Groups with finitely generated integral homologies

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Abstract

Suppose A is an abelian normal subgroup of a finitely generated group G such that G/A is abelian and $H_i(G, \mathbb{Z})$ is finitely generated for all i . We show that A is of finite (Prüfer) rank. This generalises the main result of [5] that deals with the same problem for split extension metabelian groups.

1 Introduction

In [5] J. R. J. Groves shows that if G is a finitely generated group, a split extension of an abelian group A by an abelian group Q and the homology group $H_i(G, \mathbb{Z})$ is finitely generated for all i then A is of finite rank i.e. $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$ and the torsion part of A is finite. We generalise this result to the non-split case.

Theorem A *Suppose that A is a normal abelian subgroup of a finitely generated group G such that G/A is abelian and $H_i(G, \mathbb{F}_p)$ is finite for all i and all primes p. Then A is of finite (Prüfer) rank.*

Corollary B *Suppose that A is a normal abelian subgroup of a finitely generated group G such that G/A is abelian and $H_i(G, \mathbb{Z})$ is finitely generated for all i. Then A is of finite (Prüfer) rank.*

Our proofs substantially use the method and the main tools from [5]: the geometric invariant for modules over finitely generated abelian groups defined in [2], Cartan's formula for $H_*(A, \mathbb{F}_p)$ for abelian groups A and the finite field $\mathbb{F}_p$ with p elements (in the case $p = 2$ the formula holds only for groups A of exponent 2) and close examination of the LHS spectral sequence in homology. The new ingredients are a lemma that gives sufficient conditions when $H_0(Q, H_j(A, \mathbb{F}_p))$ finite implies that $H_i(Q, H_j(A, \mathbb{F}_p))$ is finite for all $i \geq 0$ and a generalisation of Cartan's formula for the homologies of abelian groups with trivial coefficients $\mathbb{F}_2$.

2 Preliminaries on the geometric invariant for modules over finitely generated abelian groups and on homologies of abelian group

The geometric invariant $\Sigma_A(Q)$ for a finitely generated $\mathbb{Z}[Q]$-module A was first defined in [2]. By definition

$$\Sigma_A(Q) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in Hom(Q, \mathbb{R}) \setminus \{0\}, A \text{ is finitely generated over } \mathbb{Z}[Q_\chi]\},$$

$$\Sigma_A^c(Q) = S(Q) \setminus \Sigma_A(Q)$$

where $Q_\chi = \{g \in Q \mid \chi(g) \geq 0\}$ and $S(Q) = \{[\chi] \mid \chi \in Hom(Q, \mathbb{R}) \setminus \{0\}\} \simeq \mathbb{R}^n$ where n is the torsion-free rank of Q. A is said to be m-tame if every m-point subset of $\Sigma_A^c(Q)$ is contained in an open half subspace of $S(Q)$. One important property of tameness is that whenever A is m-tame the m-fold tensor power of A over $\mathbb{Z}$ is finitely generated over $\mathbb{Z}[Q]$ via the diagonal Q-action [1, Section 3.5].

We discuss now an important result of H. Cartan [4] that will play an important role in the proof of our main theorem. Suppose A is an abelian group. By definition $\widetilde{S}^j({}_pA)$ is the set of elements in the j-th tensor power of ${}_pA = \{a \in A \mid pa = 0\}$ over $\mathbb{F}_p$ which is invariant under the action of the symmetric group on j elements. Note $\widetilde{S}({}_pA) = \oplus_{i \geq 0} \widetilde{S}^i({}_pA)$ is a graded algebra with multiplication given by the shuffle product $*$ of the tensor algebra of ${}_pA$ i.e.

$$(a_1 \otimes \ldots \otimes a_s) * (a_{s+1} \otimes \ldots \otimes a_{s+k}) = \sum_\sigma a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(s+k)}$$

where $a_i \in_p A$ and the sum is over all permutation σ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(s)$ and $\sigma(s+1) < \ldots < \sigma(s+k)$.

In general homology groups do not have multiplicative structure as cohomology but in the case of an abelian group A and commutative ring with unity k there is a Pontryagin product in $H_*(A, k) = \oplus_{i \geq 0} H_i(A, k)$ which makes it a strictly anticommutative ring equipped with divided powers, for details see [3, Ch. 5, Section 6]. We remind the reader the axioms of divided powers. There is a family of functions ${}^{(i)} : H_{2j}(A, k) \to H_{2ij}(A, k)$ for all $i, j - 1 \geq 0$.

1. $x^{(0)} = 1, x^{(1)} = x$
2. $x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)}$
3. $(x^{(i)})^{(j)} = e_{i,j}x^{(ij)}$ for all $i, j > 0$ where $e_{i,j} = \prod_{2 \le t \le j} \binom{ti-1}{i-1}$.
4. $(x+y)^{(i)} = \sum_{j+k=i} x^{(j)}y^{(k)}$
5. For $i \geq 2$

$$(xy)^{(i)} = x^i y^{(i)} \text{ whenever } x, y \text{ are even elements,}$$

$$(xy)^{(i)} = 0 \text{ if } x, y \text{ are odd elements.}$$

Proposition 1 *[4, Ch. 9 , Ch. 10] Suppose p is a prime and $\mathbb{F}_p$ is the field with p elements.*

1. If p is odd

$$H_*(A, \mathbb{F}_p) \simeq (\wedge_{\mathbb{F}_p}(A/pA)) \otimes_{\mathbb{F}_p} \widetilde{S}({}_pA)$$

where A/pA has weight 1 and ${}_pA$ has weight 2.

2. If $p = 2$ and A is of exponent 2

$$H_*(A, \mathbb{F}_2) \simeq \widetilde{S}(A)$$

where A has weight 1.

In both cases the isomorphism is a natural isomorphism of graded divided powers algebras i.e. preserves grading, the multiplicative and the divided power structures.

Whenever possible we prefer using homologies with coefficients in $\mathbb{F}_p$ than $\mathbb{Z}$. The Pontryagin product gives natural embedding of the exterior algebra of $H_1(A, \mathbb{Z}) \simeq A$ in $H_*(A, \mathbb{Z})$ and it is an isomorphism if A is $\mathbb{Z}$-torsion-free [3, Ch. 5, Thm 6.4]. The problem is that this embedding does not naturally split as in the case of coefficients $\mathbb{F}_p$ for p odd. In [6, Thm C] some results linking the integral homology groups and finite generations of tensor products are established but they are not directly applicable to the proof of Theorem A.

3 More on the homology with coefficients in $\mathbb{F}_2$

In [3, Thm 6.6] it is stated that there is a non-natural isomorphism

$$H_*(A, \mathbb{F}_2) \simeq \wedge(A/2A) \otimes \widetilde{S}({}_2A) \qquad (*)$$

and that the above isomorphism could be proved as in the proof of [3, Thm 6.4]. The generating space ${}_2A = \{a \in A \mid 2a = 0\}$ of the symmetric algebra comes from a non-natural splitting of the exact sequence

$$0 \to \wedge^2(A/2A) \to H_2(A, \mathbb{F}_2) \to {}_2A \to 0$$

Still it is not made clear in [3] how to combine the non-naturality of $(*)$ with the ideas of the proof of [3, Thm 6.4] that deals with a natural description of homology groups.

In this section we show how such an isomorphism could be proved and in fact we give a natural description of $H_*(A, \mathbb{F}_2)$ in terms of a filtration with quotients isomorphic to the direct summands of $(*)$. Naturality is important for two purposes. First it is needed in the proof of the fact that the filtration is exhausting. Secondly all the applications are for $\mathbb{Z}[Q]$-modules A and we are interested not only in the underlying additive structure of the homology groups $H_*(A, \mathbb{F}_2)$ but in their structure as $\mathbb{Z}[Q]$-modules. As before $H_*(A, \mathbb{F}_2)$ is equipped with strictly anticommutative Pontryagin product and divided power structure.

Note that

$$H_1(A, \mathbb{F}_2) \simeq A/2A,$$

and by the exact universal coefficient sequence

$$0 \to H_2(A, \mathbb{Z}) \otimes \mathbb{F}_2 \to H_2(A, \mathbb{F}_2) \to Tor_1^{\mathbb{Z}}(H_1(A, \mathbb{Z}), \mathbb{F}_2) \to 0$$

As $H_2(A, \mathbb{Z}) \simeq \wedge^2 A$ and $Tor_1^{\mathbb{Z}}(H_1(A, \mathbb{Z}), \mathbb{F}_2) \simeq {}_2A = \{a \in A \mid 2a = 0\}$ we have

$$\frac{H_2(A, \mathbb{F}_2)}{\wedge^2 H_1(A, \mathbb{F}_2)} \simeq {}_2A$$

Theorem 1 *Let*

$$F^i(H_*(A, \mathbb{F}_2)) = \sum_{k \geq 0, j_1 + \ldots + j_t \leq i} H_1(A, \mathbb{F}_2)^k H_2(A, \mathbb{F}_2)^{(j_1)} \ldots H_2(A, \mathbb{F}_2)^{(j_t)},$$

where $H_2(A, \mathbb{F}_2)^{(j)}$ is the subspace spanned by all elements $\lambda^{(j)}$ for $\lambda \in H_2(A, \mathbb{F}_2)$. Then $\cup_{i \geq 1} F^i(H_*(A, \mathbb{F}_2))$ is an exhausting filtration of the graded algebra $H_*(A, \mathbb{F}_2)$ with quotients $F^i(H_*(A, \mathbb{F}_2))/F^{i-1}(H_*(A, \mathbb{F}_2)) \simeq \wedge(A/2A) \otimes \widetilde{S}^i({}_2A)$

Proof. Note that to prove that

$$\cup_{i \geq 1} F^i(H_*(A, \mathbb{F}_2)) = H_*(A, \mathbb{F}_2) \qquad (**)$$

it is sufficient to consider the following cases:

1. A is cyclic;

2. if $(**)$ holds for the finitely generated abelian groups A_1 and A_2 then $(**)$ holds $A = A_1 \oplus A_2$;

Then the filtration is exhausting for all finitely generated abelian groups. Finally as every group A is the direct limit of its finitely generated subgroups and our filtration commutes with direct limits $(**)$ holds.

If A is cyclic and finite then $(**)$ is proved in [5, p. 124]. If A is infinite then $H_i(A,\mathbb{F}_2)=0$ for all $i\geq 2$.

Now we prove that the filtration is exhausting for $A=A_1\oplus A_2$ provided the same holds for A_1 and A_2. Consider the commutative diagram

$$\begin{array}{ccc} (\cup_{i\geq 1}F^i(H_*(A_1,\mathbb{F}_2)))\otimes(\cup_{i\geq 1}F^i(H_*(A_2,\mathbb{F}_2))) & \stackrel{\beta_1\otimes\beta_2}{\longrightarrow} & (\cup_{i\geq 1}F^i(H_*(A,\mathbb{F}_2))) \\ \downarrow\alpha_1\otimes\alpha_2 & & \downarrow\alpha \\ H_*(A_1,\mathbb{F}_2)\otimes H_*(A_2,\mathbb{F}_2) & \stackrel{\varphi}{\longrightarrow} & H_*(A,\mathbb{F}_2) \end{array}$$

The maps α_1,α_2 and α are the obvious inclusions. By assumptions α_1 and α_2 are isomorphisms. The map

$$\beta_i:\cup_{s\geq 1}F^s(H_*(A_i,\mathbb{F}_2))\to\cup_{s\geq 1}F^s(H_*(A,\mathbb{F}_2))$$

is induced by the map

$$H_j(A_i,\mathbb{F}_2)\to H_j(A,\mathbb{F}_2)$$

for $j=1,2$. The map φ is the inclusion given by the Kuneth formula and as $\mathbb{F}_2$ is a field φ is an isomorphism. Then $\varphi(\alpha_1\otimes\alpha_2)$ is an isomorphism and hence α is surjective. By construction α is injective and so α is an isomorphism, as required. Then $\beta_1\otimes\beta_2$ is an isomorphism too.

Claim $(\beta_1\otimes\beta_2)(\sum_{i+j=m}F^i(H_*(A_1,\mathbb{F}_2))\otimes F^j(H_*(A_2,\mathbb{F}_2)))=F^m(H_*(A,\mathbb{F}_2))$

Proof. Assume x_1,y_1 are elements of degree one and x_2,y_2 are elements of degree two. Then by the axioms of divided powers for $j\geq 2$ the element $(x_2+y_2+x_1y_1)^{(j)}$ equals

$$\sum_{\alpha_1+\alpha_2+\alpha_3=j}x_2^{(\alpha_1)}y_2^{(\alpha_2)}(x_1y_1)^{(\alpha_3)}=\sum_{\alpha_1+\alpha_2=j}x_2^{(\alpha_1)}y_2^{(\alpha_2)}+\sum_{\alpha_1+\alpha_2=j-1}x_2^{(\alpha_1)}y_2^{(\alpha_2)}x_1y_1$$

This together with the Kuneth formula

$$H_2(A,\mathbb{F}_2)\simeq\oplus_{0\leq i\leq 2}H_i(A_1,\mathbb{F}_2)\otimes H_{2-i}(A_2,\mathbb{F}_2)$$

implies

$$H_2(A,\mathbb{F}_2)^{(j)}\subset\sum_{j_0+j_1+j_2=j}H_2(A_1,\mathbb{F}_2)^{(j_0)}H_1(A_1,\mathbb{F}_2)^{j_1}\otimes H_1(A_2,\mathbb{F}_2)^{j_1}H_2(A_2,\mathbb{F}_2)^{(j_2)}$$

$$\subseteq\wedge H_1(A,\mathbb{F}_2)\sum_{j_0+j_2\leq j}H_2(A,\mathbb{F}_2)^{(j_0)}H_2(A,\mathbb{F}_2)^{(j_2)}$$

This together with the definition of the filtration implies

$$F^m(H_*(A,\mathbb{F}_2)) \subseteq (\beta_1 \otimes \beta_2)(\sum_{i+j=m} F^i(H_*(A_1,\mathbb{F}_2)) \otimes F^j(H_*(A_2,\mathbb{F}_2)))$$

The inverse inclusion is obvious.

□

By the claim and the fact that $\beta_1 \otimes \beta_2$ is an isomorphism we have

$$F^m(H_*(A,\mathbb{F}_2))/F^{m-1}(H_*(A,\mathbb{F}_2)) \simeq$$

$$\oplus_{0\le i\le m} F^i(H_*(A_1,\mathbb{F}_2))/F^{i-1}(H_*(A_1,\mathbb{F}_2)) \otimes F^{m-i}(H_*(A_1,\mathbb{F}_2))/F^{m-i-1}(H_*(A_1,\mathbb{F}_2))$$

By assumption

$$F^i(H_*(A_1,\mathbb{F}_2))/F^{i-1}(H_*(A_1,\mathbb{F}_2)) \simeq \wedge(A_1/2A_1) \otimes \widetilde{S}^i({}_2(A_1))$$

$$F^{m-i}(H_*(A_1,\mathbb{F}_2))/F^{m-i-1}(H_*(A_1,\mathbb{F}_2)) \simeq \wedge(A_2/2A_2) \otimes \widetilde{S}^{m-i}({}_2(A_2))$$

Furthermore

$$\wedge(A_1/2A_1) \otimes \wedge(A_2/2A_2) \simeq \wedge((A_1/2A_1) \oplus (A_2/2A_2)) = \wedge(A/2A)$$

Finally it remains to note that for abelian groups M and N of exponent 2

$$\oplus_{0\le i\le m}(\widetilde{S}^i(M) \otimes \widetilde{S}^{m-i}(N)) \simeq \widetilde{S}^m(M \oplus N)$$

and apply this for $M = {}_2(A_1), N = {}_2(A_2)$, $M \oplus N \simeq {}_2A = \{a \in A \mid 2a = 0\}$ to obtain the required isomorphism

$$F^m(H_*(A,\mathbb{F}_2))/F^{m-1}(H_*(A,\mathbb{F}_2)) \simeq (\wedge(A/2A)) \otimes \widetilde{S}^m({}_2A)$$

The formula about M and N could be verified either by hand or by Proposition 1 (remember both M and N have exponent 2) it is transformed to a special case of the Kuneth formula

$$H_*(M,\mathbb{F}_2) \otimes H_*(N,\mathbb{F}_2) \simeq H_*(M \oplus N,\mathbb{F}_2)$$

This completes the proof of the theorem. □

To illustrate the above theorem we consider the case when A has exponent 2. By Proposition 1

$$H_*(A,\mathbb{F}_2) \simeq \widetilde{S}(A)$$

Any symmetric element of $\otimes^i A$ is a linear combination of elements of the form

$$(a_1^{\otimes^{k_1}}) * (a_2^{\otimes^{k_2}}) * \ldots * (a_s^{\otimes^{k_s}})$$

for some $s \leq i$, some pairwise different elements $a_1, \ldots, a_s$ of A where $\sum k_j = i$ and $*$ is the shuffle product. The Pontrjagin product is the shuffle product $*$ and the divided power structure is given by

$$(a \otimes a)^{(i)} = a^{\otimes^{2i}}$$

Then

$$a^{\otimes^k} = a^\epsilon * (a \otimes a)^{([k/2])} \in H_1(A, \mathbb{F}_2)^\epsilon H_2(A, \mathbb{F}_2)^{([k/2])}$$

where $\epsilon = k - 2[k/2]$ and the element $(a_1^{\otimes^{k_1}}) * (a_2^{\otimes^{k_2}}) * \ldots * (a_s^{\otimes^{k_s}})$ belongs to

$$(\wedge H_1(A, \mathbb{F}_2)) H_2(A, \mathbb{F}_2)^{([\frac{k_1}{2}])} H_2(A, \mathbb{F}_2)^{([\frac{k_2}{2}])} \ldots H_2(A, \mathbb{F}_2)^{([\frac{k_s}{2}])}.$$

Thus

$$F^j(H_*(A, \mathbb{F}_2)) \cap \widetilde{S}^i(A) = \sum_{[k_1/2]+\ldots+[k_s/2] \leq j} (a_1^{\otimes^{k_1}}) * (a_2^{\otimes^{k_2}}) * \ldots * (a_s^{\otimes^{k_s}})$$

4 Some results about homology

Lemma 1 *Suppose Q is a finitely generated abelian group, B is a $\mathbb{Z}[Q]$-module equipped with a finite filtration of $\mathbb{Z}[Q]$-submodules*

$$B = B_1 \supset B_2 \supset \ldots \supset B_k \supset B_{k+1} = 0$$

such that B_j/B_{j+1} is a cyclic R_j-module for some commutative Noetherian ring R_j, Q embeds in R_j and the action of Q on B_j/B_{j+1} via the embedding of Q in R_j is the original action of Q. Suppose further that $H_0(Q, B)$ is finite. Then $H_i(Q, B)$ is finite for all i.

Proof. We induct on the length of the filtration. Assume first that $k = 1$ and so B is cyclic and a commutative Noetherian ring. Let $\mathcal{F}$ be a resolution of the trivial module $\mathbb{Z}$ over R_1 with all modules finitely generated. Consider the complex $B \otimes_{\mathbb{Z}[Q]} \mathcal{F}$. Its modules could be viewed as (left) B-modules via the multiplication in B. Note this B-action is compatible with the differentials because its action could be extended to an action of R_1 and the latter ring is commutative. As B is a Noetherian ring and all modules in $B \otimes_{\mathbb{Z}[Q]} \mathcal{F}$ are finitely generated over B we deduce that all homology groups

$$H_i(B \otimes_{\mathbb{Z}[Q]} \mathcal{F}) \simeq H_i(Q, B)$$

are finitely generated B-modules. Furthermore the action of Q on $H_i(Q, B)$ is trivial and so $H_i(Q, B)$ is a finitely generated $H_0(Q, B)$-module. As by assumption $H_0(Q, B)$ is finite we are done.

If $k \geq 2$ consider the short exact sequence of modules

$$0 \to B_2 \to B \to B/B_2 \to 0$$

It induces a long exact sequence in homology

$$\ldots \to H_i(Q, B_2) \to H_i(Q, B) \to H_i(Q, B/B_2) \to \ldots$$

$$\to H_1(Q, B/B_2) \to H_0(Q, B_2) \to H_0(Q, B) \to H_0(Q, B/B_2) \to 0$$

As $H_0(Q, B)$ is finite $H_0(Q, B/B_2)$ is finite and by induction $H_i(Q, B/B_2)$ is finite for all i. In particular $H_1(Q, B/B_2)$ is finite and hence $H_0(Q, B_2)$ is finite. Again by induction $H_i(Q, B_2)$ is finite for all i and using the long exact sequence we see that $H_i(Q, B)$ is finite for all i as required.

□

Lemma 2 *Suppose A is a finitely generated $\mathbb{Z}[Q]$-module and for some prime p and some j the homology group $H_0(Q, H_j(A, \mathbb{F}_p))$ is finite. Then $H_i(Q, H_j(A, \mathbb{F}_p))$ is finite for all i.*

Proof. By Proposition 1 (for p odd) and Theorem 1 (for $p = 2$) there is a filtration of $H_j(A, \mathbb{F}_p)$ with quotients isomorphic to some $B_{\alpha,\beta} = \wedge^\alpha(A/pA) \otimes \widetilde{S}^\beta({}_pA)$ for $\alpha + 2\beta = j$. $B_{\alpha,\beta}$ is a module over $\Pi_\alpha \otimes \Pi_\beta$ where Π_k is the invariant subring of $\mathbb{F}_p[Q^k]$ under the action of the symmetric group S_k that permutes the factors of Q^k. Note Π_k is a finitely generated algebra that contains the diagonal subgroup of Q^k. Then $\Pi_\alpha \otimes \Pi_\beta$ is a Noetherian commutative ring containing the diagonal subgroup of $Q^{\alpha+\beta}$ and we can apply the previous lemma. □

Proposition 2 *Suppose some extension G of A by Q is finitely generated, p is a prime and $H_t(G, \mathbb{F}_p)$ is finite for all t. Then $H_i(Q, H_t(A, \mathbb{F}_p))$ is finite for all t and i.*

Proof. We prove the proposition by induction on t. The case $t = 1$ is very easy, as $H_1(A, \mathbb{F}_p) \simeq A/pA$ and A is finitely generated over $\mathbb{Z}[Q]$. Thus $H_0(Q, H_1(A, \mathbb{F}_p))$ is finite. By Lemma 2 $H_i(Q, H_1(A, \mathbb{F}_p))$ is finite for all i.

For the inductive step assume $H_i(Q, H_j(A, \mathbb{F}_p))$ is finite for all $j \leq t-1$ and all i and consider the Lyndon-Hochshild-Serre spectral sequence over the trivial module $\mathbb{F}_p$

$$E^2_{i,j} = H_i(Q, H_j(A, \mathbb{F}_p))$$

with differentials

$$d^r : E^r_{i,j} \to E^r_{i-r,j+r-1}$$

By induction $E^2_{i,k}$ is finite for $k \leq t-1$ and all $i \geq 0$. This together with the fact that d^r has bidegree $(-r, r-1)$ and our spectral sequence is a first quadrant spectral sequence, in particular $E^\infty_{0,t} = E^{t+1}_{0,t}$, implies that

$$E^\infty_{0,t} \text{ is finite if and only if } E^2_{0,t} \text{ is finite}$$

At the same time since $H_{i+j}(G, \mathbb{F}_p)$ is finite we have that $E^\infty_{i,j}$ is finite for every i, j. Thus $E^2_{0,t} = H_0(Q, H_t(A, \mathbb{F}_p))$ is finite. Finally by Lemma 2 $H_i(Q, H_t(A, \mathbb{F}_p))$ is finite for all i.

□

5 Proof of Theorem A

Assume that A is not of finite rank. By [5, Section 3.1, Section 5.1] there exists a prime number p such that A/pA is infinite and A has an epimorphic image M which is a just-infinite cyclic $\mathbb{F}_p[Q]$-module of exponent p.

We claim that for p odd $H_0(Q, \wedge^i M)$ is finite for all i and for $p = 2$ the image of $H_0(Q, \wedge^i M)$ in $H_0(Q, H_i(M, \mathbb{F}_2)) \simeq H_0(Q, \widetilde{S}^i(M))$ is finite.

Indeed by Proposition 1 for p odd $\wedge^i_{\mathbb{F}_p}(A/pA)$ is a direct summand of $H_i(A, \mathbb{F}_p)$. Thus the embedding of the exterior algebra of $H_1(A, \mathbb{F}_p) \simeq A/pA$ in $H_*(A, \mathbb{F}_p)$ is natural and split and compatible with the multiplicative structure on the strictly anticommutative algebra $H_*(A, \mathbb{F}_p)$. Hence the action of Q on the homology group induces on the exterior algebra of A/pA the diagonal Q-action. Then $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} (\wedge^i(A/pA))$ embeds in $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} H_i(A, \mathbb{F}_p)$. By Proposition 2 $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} H_i(A, \mathbb{F}_p)$ is finite and so $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} (\wedge^i(A/pA))$ is finite. As M is a surjective image of A/pA we deduce that $H_0(Q, \wedge^i M)$ is finite.

If $p = 2$ consider the commutative diagram

$$\begin{array}{ccc} H_0(Q, H_i(A, \mathbb{F}_2)) & \to & H_0(Q, H_i(M, \mathbb{F}_2)) \simeq \mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^i(M) \\ \uparrow & & \uparrow \\ \mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} (\wedge^i_{\mathbb{F}_2}(A/2A)) & \xrightarrow{\varphi} & \mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \wedge^i M \end{array}$$

As by Proposition 2 $H_0(Q, H_i(A, \mathbb{F}_2))$ is finite and φ is surjective we deduce that the image of $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \wedge^i M$ in $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^i(M)$ is finite, as required.

From now on we will forget the existence of the group G and will deal only with just -infinite cyclic $\mathbb{Z}[Q]$-modules M of additive exponent p with the property that for p odd $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} (\wedge^i_{\mathbb{Z}} M)$ is finite for all i and for $p = 2$ the image of $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} (\wedge^i_{\mathbb{Z}} M)$ in $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^i(M)$ is finite for all i. By [5, Section 5.2] there exists a series

$$Q = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_t$$

of multiplicative subgroups of the field of fraction K of M and a series

$$M = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t$$

of additive subgroup of K defined by $M_i = Q_i M$ with the following properties:

1. $Q_{i+1} = Q_i \times < \alpha_i >$;
2. there exists $r_i \in \mathbb{Z}[Q_i]$ with $(r_i - \alpha_i) M_{i+1} = 0$;
3. M_i is fully tame i.e. M_i is n_i-tame as a module over $\mathbb{F}_p[Q_i]$ where n_i is the torsion free rank of Q_i.

Lemma 3 *If p is odd and $\mathbb{F}_p \otimes_{\mathbb{Z}[Q_i]} (\wedge^j M_i)$ is finite then $\mathbb{F}_p \otimes_{\mathbb{Z}[Q_{i+1}]} (\wedge^j M_{i+1})$ is finite.*

If $p = 2$ and the image of $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} (\wedge^j_{\mathbb{Z}} M_i)$ in $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^j(M_i)$ is finite then the image of $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} (\wedge^j_{\mathbb{Z}} M_{i+1})$ in $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^j(M_{i+1})$ is finite.

Proof. First let p be odd. Suppose $f_1, \ldots, f_s$ are elements of $\wedge^j M_i$ such that

$$\wedge^j_{\mathbb{F}_p} M_i = \mathbb{F}_p f_1 + \ldots + \mathbb{F}_p f_s + Aug(\mathbb{F}_p[Q_i])(\wedge^j_{\mathbb{F}_p} M_i)$$

where Aug denotes the augmentation ideal. We claim that

$$\wedge^j_{\mathbb{F}_p} M_{i+1} = \mathbb{F}_p f_1 + \ldots + \mathbb{F}_p f_s + Aug(\mathbb{F}_p[Q_{i+1}])(\wedge^j_{\mathbb{F}_p} M_{i+1})$$

Let f be an element from $\wedge^j_{\mathbb{F}_p} M_{i+1}$. As M_{i+1} is a localisation of M_i for some large positive integer β we have $\alpha_i^\beta f \in \wedge^j_{\mathbb{F}_p} M_i$. Then

$$\alpha_i^\beta f = z_1 f_1 + \ldots + z_s f_s + w$$

where $z_i \in \mathbb{F}_p$ and $w \in Aug(\mathbb{F}_p[Q_i])(\wedge^j M_i)$ and hence

$$f = \alpha_i^{-\beta}(z_1 f_1 + \ldots + z_s f_s + w) \in z_1 f_1 + \ldots + z_s f_s + Aug(\mathbb{F}_p[Q_{i+1}])(\wedge^j_{\mathbb{F}_p} M_{i+1}),$$

as required.

If $p = 2$ consider the commutative diagram

$$\begin{array}{ccc} \varphi_i : \wedge^j_{\mathbb{F}_2} M_i & \to & \widetilde{S}^j(M_i) \\ \downarrow & & \downarrow \\ \varphi_{i+1} : \wedge^j_{\mathbb{F}_2} M_{i+1} & \to & \widetilde{S}^j(M_{i+1}) \end{array}$$

Then there exist elements $f_1, \ldots, f_s \in \wedge^j_{\mathbb{F}_2} M_i$ such that

$$\varphi_i(\wedge^j_{\mathbb{F}_2} M_i) \subseteq \varphi_i(\mathbb{F}_2 f_1 + \ldots + \mathbb{F}_2 f_s) + Aug(\mathbb{Z}[Q_i])(\widetilde{S}^j(M_i))$$

An obvious modification of the first part of the proof gives

$$\varphi_{i+1}(\wedge^j_{\mathbb{F}_2} M_{i+1}) \subseteq \varphi_{i+1}(\mathbb{F}_2 f_1 + \ldots + \mathbb{F}_2 f_s) + Aug(\mathbb{F}_2[Q_{i+1}])(\widetilde{S}^j(M_{i+1}))$$

□

Thus to prove the main theorem it is sufficient to work with $M = M_t$ and $Q = Q_t$, so we can assume that

1. M is a cyclic $\mathbb{F}_p[Q]$-module
2. M is n-tame where n is the torsion free rank of Q
3. As t could be chosen arbitrary large we can assume that $n+1$ is a multiple of the order of the torsion part of Q.
4. If p is odd $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} \wedge^j M$ is finite for all j. If $p = 2$ the image of $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} (\wedge^j_{\mathbb{Z}} M)$ in $\mathbb{F}_2 \otimes_{\mathbb{F}_2[Q]} \widetilde{S}^j(M)$ is finite for all j.

Then by [5, Proposition 4.3]

$$\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} (\otimes^{n+1}_{\mathbb{F}_p} M) \text{ is infinite}$$

At the same time as shown in [5, Section 5.4] the forth property of M together with the n-tameness of M implies that $\mathbb{F}_p \otimes_{\mathbb{F}_p[Q]} (\otimes^{n+1}_{\mathbb{F}_p} M)$ is finite, a contradiction.

Acknowledgements

The author is supported by a research grant 98/00482-3 from FAPESP, Brazil.

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