On the combinatorics of the Fibonacci Numbers

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Abstract

In this paper following some ideas introduced by Andrews in [4] and results given by Santos in [7] we give new formula and combinatorial interpretation for the Fibonacci Numbers.

1 Introduction

In [9] Lucy Slater presented a list of 130 q-series identities including the 3 listed below that are the ones of numbers 18, 14 and 20 respectively been the first two the famous Rogers Ramanujan identities.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n}) \tag{1.3}$$

where

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}),$$

n a nonnegative integer.

To describe an idea about how to look for a combinatorial interpretation for identities of this type we use (1.1) as a prototypical example.

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In [4] Andrews considers in as simple a manner as possible a two-variable generalization $f(q,t)$ that has the following properties:

(i) $f(q,t)=\sum_{n=0}^{\infty} P_n(q)t^n$, where $P_n(q)$ are polynomials.

(ii) $\lim_{n\to\infty} P_n(q)=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$

(iii) $f(q,t)$ satisfies a first-order nonhomogeneous q-difference equation.

By stating things this in this generality no one could guess what to do next. However in practice $f(q,t)$ is generally easily produced. A parameter t is inserted into (1.1) in such a way that one essentially obtains the $(n+1)st$ term from the nth term by replacing t by tq. In this instance

$$f(q,t)=\sum_{n=0}^{\infty}\frac{t^{2n}q^{n^2}}{(1-t)(1-tq)\dots(1-tq^n)}$$

The factor $(1-t)$ in the denominator is essential to guarantee (ii).

We now check that our three conditions have been verified. First

$$\begin{aligned} f(q,t) &= \frac{1}{1-t}+\sum_{n=1}^{\infty}\frac{t^{2n}q^{n^2}}{(t;q)_{n+1}} \\ &= \frac{1}{1-t}+\sum_{n=1}^{\infty}\frac{t^{2n+2}q^{n^2+2n+1}}{(1-t)(tq;q)_{n+1}} \\ &= \frac{1}{1-t}+\frac{t^2q}{1-t}f(q;tq) \end{aligned}$$

or

$$(1-t)f(q,t)=1+t^2qf(q,tq).$$

Thus (iii) is satisfied. Next we note

$$\begin{aligned} f(q,t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{2n}q^{n^2}t^m \begin{bmatrix} n+m \\ m \end{bmatrix} \qquad \text{(by Andrews [1], Theorem 3.3 p. 36]} \\ &= \sum_{n=0}^{\infty} t^N \sum_{0\le 2n\le N} q^{n^2}\begin{bmatrix} N-n \\ n \end{bmatrix}. \end{aligned} \tag{1.4}$$

Hence we have (i) since by (1.4)

$$P_N(q) = \sum_{0 \le 2n \le N} q^{n^2} \left[\begin{array}{c} N-n \\ n \end{array} \right] \tag{1.5}$$

For (ii) we may use Abel's lemma (Whittaker and Watson [[11], p.57] or Andrews [[5], p. 190]:

$$\begin{aligned} \lim_{n\to\infty} P_n(q) &= \lim_{t\to 1^-} (1-t) f(q,t) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \end{aligned}$$

A natural question at this point may be: So what? We started with (1.1) and obtained $f(q,t)$; however we appear not to have anything new of any real significance. We might, of course, attempt a justification by pointing out that the polynomials $P_n(q)$, in this case, were important in the treatment of Regime I of the Hard Hexagon Model been $f(q,t)$ the generating function for those polynomials.

In this paper we are going to explore the combinatorics of $P_n(q)$ obtained from the $f(q,t)$ associated with identity (1.3) but it is important to say that $f(q,t)$ is of interest for other values of t besides 1. In particular

$$f(q,-1) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \frac{1}{2} f_0(q),$$

where $f_0(q)$ is one of Ramanujan's fifth-order mock theta functions (cf. Watson [10]).

Before starting our study of another $P_n(q)$ from which we got new combinatorial interpretation for the Fibonacci numbers we have to mention that by analyzing these functions at other points, as mentioned above, one can not only give new proofs for the identities in Slater's list but also find new ones with the help of Abel's lemma, Bailey's lemma and Jacobi's Triple Product together with a symbolic algebra package (cf. Santos [7]).

A fuller discussion of the combinatorics of this construction is given by Andrews [4].

2 Some definitions for our proof

When dealing with the expression

$$(1 + x + x^2)^n \tag{2.1}$$

we call the coefficients of x^j in the expanded form of (2.1) of trinomial coefficients.

It is easy to show that if

$$(1 + x + x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{2.2}$$

then

$$\begin{aligned} \binom{n}{j}_2 &= \sum_{h \geq 0} \frac{n!}{h!(h+j)!(n-j-2h)!} && (2.3) \\ &= \sum_{h \geq 0} (-1)^h \binom{n}{h} \binom{2n-2h}{n-j-h} && (2.4) \end{aligned}$$

and also

$$\binom{n}{j}_2 = \binom{n}{-j}_2 \tag{2.5}$$

and

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2 \tag{2.6}$$

The following expressions (Andrews & Baxter [6]) are q-analogs of the trinomial coefficient in the same way that the Gaussian polynomial is a q-analog of the binomial coefficient, that is, the limit of each one of them when q approaches 1 is equal to the trinomial coefficient given by (2.3) and (2.4).

$$
\begin{aligned}
T_0(m,A,q) &= \sum_{j=0}^{m}(-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}, && (2.7)\\
T_1(m,A,q) &= \sum_{j=0}^{m}(-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}, && (2.8)\\
T_1(m,A,q) &= T_1(m-1,A,q) + q^{m+A}T_0(m-1,A+1,q) \\
&\quad + q^{m-A}T_0(m-1,A-1,q) && (2.9)\\
T_0(m,A,q) &= T_0(m-1,A-1,q) + q^{m+A}T_1(m-1,A,q) \\
&\quad + q^{2m+2A}T_0(m-1,A+1,q) && (2.10)
\end{aligned}
$$

We need also (Andrews & Baxter [6]) the identity:

$$
\begin{aligned}
&T_1(m,A,q) - q^{m-A}T_0(m,A,q) - T_1(m,A+1,q) \\
&+q^{m+A+1}T_0(m,A+1,q) = 0 \qquad (2.11)
\end{aligned}
$$

If we define

$$U(m,A,q) = T_0(m,A,q) + T_0(m,A+1,q) \qquad (2.12)$$

then the following two results (Andrews [3], pp. 13-15) are true:

$$
\begin{aligned}
&U(m,A,q) = (1+q^{2m-1})U(m-1,A,q) \\
&+q^{m-A}T_1(m-1,A-1,q) + q^{m+A+1}T_1(m-1,A+2,q) \qquad (2.13)\\
&U(m,A,q) = (1+q+q^{2m-1})U(m-1,A,q) \\
&-qU(m-2,A,q) + q^{2m-2A}T_0(m-2,A-2,q) \\
&+q^{2m+2A+2}T_0(m-2,A+3,q) \qquad (2.14)
\end{aligned}
$$

The following limiting value of our q-analog (2.12) is necessary:

$$\lim_{m\to\infty} U(m,A,q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \quad \text{([3], eq. 4.16)} \tag{2.15}$$

3 The Fibonacci Numbers from a sequence $P_n(q)$

We start by considering the function $f(q,t)$ associated with equation (1.3) that is

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n}).$$

In Santos [7] there is a list of functions $f(q,t)$ associated with 74 of the 130 identities given by Slater together with a conjecture of an explicit formula for $P_n(q)$ in terms of q-analogs of binomial or trinomial coefficients.

$$\begin{aligned}
f(q,t) &= \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(t;q^2)_{n+1}(-tq^2;q^2)_n} \qquad (3.1)\\
&= \frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{t^n q^{n^2}}{(t;q^2)_{n+1}(-tq^2;q^2)_n}\\
&= \frac{1}{1-t} + \frac{1}{(1-t)(1+tq^2)} \sum_{n=1}^{\infty} \frac{t^n q^{n^2}}{(tq^2;q^2)_n(-tq^4;q^2)_{n-1}}\\
&= \frac{1}{1-t} + \frac{1}{(1-t)(1+tq^2)} \sum_{n=0}^{\infty} \frac{t^{n+1} q^{n^2+2n+1}}{(tq^2;q^2)_{n+1}(-tq^4;q^2)_n}\\
&= \frac{1}{1-t} + \frac{tq}{(1-t)(1+tq^2)} \sum_{n=0}^{\infty} \frac{(tq^2)^n q^{n^2}}{(tq^2;q^2)_{n+1}(-tq^4;q^2)_n}\\
&= \frac{1}{1-t} + \frac{tq}{(1-t)(1+tq^2)} f(q,tq^2).
\end{aligned}$$

From this we have

$$(1-t)(1+tq^2)f(q,t) = 1 + tq^2 + tqf(q,tq^2).$$

In order to obtain a recurrence relation from this functional equation we make the following substitution:

$$f(q,t)=\sum_{n=0}^{\infty}P_n(q)t^n.$$

By equating coefficients of the same power in both sides we get the recurrence

$$\begin{aligned}&P_0(q)=1;\ \ P_1(q)=1+q\\&P_n(q)=(1-q^2+q^{2n-1})P_{n-1}(q)+q^2P_{n-2}(q).\end{aligned}\tag{3.2}$$

In [7] Santos gave the following explicit formula as a conjecture for $P_n(q)$.

$$\begin{aligned}C(n)\ &=\ \sum_{j=-\infty}^{\infty}q^{10j^2+j}U(n,5j)\\&-\ \sum_{j=-\infty}^{\infty}q^{10j^2+11j+3}U(n,5j+2).\end{aligned}\tag{3.3}$$

We note that having proved this conjecture we can get identify (1.3) by taking the $\lim\limits_{n\to\infty}C(n)$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}\ &=\ \lim_{t\to1^-}(1-t)f(q,t)=\\ \lim_{n\to\infty}C(n)\ &=\ \lim_{n\to\infty}\left[\sum_{j=-\infty}^{\infty}q^{10j^2+j}U(n,5j)-\sum_{j=-\infty}^{\infty}q^{10^2j+11j+3}U((n,5j+2)\right]\\ &\overset{\text{by }(2.15)}{=}\ \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left[\sum_{j=-\infty}^{\infty}q^{10^2j+j}-\sum_{j=-\infty}^{\infty}q^{10^2j+11j+3}\right]\\ &=\ \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{5n^2+n}{2}}\\ &=\ \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\pi(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n})\end{aligned}$$

where we have used Jacobi's Triple Product in the last equality.

It is easy to see that when we replace q by 1 in (3.2) we get

$$P_0(1) = 1 \ ; \ P_1(1) = 2 \ ; \ P_n(1) = P_{n-1}(1) + P_{n-2}(1)$$

which is the Fibonacci sequence

$$F_0 = 1, \ F_2 = 2, \ F_n = F_{n-1} + F_{n-2}, \ n \geq 2.$$

Now from (3.3) we can get a new formula for the Fibonacci sequence by taking the $\lim_{q \to 1} C(n)$ once we have proved that the conjecture is correct. This is done in the next theorem.

Theorem 3.1. The recurrence

$$P_m = (1 - q^2 + q^{2m-1}) P_{m-1} + q^2 P_{m-2}$$

holds for the expression below which is given by (3.3)

$$C(m) = \sum_{j=-\infty}^{\infty} q^{10j^2+j} U(m, 5j) - \sum_{j=-\infty}^{\infty} q^{10j^2+11j+3} U(m, 5j+2).$$

Proof: If (3.2) were true for $U(m, A)$ we would be done. But since this is not the case we consider the following expression:

$$U(m, A) - (1 - q^2 + q^{2m-1}) U(m-1, A) - q^2 U(m-2, A). \tag{3.4}$$

Now, replacing here $U(m, A)$ by its definition given in (2.12) together with identity (2.13), we have:

$$U(m, A) - (1 + q^{2m-1}) U(m-1, A) + q^2 U(m-1, A) -$$
$$q^2 U(m-2, A) = q^{m-A} T_1(m-1, A-1) + q^{m+A+1} T_1(m-1, A+2) +$$
$$q^2 T_0(m-1, A) + q^2 T_0(m-1, A+1) - q^2 (T_0(m-2, A) + T_0(m-2, A+1)).$$

If, on the right side of this last equation, we apply (2.9) on the 1st and 2nd terms and (2.10) on the 3rd and 4th terms, we have:

$$q^{m-A}(T_1(m-2,A-1)+q^{m+A-2}T_0(m-2,A)+q^{m-A}T_0(m-2,A-2))+$$
$$q^{m+A+1}(T_1(m-2,A+2)+q^{m+A+1}T_0(m-2,A+3)+q^{m-A-3}T_0(m-2,A+1))+$$
$$q^2(T_0(m-2,A+1)+q^{m-1-A}T_1(m-2,A)+q^{2m-2A-2}T_0(m-2,A-1))+$$
$$q^2(T_0(m-2,A)+q^{m+A}T_1(m-2,A+1)+q^{2m+2A}T_0(m-2,A+2))-$$
$$q^2T_0(m-2,A)-q^2T_0(m-2,A+1).$$

After two easy cancellations we have:

$$q^{m-A}T_1(m-2,A-1)+q^{2m-2}T_0(m-2,A)+q^{2m-2A}T_0(m-2,A-2)+$$
$$q^{m+A+1}T_1(m-2,A+2)+q^{2m+2A+2}T_0(m-2,A+3)+q^{2m-2}T_0(m-2,A+1)+$$
$$q^{m-A+1}T_1(m-2,A)+q^{2m-2A}T_0(m-2,A-1)+$$
$$q^{m+A+2}T_1(m-2,A+1)+q^{2m+2A+2}T_0(m-2,A+2).$$

We have now in order to complete our proof to show that this expression, when replaced in (3.3), is identically zero. After the substitution we have:

$$\sum_{j=-\infty}^{\infty} q^{10j^2-4j+m}T_1(m-2,5j-1)+\sum_{j=-\infty}^{\infty} q^{10j^2+j+2m-2}T_0(m-2,5j)+$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2-9j+2m}T_0(m-2,5j-2)+\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+2)+$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+2}T_0(m-2,5j+3)+\sum_{j=-\infty}^{\infty} q^{10j^2-9j+2m}T_0(m-2,5j+1)+$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+2}T_1(m-2,5j)+\sum_{j=-\infty}^{\infty} q^{10j^2-9j+2m}T_0(m-2,5j-1)+$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+2}T_1(m-2,5j+1)+\sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+2}T_0(m-2,5j+2)-$$

$$\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+1) - \sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+1}T_0(m-2,5j+2) -$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+j+2m-1}T_0(m-2,5j) - \sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+6}T_1(m-2,5j+4) -$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+21j+2m+9}T_0(m-2,5j+5) - \sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+1}T_0(m-2,5j+3) -$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+2}T_1(m-2,5j+2) - \sum_{j=-\infty}^{\infty} q^{10j^2+j+2m-1}T_0(m-2,5j+1) -$$
$$\sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+7}T_1(m-2,5j+3) - \sum_{j=-\infty}^{\infty} q^{10j^2+21j+2m+9}T_0(m-2,5j+4).$$

The 1st, 2nd and 3nd sums are canceled by the 14th, 15th and 16th when we replace j by $j+1$ in the last three sums. Now, putting together the 4th with 17th, the 10th with 12th, the 6th with 18th, and the 9th with 11th we have:

$$\begin{aligned}
& (1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+2) \\
- \; & (1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+1}T_0(m-2,5j+2) \\
+ \; & (1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+j+2m-2}T_0(m-2,5j+1) \\
- \; & (1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+1) \\
+ \; & \sum_{j=-\infty}^{\infty} q^{10j^2+11j+2m+2}T_0(m-2,5j+3) + \sum_{j=-\infty}^{\infty} q^{10j^2-4j+m+1}T_1(m-2,5j) \\
- \; & \sum_{j=-\infty}^{\infty} q^{10j^2+j+2m-1}T_0(m-2,5j) - \sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+7}T_1(m-2,5j+3) \\
+ \; & \sum_{j=-\infty}^{\infty} q^{10j^2-9j+2m}T_0(m-2,5j-1) - \sum_{j=-\infty}^{\infty} q^{10j^2+21j+2m+9}T_0(m-2,5j+4).
\end{aligned}$$

Now the 2nd plus the 3rd, the 5th plus the 8th, the 6th plus the 7th are, respectively:

$$(1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}(q^{m-3-5j}T_0(m-2,5j+1)-q^{m+5j}T_0(m-2,5j+2)),$$

$$-\sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+7}(T_1(m-2,5j+3)-q^{m-5j-5}T_0(m-2,5j+3)),$$

$$\sum_{j=-\infty}^{\infty} q^{10j^2-4j+m+1}(T_1(m-2,5j)-q^{5j+m-2}T_0(m-2,5j)).$$

Applying now (2.11) in all these expressions with A replaced by $5j+1$ in the first, by $5j+3$ in the second, by $-5j$ in the third, and in all m replaced by $m-2$ we have:

$$(1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}(T_1(m-2,5j+1)-T_1(m-2,5j+2))$$
$$+(1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+2)$$
$$-(1-q)\sum_{j=-\infty}^{\infty} q^{10j^2+6j+m+1}T_1(m-2,5j+1)$$
$$-\sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+7}(T_1(m-2,5j+4)-q^{m+5j+2}T_0(m-2,5j+4))$$
$$+\sum_{j=-\infty}^{\infty} q^{10j^2-4j+m+1}(T_1(m-2,-5j+1)-q^{m-1-5j}T_0(m-2,-5j+1))$$
$$+\sum_{j=-\infty}^{\infty} q^{10j^2-9j+2m}T_0(m-2,5j-1)-\sum_{j=-\infty}^{\infty} q^{10j^2+21j+2m+9}T_0(m-2,5j+4).$$

The first line cancels the second and the third. From the last three lines we are left only with:

$$-\sum_{j=-\infty}^{\infty} q^{10j^2+16j+m+7}T_1(m-2,5j+4)+\sum_{j=-\infty}^{\infty} q^{10j^2-4j+m+1}T_1(m-2,-5j+1).$$

which is equal to zero by replacing j by $j+1$ in the second and using the fact that $T_1(m,A,q) = T_1(m,-A,q)$.

$\square$

Knowing now that

$$P_n(q) = \sum_{j=-\infty}^{\infty} q^{10j^2+j} U(n,5j) - \sum_{j=-\infty}^{\infty} q^{10j^2+11j+3} U(n,5j+2) \tag{3.5}$$

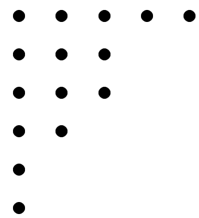
we can use (2.12) and (2.7) in order to find a new formula for the Fibonacci Numbers by taking the limite in (3.5) when q approaches 1.

$$\begin{aligned} \lim_{q\to 1} P_n(q) &= \sum_{j=-\infty}^{\infty} \left[\binom{n}{5j}_2 + \binom{n}{5j+1}_2 - \binom{n}{5j+2}_2 - \binom{n}{5j+3}_2 \right] \\ P_n(1) &= F_n. \end{aligned}$$

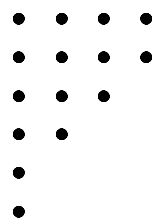
4 A new combinatorial interpretation for F_n

Definition: we say that a partition is "Frobenius even alternating" (F.E.A.) if the parity of parts on the top row reading from right to left alternates starting with even for the entire top row.

Below we have the Ferrers graph for two partitions of 15 with the corresponding Frobenius Symbol and where only the first one is F.E.A.



$$\longleftrightarrow \qquad \begin{pmatrix} 4 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$



$$\longleftrightarrow \qquad \begin{pmatrix} 3 & 2 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$

In this section we are going to prove that the coefficient of t^N in the expansion of $f(q,t)$ given by (3.1), that is $P_n(q)$ in (3.2), is the generating function for self-conjugate F.E.A. partitions with largest part $\leq N$.

Let us take (3.1) and write it in the following form:

$$\begin{aligned} f(q,t) &= \sum_{n=0}^{\infty} P_n(q) t^n \\ &= \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(t;q^2)_{n+1}(-tq^2;q^2)_n} \\ &= \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(1-t)(tq^2;q^2)_n(-tq^2;q^2)_n} \\ &= \frac{1}{(1-t)} \sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(t^2q^4;q^4)_n} \end{aligned}$$

It is easy to see that in the denominator

$$(t^2q^4;q^4)_n = (1-t^2q^4)(1-t^2q^8)\dots(1-t^2q^{4j})\dots(1-t^2q^{4n})$$

the exponent of q is a multiple of 4 and also of j where j is the position of that factor in that product. So if one divide the exponent of q by 2 the resulting number is always even and multiple of j. From this trivial observation we have that in the expansion of $\dfrac{1}{1-t^2q^{4j}}$ that is

$$1+(t^2q^{4j})^1+(t^2q^{4j})^2+\cdots+(t^2q^{4j})^i+\cdots$$

the exponent of q when divided by $2j$ is always equal to the exponent of t that is also even.

Now we can explain how to build a self-conjugate F.E.A. partition from the coefficient of t^N in $f(q,t)$.

The following example will make it clear. Let us take $t^nq^{n^2}/(t^2q^4;q^4)_n$ for $n=3$.

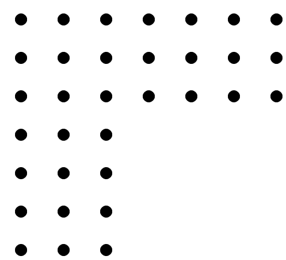
$$\begin{aligned}\frac{t^3q^9}{(t^2q^4;q^4)_3} &= \frac{t^3q^9}{(1-t^2q^4)(1-t^2q^8)(1-t^2q^{12})}\\ &= t^3q^9(1+t^2q^4+t^4q^8+t^6q^{12}+\cdots)\\ &\quad\cdot(1+t^2q^8+t^4q^{16}+t^6q^{24}+\cdots)\\ &\quad\cdot(1+t^2q^{12}+t^4q^{24}+t^6q^{36}+\cdots).\end{aligned}\tag{4.1}$$

We start by drawing a 3×3 square which is coming from q^9. The exponent of t which is 3 is the contribution of this square to the largest part of the partition we are building.

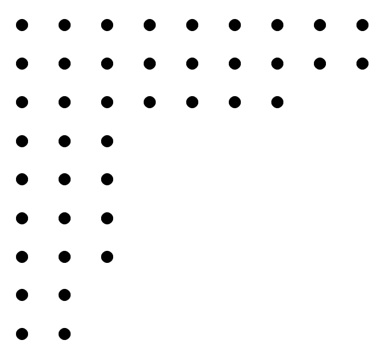
$$\begin{matrix}\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\end{matrix}$$

Let us now take the term t^4q^{24} from the third factor in (4.1). We divide the exponent of q, that is, 24 by 2 getting 12 and next divide it by 3 (it is coming from the third factor) getting 4 that is the exponent of t.

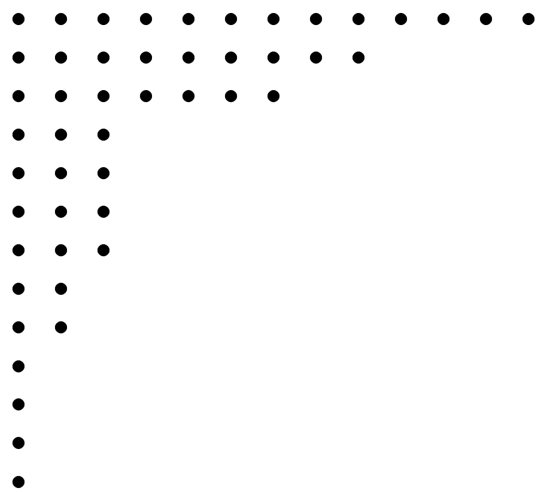
Now we place those 12 points on the right of the square as a pile of width 4 (exponent of t) and high 3 (the position of the factor). The other 12 points are placed on the symmetric position about the diagonal. The following figure shows what we get.



Let us take now, the term t^2q^8 from the second factor. We divide the exponent of q, which is 8, by 2 getting 4 and then by 2 (the position of the factor). We place those 4 points as a pile of width 2 (exponent of t) and high 2 (the position of the factor). The next figure shows the result.



Taking the term t^4q^8 from the first factor we just divide the exponent of q, which is 8, by 2 and by 1. We now place those 4 points in a pile of width 4 (exponent of t) and high 1 (position of the factor) getting the following representation:



We observe that by placing these 49 points coming from

$$t^3q^9(t^4q^{24})(t^2q^8)(t^4q^8) = t^{13}q^{49}$$

in this way we get a representation of a partition of 49 with largest part 13 that is F.E.A. and self-conjugate. It is

$$\begin{pmatrix} 12 & 7 & 4 \\ 12 & 7 & 4 \end{pmatrix}$$

So, in general, the coefficient of t^N in the expansion of

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2}}{(t^2q^4;q^4)_n}$$

is the generating function for self-conjugate partitions F.E.A. having largest part equal to N. Considering the factor $1/(1-t)$ we may conclude that $P_n(q)$, which is the coefficient of t^N in $f(q,t)$, is the generating function for self-conjugate F.E.A. partitions with largest part $\leq N$.

It is necessary to explain how to find, for a given self-conjugate F.E.A. partition, the terms from which they have been generated. This is easy. The Durfee square tell us the value of n and to find the factor we do the following: take the high of the first pile on the right of the square and its width. The width is the exponent of t, the high is the factor and the exponent of q is twice the high times the width. Repeat this for the following piles.

Recalling that $P_n(1) = F_n$ we have proved the following

Theorem 4.1. The total number of self-conjugate F.E.A. partitions with largest part $\leq N$ is equal to F_N.

Acknowledgments

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