})|$$

$$\leq |\mathcal{L}U_t| \cdot |U_{tt}| + C|\mathcal{L}^{1/2}U_t| \cdot |U_{tt}| + c_1|\mathcal{L}^{1/2}U_t| \cdot |\mathcal{L}U| \cdot |U_{tt}| + c_1|\mathcal{L}^{1/2}U| \cdot |\mathcal{L}U_t| \cdot |U_{tt}| + |F_t| \cdot |U_{tt}|$$

in view of (1.2), (1.5), so that, by (2.4), (3.1), and (3.2),

$$|U_{tt}|^2 \leq C(M, M_0, M_1)|\mathcal{L}U_t|^2 + C|F_t|^2$$

Integration in t gives (3.6), by (3.5). Q.E.D. ∎

4. Convergence and uniqueness.

¿From the estimates we have obtained it follows that (U_n) is a bounded sequence in

$$H^2(\tau; \mathcal{H}) \cap H^1(\tau; D(\mathcal{L})) \cap L^\infty(\tau; D(\mathcal{L})) \cap W^{1,\infty}(\tau; D(\mathcal{L}^{1/2})) \tag{4.1}$$

and that there exists a subsequence (U_μ) and some U in the space (4.1) such that

$$U_\mu \to U \quad \text{weak star in } L^\infty(\tau; D(\mathcal{L}))$$

$$U_\mu \to U \quad \text{strongly in } L^\infty(\tau; D(\mathcal{L}^{1/2}))$$

$$D_t U_\mu \to D_t U \quad \text{weak star in } L^\infty(\tau; D(\mathcal{L}^{1/2}))$$

$$D_t U_\mu \to D_t U \quad \text{strongly in } L^\infty(\tau; \mathcal{H})$$

Set $U = U_\mu$ in the identity (2.1) for $\mu \geq n$, and let $\mu \to \infty$ To obtain (1.1) observe that, by (1.2) and (2.12), for all $\phi \in \mathcal{H}_n$ we have

$$|\mathcal{B}(U_\mu, U_\mu, \phi) - \mathcal{B}(U, U, \phi)| \leq |\mathcal{B}(U_\mu - U, U_\mu, \phi)| + |\mathcal{B}(U, U_\mu - U, \phi)|$$

$$\leq c_1 |\mathcal{L}^{1/2}(U_\mu - U)| \cdot |\mathcal{L}^{3/4} U_\mu| \cdot |\phi| + c_1 |\mathcal{L}^{1/2} U| \cdot |\mathcal{L}^{1/2}(U_\mu - U)|^{1/2} \cdot |\mathcal{L}(U_\mu - U)|^{1/2} |\phi| \to 0$$

uniformly in t, in view of (2.7), (3.1), and then

$$(U_t, \phi) + \mathcal{A}(U, \phi) + \mathcal{B}(U, U, \phi) + \mathcal{R}(U, \phi) = (F, \phi)$$

easily follows for the limit function U and all $\phi \in \mathcal{H}$.

To prove the uniqueness assume, to the contrary, that there are two different solutions U and V. Then $W = U - V$ satisfies, for all $\phi \in \mathcal{H}$,

$$(W_t, \phi) + \mathcal{A}(W, \phi) + \mathcal{B}(U, W, \phi) + \mathcal{B}(W, V, \phi) + \mathcal{R}(W, \phi) = 0$$

Set $\phi = W$ and we get by (1.2), (2.12), (2.4) and (2.7), (3.1),

$$\frac{1}{2} \frac{d}{dt} |W|^2 + k_1 |\mathcal{L}^{1/2} U|^2 \leq |\mathcal{B}(W, V, W)| \leq c_1 |\mathcal{L}^{1/2} W| \cdot |\mathcal{L}^{3/4} V| \cdot |W| \leq$$

$$\leq c_1 |\mathcal{L}^{1/2} W| \cdot |\mathcal{L}^{1/2} V|^{1/2} \cdot |\mathcal{L} V|^{1/2} \cdot |W| \leq M^{1/4} \cdot C(M, M_0, M_1) |\mathcal{L}^{1/2} W|^2$$

whence for small M we obtain, for some $c > 0$,

$$\frac{d}{dt} |W|^2 + c|W|^2 \leq 0 \tag{4.2}$$

which leads to the contradiction, as (4.2) yields $W \equiv 0$ by Gronwall's lemma and periodicity of W. This proves the uniqueness. Q.E.D. ■

Remark 4.1 Observe that in fact we have used (1.2)-(1.3) only for three pairs of parameters (θ, ρ), namely, $(1/4, 1), (1/2, 1)$, and $(1/2, 3/4)$. Thus, we can weaken the assumption (1.2)-(1.3) of theorem 1.1 appropriately.

5. Application.

As an example of application of theorem 1.1 we shall consider a model of magneto-micropolar fluid. In the incompressible case the governing system of equations for this model is [7]:

$$\text{div } u = 0, \qquad \text{div } h = 0 \tag{5.1}$$

$$\frac{\partial u}{\partial t} - (\nu + \nu_r) \triangle u + (u \cdot \nabla)u + \nabla(p + \frac{1}{2}h \cdot h) = 2\nu_r \text{rot}\omega + rh \cdot \nabla h + f \tag{5.2}$$

$$j\frac{\partial \omega}{\partial t} - \alpha\triangle\omega - \beta\nabla\text{div}\omega + j(u \cdot \nabla)\,\omega + 4\nu_r\omega = 2\nu_r \text{ rot}u + g \tag{5.3}$$

$$\frac{\partial h}{\partial t} - \gamma\triangle h + u \cdot \nabla h - h \cdot \nabla u = 0 \tag{5.4}$$

where u is the velocity, p is the pressure, ω is the microrotation (angular velocity of rotation of particles) and h is the magnetic field. Moreover, f and g are external fields and $\nu, \nu_r, j, \alpha, \beta, \gamma$ are positive constants (ν is the usual Newtonian viscosity, ν_r is the microrotation viscosity). We assume that the density of the fluid is equal to one. We also set $r = j = 1$ for simplicity. Let Ω be a bounded set in R^3 with smooth boundary.

By $\mathcal{H}$ we denote the Hilbert space $H \times L^2(\Omega)^3 \times H$ where H is the closure in the norm of $L^2(\Omega)^3$ of the set of divergence free, smooth functions with compact support in Ω. The norm in $\mathcal{H}$ will be denoted by $[\cdot]$.

We introduce the following operators:

$$\mathcal{L}(U) = (-(\nu + \nu_r)\mathcal{P}\triangle u_1, -\alpha\triangle\omega_1 - (\alpha + \beta)\nabla\text{div}\omega_1, -\gamma\mathcal{P}\triangle h_1) = (A_1u_1, A_2\omega_1, A_3h_1)$$

for $U = (u_1, \omega_1, h_1) \in D(\mathcal{L})$, $V = (u_2, \omega_2, h_2) \in D(\mathcal{L})$,

$$B(U, V) = (\mathcal{P}(u_1 \cdot \nabla u_2) - (\mathcal{P}h_1 \cdot \nabla h_2), u_1 \cdot \nabla\omega_2, (\mathcal{P}u_1 \cdot \nabla h_2) - (\mathcal{P}h_1 \cdot \nabla u_2))$$

$$R(U) = (-2\nu_r\text{rot}\omega_1, -2\nu_r\text{rot}u_1 + 4\nu_r\omega_1, 0)$$

The operator $\mathcal{P}$ above is the orthogonal projection in $L^2(\Omega)^3$ on the subspace H. In this notation the system of equations (5.1)-(5.4) takes the form (1.1) with $F = (\mathcal{P}f, g, 0)$

We check the assumptions of theorem 1.1.

Operator $\mathcal{L}$ is self-adjoint, positive with $D(\mathcal{L}) = W^{2,2}(\Omega)^3 \times W^{2,2}(\Omega)^3 \times W^{2,2}(\Omega)^3) \cap (W_0^{1,2}(\Omega)^3 \times W_0^{1,2}(\Omega)^3 \times W_0^{1,2}(\Omega)^3 \cap \mathcal{H}$ where $W^{2,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ are Sobolev spaces. The norms $[\mathcal{L}U]$ and $||U||_{W^{2,2}}$ are equivalent on $D(\mathcal{L})$. We denote $X^\theta = D(\mathcal{L}^\theta)$ and $Y_i^\theta = D(A_i^\theta),\ i = 1, 2, 3.$ Observe that $Y_2^\theta = D(-\triangle^\theta)$, $0 \leq \theta \leq 1$.

Operator B satisfies (1.4), cf. [7]. ¿From properties of the Laplace and the Stokes operators [3], [4], [5] we have, in particular, $Y_i^{1/4} \subset L^3(\Omega)$, $Y_i^{1/2} \subset L^6(\Omega)$, $Y_i^{3/4} \subset W^{1,3}(\Omega)$, with continuous imbeddings. Thus, for $u \in Y_1^{1/4}$, $v \in Y_i$, $i = 1, 2, 3$,

$$|u \cdot \nabla v| \leq c|u|_{L^3} \cdot |\nabla v|_{L^6} \leq c_1 |A_1^{1/4} u| \cdot |A_i v|$$

and, similarly, for $u \in Y_1^{1/2}$, $v \in Y_i^{3/4}$, $i = 1, 2, 3$,

$$|u \cdot \nabla v| \leq c|u|_{L^6} \cdot |\nabla v|_{L^3} \leq c_1 |A_1^{1/2} u| \cdot |A_i^{3/4} v|.$$

Finally, after simple calculations, we obtain

$$|B(U, V)| \leq c_1 [\mathcal{L}^{1/4} U] \cdot [\mathcal{L} V]$$

and

$$|B(U, V)| \leq c_1 [\mathcal{L}^{1/2} U] \cdot [\mathcal{L}^{3/4} V]$$

for $U \in X^{1/4}$, $V \in X^1$ and $U \in X^{1/2}$, $V \in X^{3/4}$, respectively.

Moreover, operator R satisfies (1.5) and also (1.6) holds , cf. [6].

From theorem 1.1, cf. Remark 4.1, follows existence of τ-periodic solution for the system of equations of magneto-micropolar fluids.

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