On a new combinatorial interpretation for a theorem of Euler

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Abstract

In this paper we describe a new set of partitions that is equinumerous with the set of partitions into odd parts. A new combinatorial interpretation for the Rogers-Ramanujan identities is given as an application.

1 Introduction

By a simple geometric argument we present a bijective proof for the following theorem:

Theorem 1. The number of partitions of N into odd parts equals the number of partitions of N of the form

$$p_1 + p_2 + \cdots + p_s \qquad (p_i \geq p_{i+1})$$

in which the largest part is at least

$$sp_s + (s-1)(p_{s-1} - p_s) + (s-2)(p_{s-2} - p_{s-1}) + \cdots + 2(p_2 - p_3). \tag{1.1}$$

It is a well known result, given by Euler, that for $|q| < 1$

$$\prod_{n=1}^{\infty}(1 + q^n) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})} \tag{1.2}$$

If we denote by $p(\mathcal{D}, n)$ the number of partitions of n into distinct parts and by $p(\mathcal{O}, n)$ the number of partitions of n into odd parts we have from (1.2) that

$$\sum_{n=0}^{\infty} p(\mathcal{D}, n)q^n = \sum_{n=0}^{\infty} p(\mathcal{O}, n)q^n$$

which is the following result:

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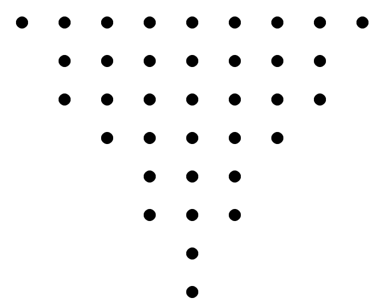
Theorem 2. (Euler) The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Sylvester gave a nice bijective proof for this result that can be explained by modifying the Ferrers graph so that the central dots in each part all lie in the same column. To illustrate this we take the following partitions of 36 into odd parts $9+7+7+5+3+3+1+1$.



(figure 1.1)

Now we draw a path starting at the bottom of the central column up to the top and turn right as shown in figure (1.2). The number of dots in this path, that is 12, represent the first part in our partition into distinct parts. To get the second part we start, now, at the bottom of the first column on the left of the central column and at the top turn left.

The number of dots, 9, in this path is the second part. To get the third part, 7, we repeat the process now starting at the bottom of the first column on the right of the central column turning right on the second row. The figure (1.2) shows how to get the remaining parts 4, 3, 1, by alternating sides and repeating this process.

We have to show how to reverse this process in order to get a bijection. It is easy to see that if we have a partition into s distinct parts then when s is odd the smallest part corresponds to a column of dots on the right and when s is even the smallest part corresponds to a row of dots on the left.

We have to observe also that each time we represent a part by an angle of dots on the right, the next larger part is represented by an angle on the left with a column of dots that is one longer than that of the column that has just been placed and that each time we represent a part by an angle of dots on the left, the next larger part is represented by an angle on the right with a row of dots that is one longer than that of the row that has just been placed.

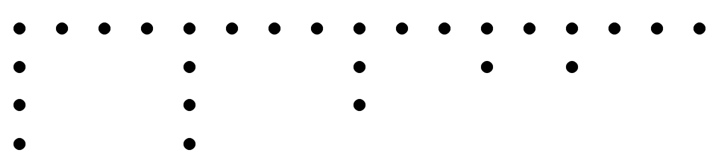
By doing this the bijection is established.

There is another bijection, given by Glaisher in which to go from the set of partitions into odd parts to the set of partitions into distinct parts we represent as a sum of powers of 2 the number of times each odd part appears multiplying each of those powers by the odd number being counted. It is not difficult to reserve this process and the following example shows how for one particular partition of 96.

$$\begin{aligned} & 15+15+15+11+11+7+5+5+5+3+1+1+1+1 \\ = \; & 3.15+2.11+1.7+3.5+1.3+4.1 \\ = \; & (2^1+2^0).15+2^1.11+2^0.7+(2^1+2^0).5+2^0.3+2^2.1 \\ = \; & 2^1.15+2^0.15+2^1.11+2^0.7+2^1.5+2^0.5+2^0.3+2^2.1 \\ = \; & 30+15+22+7+10+5+3+4 \\ = \; & 30+22+15+10+7+5+4+3 \end{aligned}$$

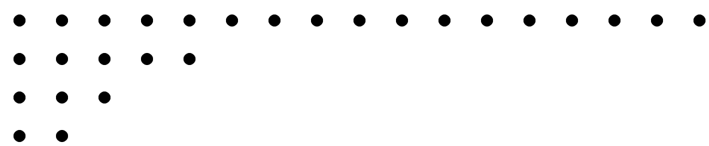
2 The proof of our theorem

Proof: In order to give a bijection between those two sets of partitions we start with a partition into odd parts and represent each odd part by a symmetric right angle of dots displaying them one after the other from left to right. The figure (2.1) shows this for the partition $7+7+5+3+3+1+1$ of 27.



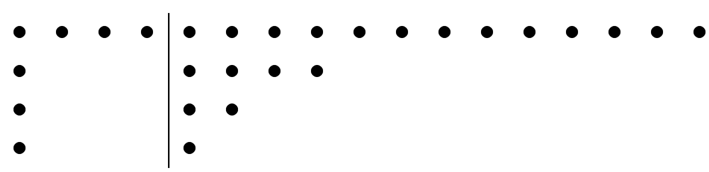
(figure 2.1)

Now we just move those points not on the first row nor on the first column to get a graph as the one in figure (2.2) and the parts of the new partition are given by the numbers of dots in each row that are in this case $17+5+3+2$.



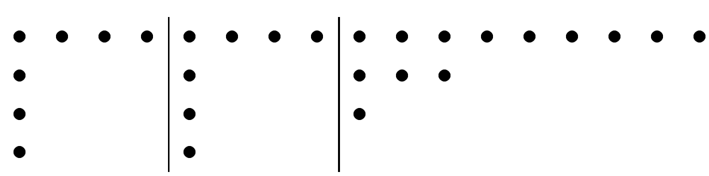
(figure 2.2)

We have to explain how to reverse this process to see that this is a bijection. Given a partition satisfying (1.1) as the one in figure (2.2) we have to move all the dots not on the first row nor on the first column to the right to a position so that the number of dots on the left forms a symmetric right angle. These dots that are on the left of the vertical line in figure (2.3) represent the first odd part (7 in this case).



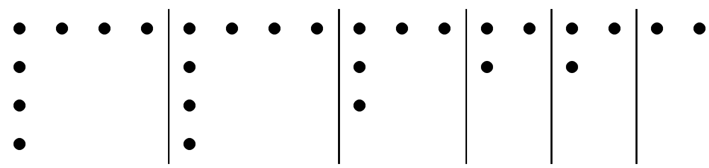
(figure 2.3)

Now we have to apply the same idea only to those dots on the right of the vertical line of figure (2.3). Figure (2.4) shows what we get after doing that. The second odd part is the number of dots between the two vertical lines in figure (2.4) which is 7.



(figure 2.4)

In figure (2.5) we have what we get for the next three odd parts, 5, 3 and 3, following the same procedure.



(figure 2.5)

When after the last vertical line there are dots only on the first row each one of them is a part. For the example we are using we get the partition 7+7+5+3+3+1+1 and the bijection is established.

□

In the table below we list the partitions of 10 into distinct parts, odd parts and the ones just described by the bijection given above.

distinct parts	odd parts	as in theorem 1
6+4	9+1	6+1+1+1+1
7+3	7+1+1+1	7+1+1+1
5+4+1	7+3	6+2+1+1
8+2	5+1+1+1+1+1	8+1+1
6+3+1	5+3+1+1	7+2+1
4+3+2+1	5+5	6+2+2
5+3+2	3+3+3+1	7+3
7+2+1	3+3+1+1+1+1	8+2
9+1	3+1+1+1+1+1+1+1	9+1
10	1+1+1+1+1+1+1+1+1+1	10

It is important to mention that the idea used here to prove theorem 1 can be applied not only for odd numbers, that are congruent to 1 modulo 2, but for any congruence class modulo any integer.

A direct application of theorem 1 can give us new combinatorial interpretations for many of the identities that are in Slater's list [4]. We explain, next, how to apply it to the Rogers-Ramanujan identities that are, in its analytic form, given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

where $(q;q)_n = (1-q)(1-q^2)\dots(1-q^n)$; $|q| < 1$.

We recall that to explain combinatorialy the identity

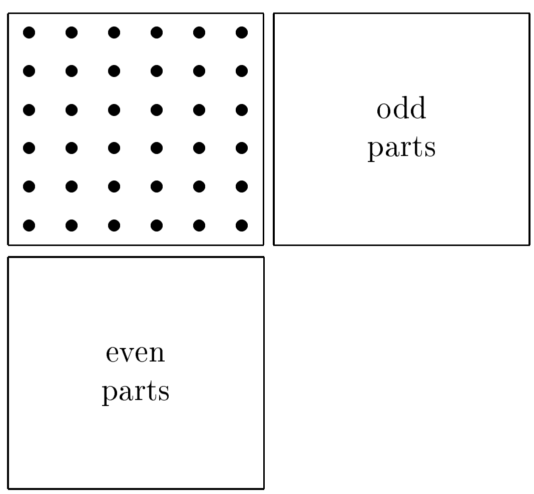
$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

one has just to remember that the number of partitions into at most n parts equals the number of partitions in which no part is greater than n and considering the Durfee squares add over all possible sides of the squares. In [2] one can find this in great details.

By using our Theorem 1 we can do something similar by considering the odd and even powers of q in the factors of $(q;q)_n$ and building a partition in a way that can be better explained by an example. Let's take $n = 6$ and consider the term $q^{6^2}/(q;q)_6$. We write it as

$$\frac{q^{6^2}}{(q;q)_6} = \frac{q^{36}}{(q;q^2)_3(q^2;q^2)_3} = \frac{q^{36}}{(1-q)(1-q^3)(1-q^5)(1-q^2)(1-q^4)(1-q^6)} =$$
$$q^{36}(1+q+q^2+\cdots)(1+q^3+q^6+q^9+\cdots)(1+q^5+q^{10}+q^{15}+\cdots)$$
$$(1+q^2+q^4+q^6+\cdots)(1+q^4+q^8+q^{12}+\cdots)(1+q^6+q^{12}+q^{18}+\cdots)$$

and consider the figure



(figure 2.6)

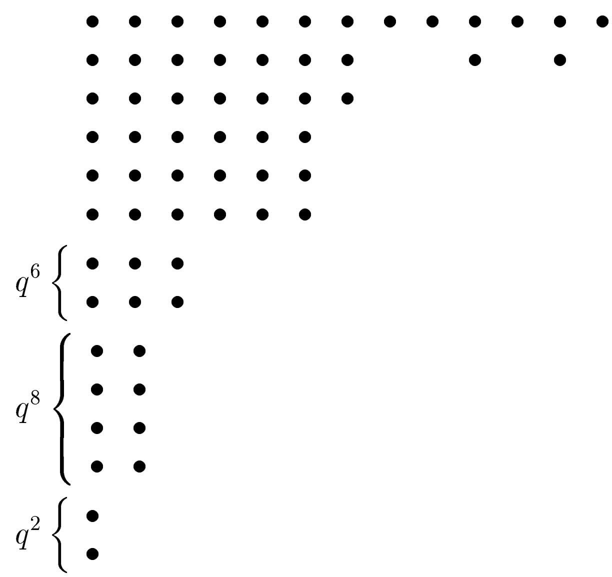
where we have a square with 36 dots, the odd parts are been placed to the right of this square by the way described in theorem 1 and the even ones below that square in the following way:

A contribution from the third factor $(1+q^6+q^{12}+q^{18}+\cdots)$ like q^6 is placed by dividing 6 by 3 and placing it in 3 columns of 2 dots as shown in figure (2.7).

A contribution from the second factor $(1+q^4+q^8+q^{12}+\cdots)$ like q^8 is placed by dividing 8 by 2 and placing it in 2 columns of 4 as shown in figure (2.7).

A contribution from the first factor $(1+q^2+q^4+q^6+q^8+\cdots)$ like q^2 is placed by dividing 2 by 1 and placing it in just 1 column of 2 as shown in figure (2.7).

In general we divide the exponent by the position "j" of the factor from which the term was taken placing the result in j columns.



(figure 2.7)

It is not difficult to see that knowing one distribution for even powers one can tell immediately from which factors and powers they came from.

On the right of the square in figure (2.7) we have the contribution q^5 from $(1+q^5+q^{10}+\cdots)$, the q^6 from $(1+q^3+q^6+\cdots)$ and no ones by taking 1 in $(1+q+q^2+\cdots)$.

Now by observing the resulting figure and using the Frobenius Symbol

$$\begin{pmatrix} s-1+a_1 & s-2+a_2 & s-3+a_3 & \cdots & a_s \\ b_1 & b_2 & b_3 & \cdots & b_s \end{pmatrix}$$

one can read the following theorem in which we have, for the case of the second identity, to add one more line of dots below the square because of the exponent n^2+n.

Theorem 3. The number of partitions of n into parts that are $\equiv \pm 1 \bmod 5 (\pm 2 \bmod 5)$ equals the number of partitions of n in which the numbers on the bottom row of the Frobenius Symbol reading from right to left alternates beginning with even (odd) and having as its first $\lfloor \frac{s}{2} \rfloor$ values the sequence $0, 1, 2, \ldots, \lfloor \frac{s}{2} \rfloor (1, 2, \ldots, \lfloor \frac{s}{2} \rfloor + 1)$, the top row has for its last $\lfloor \frac{s-1}{2} \rfloor$ values the sequence $0, 1, 2, \ldots, \lfloor \frac{s-1}{2} \rfloor$ and $a_1, a_2, \ldots, a_{\lfloor \frac{s+1}{2} \rfloor}$ satisfies the restriction (1.1) given in theorem 1.

For the first R–R identity the 14 partitions of 15 are listed below.

$$\begin{pmatrix} 14 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 11 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 9 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 7 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 8 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 0 \\ 6 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 1 & 0 \\ 4 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 & 0 \\ 4 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 3 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

References

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[5] J.J. Sylvester, 1882. A constructive theory of partitions, arranged in three acts, an interact, and an exodion. American Journal of Mathematics 5: 251-330. Reprinted in The Collected Mathematical Papers of J.J. Sylvester. Vol. 4, pp. 1-81. Cambridge University Press, 1912. Preprint. New York: Chelsea, 1973.