Reproductive Weak solutions for generalized Boussinesq models in exterior domains[1],[2]

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Abstract. We established the existence of reprodutive weak solutions of a generalized Boussinesq model for thermally driven convection in exterior domains.

1 Introduction

In this work we study the existence of reprodutive weak solutions for the equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are temperature dependent. This study is done in a exterior domain $\Omega \subseteq \mathbb{R}^3$, in a the time interval $[0, \infty)$. The problem we have interest is (see [1], for instance):

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \mathrm{div}(\nu(\theta)\nabla \mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\theta\mathbf{g} + \nabla p &= 0, \text{ in } \widehat{\Omega} \\
\mathrm{div}\ \mathbf{u} &= 0, \text{ in } \widehat{\Omega} \qquad \text{(P1)} \\
\frac{\partial \theta}{\partial t} - \mathrm{div}(k(\theta)\nabla\theta) + \mathbf{u}\cdot\nabla\theta &= 0, \text{ in } \widehat{\Omega}
\end{aligned}
$$

where,

[1]**Supported by research grant 1997/3711-0, FAPESP-BRAZIL.**
[2]**Supported by research grant PEI B-007, Universidad de Antofagasta, Chile.**
[3]**Supported by research grant 300116/93-4, CNPq-BRAZIL.**

- $\widehat{\Omega} = \Omega \times (0, \infty)$, $\mathbf{u}(x,t) \in I\!\!R^3$ denotes the velocity of the fluid at point $x \in \Omega$ and time $t \in [0, \infty)$;

- $\theta(x,t) \in I\!\!R,$ denotes the temperatura;

- $p(x,t) \in I\!\!R$ denotes the hydrostatic pressure;

- $\mathbf{g}(t,x) \ \in I\!\!R^3$ is a gravitational field;

- $\nu(\cdot)$ is the kinematic viscosity;

- $k(\cdot)$ is thermal conductivity;

- $\alpha > 0$ is a positive constant associated to the coefficient of volume expansion.

Without loss of generality, we have considered the reference temperature as zero.

The symbols ∇, Δ and div denote the gradient, Laplacian and divergence operators, respectively.We also denote $\dfrac{\partial \mathbf{u}}{\partial t}$ by $\mathbf{u}_t$. The i^{th} component of $\mathbf{u}\cdot\nabla\mathbf{u}$ is given by $[(\mathbf{u}\cdot\nabla)\mathbf{u}]_i = \sum_j u_j \dfrac{\partial u_i}{\partial x_j}$ and $\mathbf{u}\cdot\nabla\theta = \sum_j u_j \dfrac{\partial \theta}{\partial x_j}$.

The first equation in problem (P1) corresponds to the balance of linear momentum; the second one says that fluid is incompressible and the third equation is the balance of temperature.

We assume that $\Omega = K^c$, where $K \subseteq I\!\!R^3$ is compact, and its boundary ∂K is de class C^2. We observe that $\Gamma = \partial\Omega = \partial K$.

Thus, the boundary conditions and conditions at infinity are

$$\begin{aligned} \mathbf{u}(x,t) &= 0, \ x \in \Gamma, \ t \in [0,\infty) && (1)\\ \theta(x,t) &= \theta_0 > 0, \ x \in \Gamma, \ t \in [0,\infty) && (2)\\ \lim_{|x|\to\infty} \mathbf{u}(x,t) &= 0, \ \lim_{|x|\to\infty} \theta(x,t) = 0. && (3) \end{aligned}$$

Let $\{\mathbf{u}, \theta\}$ be a weak solution of problem (P1) (the exact definition will be given later on). If the functions $\mathbf{u}$ and θ satisfy

$$\mathbf{u}(x,0) = \mathbf{u}(x,T), \ \theta(x,0) = \theta(x,T), \qquad (4)$$

then we say that the system has the *reproductive property* (see Kaniel and Shinbrot [6] for the case of Navier-Stokes equations). We observe that the above property is a generalization of the notion of periodicity. We will show that problem (P1) has always a reproductive weak solution.

Problem (P1) was considered by Lorca and Boldrini [10], [11] in a bounded domain with Dirichlet's conditions. The reduced model, where ν and k are positive constants, was discussed by many authors, see for instance, Korenev [7], Rojas-Medar and Lorca [17], [18] (in a bounded domain) and Hishida [5], $\overline{O}$eda [14], [15] (in a exterior domain). The related stationary model was considered by Notte-Cuello and Rojas-Medar [13].

2 Preliminaries

The functions in this paper are either $\mathbb{R}$ or $\mathbb{R}^3$ valued, and we will not distinguish these two situations in our notation, since that will be clear from the context.

The extending domain method was introduced by Ladyzhenskaya [8] to study the Navier-Stokes equations in unbounded domains. As observed by Heywood [3] the method is useful in certain class of unbounded domains. Certainly, our domain is in this class. The basic idea is the following: The exterior domain Ω can be approximated by interior domains $\Omega_m = B_m \cap \Omega$, where B_m is a ball with radius m and center at 0, as $m \to \infty$.

In each interior domain Ω_m, we will prove the existence of a weak solution, by using the Galerkin method together with the Brouwer's fixed point theorem as in Heywood [3]. Next, by using the estimates given in Ladyzhenskaya's book [8] together with diagonal argument and Rellich's compactness theorem, we obtain the desirable weak solution to problem (P1) and conditions (1) through (4).

Let D denote Ω or Ω_m, $\widehat{D}' = D \times [0,T]$ and $\widehat{D \cup \Gamma} = (D \cup \Gamma) \times [0,T]$. And consider the following notation

$$
\begin{aligned}
W^{r,p}(D) &= \{\mathbf{u}\ ;\ D^\alpha \mathbf{u} \in L^p(D), |\alpha| \leq r\}, \\
W_0^{r,p}(D) &= \text{Completion of } C_0^k(D) \text{ in } W^{r,p}(D), \\
C_{0,\sigma}^\infty(D) &= \{\mathbf{v} \in C_0^\infty(D)\ ;\ \operatorname{div} \mathbf{v} = 0\}, \\
J(D) &= \text{Completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\nabla \phi\|, \\
H(D) &= \text{ Completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\phi\|,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{W}_\sigma(\widehat{D}) &= \{\varphi \in C_0^\infty(\widehat{D}'); \mathrm{div}\ \varphi = 0\},\\
\widehat{W}(\widehat{D}) &= \{\psi \in C_0^\infty(\widehat{D \cup \Gamma}'); \ \varphi(\Gamma) = 0\},\\
\widehat{W_{\sigma,\pi}}(\widehat{D}) &= \{\varphi \in C_{0,\sigma}^\infty(\widehat{D}); \ \varphi(x,T) = \varphi(x,0)\},\\
\widehat{W}_\pi(\widehat{D}) &= \{\psi \in \widehat{W}(\widehat{D}); \ \psi(x,T) = \psi(x,0)\},\\
L_\pi^p(0,T;J(\Omega_k)) &= \{\mathbf{u} \in L_\pi^p(0,T;J(\Omega_k)); \ \mathbf{u}(x,T) = \mathbf{u}(x,0) \ x \in \Omega_k \text{ a.e.}\},\\
L_\pi^p(0,T;H_0^1(\Omega_k)) &= \{\mathbf{w} \in L_\pi^p(0,T;H_0^1(\Omega_k)); \ \mathbf{w}(x,T) = \mathbf{w}(x,0) \ x \in \Omega_k \text{ a.e.}\},\\
L_\pi^p(0,T;L^6(\Omega_k)) &= \{\mathbf{f} \in L_\pi^p(0,T;L^6(\Omega_k)); \ \mathbf{f}(x,T) = \mathbf{f}(x,0) \ x \in \Omega_k \text{ a.e.}\}.
\end{aligned}
$$

The norm $\|\cdot\|$ is the L^2-norm and $\|\cdot\|_p$ denotes the L^p-norm for $1 \le p \le \infty$. We observe that $J(D)$ is equivalent to

$$\{\phi \in W^{1,2}(D) \ ; \ \phi|_{\partial\Omega} = 0, \ \mathrm{div}\, \phi = 0\},$$

as was proved by Heywood[4].

When $p = 2$, as it usual, we denote $W^{r,p}(D) \equiv H^r(D)$ and $W_0^{r,p}(D) \equiv H_0^r(D)$.

We make use of some inequalities with constants that depend only on the dimension and are independent of the domain (see [8] chapter I).

Lemma 1 *Suppose the space dimension is 3, with D bounded or unbounded. Then*

(a) For $\mathbf{u} \in W_0^{1,2}(D)$ (or $J(D)$ or $H_0^1(D)$), we have

$$\|\mathbf{u}\|_{L^6(D)} \le C_L \|\nabla \mathbf{u}\|_{L^2(D)}$$

where $C_L = (48)^{1/6}$.

(b) (Hölder's inequality). If each integral makes sense. Then we have

$$|((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w})| \le 3^{\frac{1}{p}+\frac{1}{r}} \|\mathbf{u}\|_{L^p(D)} \|\nabla \mathbf{v}\|_{L^q(D)} \|\mathbf{w}\|_{L^r(D)}$$

where $p, q, r > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Lemma 2 *Suppose that D is a bounded domain in $I\!R^n$ and its boundary $\partial\Omega$ is of class C^2. Let us take an orthonormal basis $\{\omega^j\}_{j=1}^\infty$ of $L^2(D)$. Then for any $\varepsilon > 0$, there exists a number N_ε such that*

$$\|\mathbf{u}\|^2_{L^2(D)} \le \sum_{j=1}^{N_\varepsilon} (\mathbf{u}, \omega^j)^2 + \varepsilon \|\mathbf{u}\|^2_{W^{1,m}} \text{ for all } \mathbf{u} \in W_0^{1,m}(D), \qquad (5)$$

where $m > \frac{2n}{n+2}$ $(n \ge 2)$, $m \ge 1$ $(n = 1)$ and N_ε is independent of $\mathbf{u}$.
The following assumptions will be needed throughout this paper.

(A1) $w_0 \subset K$ (w_0 is a neighborhood of the origin 0) and $K \subseteq B = B(0, d)$ which is a ball with radius d and center at 0.

(A2) $\partial\Omega = \Gamma = \partial K \in C^2$.

(A3) $\mathbf{g}(x)$ is a bounded and continuous vector function in $\mathbb{R}^3 \backslash w_0$. Moreover $\mathbf{g} \in L^p(\Omega)$ for $p \ge 6/5$.

We assume that the functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfy

$$0 < \nu_0(T_0) \le \nu(\tau) \le \nu_1(T_0),$$
$$0 < \kappa_0(T_0) \le \kappa(\tau) \le \kappa_1(T_0),$$

for all $\tau \in \mathbb{R}$, where

$$\nu_0(T_0) = \inf\{\nu(t); |t| \le \sup_{\partial\Omega} |T_0|\}/2, \nu_1(T_0) = \sup\{\nu(t); |t| \le \sup_{\partial\Omega} |T_0|\},$$

with analogous definitions for $\kappa_0(T_0)$ and $\kappa_1(T_0)$, and ν, κ, are continuous functions.

To transform the boundary condition on T to a homogeneous boundary condition, we introduce an auxiliary function S (see Gilbarg and Trudinger [2] pp. 137).

Lemma 3 *There exists a function S which satisfies the following properties*

(i) $S(\Gamma) = T$;

(ii) $S \in C_0^2(\mathbb{R}^3)$;

(iii) for any $\epsilon > 0$ and $p \ge 1$, we can redefine S, if necessary, such that $\|S\|_{L^p} < \epsilon$.

Now, making $\varphi = \theta - S$ we obtain

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \operatorname{div}(\nu(\varphi+S)\nabla \mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\varphi\mathbf{g} - \alpha S\mathbf{g} + \nabla p &= 0,\\
\operatorname{div}\,\mathbf{u} &= 0,\\
\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\varphi+S)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi - \operatorname{div}(\kappa(\varphi+S)\nabla S) + \mathbf{u}\cdot\nabla S &= 0,
\end{aligned}$$

in Ω, with boundary conditions

$$\begin{gathered}
\mathbf{u} = 0 \text{ and } \varphi = 0 \text{ on } \partial\Omega,\\
\lim_{|x|\to\infty} \mathbf{u}(t,x) = 0\,; \quad \lim_{|x|\to\infty} \varphi(t,x) = 0\,.
\end{gathered}$$

Definition 1 *The solution* $(\mathbf{u},\varphi) \in (L^2(0,T;J(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega))) \times (L^2(0,T;H^1_0(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega))$ *is called a reproductive weak solution of problem (P1) and conditions (1) through (4), if it satisfies*

$$\int_0^T \{(\mathbf{u},\mathbf{v}_t) + (\nu(\varphi+S)\nabla\mathbf{u},\nabla\mathbf{v}) + B(\mathbf{u},\mathbf{v},\mathbf{u}) - \alpha(\varphi\mathbf{g},\mathbf{v}) - \alpha(S\mathbf{g},\mathbf{v})\}dt = 0,$$

$$\int_0^T \{(\varphi,\psi_t) + (\kappa(\varphi+S)\nabla\varphi,\nabla\psi) + b(\mathbf{u},\psi,\varphi) + (\kappa(\varphi+S)\nabla S,\nabla\psi) + b(\mathbf{u},\psi,S)\}dt = 0\,,$$

for all $v \in \widehat{D}_{\sigma,\pi}(\widehat{\Omega})$ *and all* $\psi \in \widehat{D}_\pi(\widehat{\Omega})$. *Where*

$$B(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) = \int_\Omega \sum_{i,j=1}^3 u_j(t,x)(\partial v_i/\partial x_j)(t,x)w_i(t,x)\,dx\,,$$

$$b(\mathbf{u},\varphi,\psi) = (\mathbf{u}\cdot\nabla\varphi,\psi) = \int_\Omega \sum_{i,j=1}^3 u_j(t,x)(\partial \varphi_i/\partial x_j)(t,x)\psi_i(t,x)\,dx\,.$$

Theorem 4 *(Existence) Under Assumptions (A1), (A2) and (A3), there exists a weak reproductive solution for problem (P1) and conditions (1) through (4).*

3 Auxiliary problem.

Following the extending domain method, we first present a lemma which ensures the existence of weak solutions for interior problems in domains $\Omega_m = B_m \cap \Omega$. A interior problem, P_m, is stated as follows:

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} - \operatorname{div}\,(\mu(\eta+S)\nabla\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v} - \alpha\eta\mathbf{g} - \alpha S\mathbf{g} + \nabla p &= 0,\\
\operatorname{div}\mathbf{v} &= 0,\\
\frac{\partial \eta}{\partial t} - \operatorname{div}\,(k(\eta+S)\nabla\eta) + \mathbf{v}\cdot\nabla\eta - \operatorname{div}\,(k(\eta+S)\nabla S) + \mathbf{v}\cdot\nabla S &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= 0, \text{ on } \partial\Omega_m = \partial\Omega\cap\partial B_m,\\
\eta &= 0 \text{ on } \partial\Omega_m = \partial\Omega\cap\partial B_m,
\end{aligned}
$$

$$
\mathbf{v}(\cdot,T) = \mathbf{v}(\cdot,0),\ \eta(\cdot,T) = \eta(\cdot,0).
$$

Definition 2 $(\mathbf{v},\eta) \in (L^2(0,T;J(\Omega_m))\cap L^2_\pi(0,T;L^6(\Omega)))\times(L^2(0,T;H^1_0(\Omega_m))\times L^2_\pi(0,T;L^6(\Omega)))$ *is called a reproductive weak solution for* (P_m) *if it satisfies*

$$
\int_0^T (\mathbf{v},\mathbf{w}_t) + (\nu(\eta+S)\nabla\mathbf{v},\nabla\mathbf{w}) + B(\mathbf{v},\mathbf{w},\mathbf{v}) - \alpha(\eta\mathbf{g},\mathbf{w}) - \alpha(S\mathbf{g},\mathbf{w}) = 0,
$$

$$
\int_0^T (\eta,\psi_t) + (\kappa(\eta+S)\nabla\eta,\nabla\psi) + b(\mathbf{v},\psi,\eta) + (\kappa(\eta+S)\nabla S,\nabla\psi) + b(\mathbf{v},S,\psi) = 0\,,
$$

for all $\mathbf{w}\in\widehat{D}_{\sigma,\pi}(\widehat{\Omega}_m)$*, and for all* $\psi\in\widehat{D}_\pi(\widehat{\Omega}_m)$*.*

Lemma 5 *Under Assumptions (A1), (A2), and (A3) we can construct a weak solution* $(\overline{\mathbf{u}}^m,\overline{\eta}^m)$ *of* (P_m)*.*

To prove the existence of reproductive weak solutions for the system (P_m) we use the Galerkin method together with Brouwer's fixed point theorem as in Lions [9](see also Heywood [3]).

First, we prove the a priori estimates for weak solutions of (P_m).

Lemma 6 *Let* $(\mathbf{v}^m,\eta^m)$ *a weak solution of* (P_m)*. Then, they satisfy the following estimate*

$$
\frac{d}{dt}(\|\mathbf{v}^m\|^2+\|\eta^m\|^2) + \frac{\nu_0}{2}(\gamma - \frac{9C_L^2}{k_0}\|S\|_3^2)\|\nabla\mathbf{v}^m\|^2 + \frac{k_0}{2}\gamma\|\nabla\eta^m\|^2 \le f(t), \quad (6)
$$

where $\gamma = 1 - 3\alpha C_L^2/\sqrt{k_0\nu_0}\|\mathbf{g}\|_{\frac{3}{2}}^2$ *and* $f(t) = 9C_L^2/k_0\|S_t\|^2 + k_1^2/k_0\|\nabla S\|^2 + 9\alpha^2 C_L^2/2\nu_0\|\mathbf{g}\|^2\|S\|_3^2$.

Proof. Multiplying $(P_m)_i$ and $(P_m)_{iii}$ by $\mathbf{v}^m$ and η^m, respectively, after on integrate on Ω_m, we get

$$\frac{d}{dt}\|\mathbf{v}^m\|^2 + (\nu(\eta^m + S)\nabla\mathbf{v}^m, \nabla\mathbf{v}^m) = (\alpha\eta^m\mathbf{g}, \mathbf{v}^m) + (\alpha S\mathbf{g}, \mathbf{v}^m),$$

$$\frac{d}{dt}\|\eta^m\|^2 + (k(\eta^m + S)\nabla\eta^m, \nabla\eta^m) = -(\mathbf{v}^m\cdot\nabla S, \eta^m) - (S_t, \eta^m) - (k(\eta^m + S)\nabla S, \nabla\eta^m).$$

Now, we estimate the right-hand sides of the above equalities by using the Lemma 1

$$\begin{array}{rcl} (\alpha\eta^m\mathbf{g}, \mathbf{v}^m) & \leq & 3\alpha\|\mathbf{g}\|_{\frac{3}{2}}\|\eta^m\|_6\|\mathbf{v}^m\|_6, \\ (\alpha S\mathbf{g}, \mathbf{v}^m) & \leq & 3\alpha\|\mathbf{g}\|\|S\|_3\|\mathbf{v}^m\|_6, \\ (\mathbf{v}^m\cdot\nabla S, \eta^m) & = & (\mathbf{v}^m\cdot\nabla\eta^m, S) \leq 3\|\mathbf{v}^m\|_6\|\nabla\eta^m\|\|S\|_3, \\ (S_t, \eta^m) & \leq & 3\|S_t\|_{\frac{6}{5}}\|\eta^m\|_6, \\ (k(\eta^m + S)\nabla S, \nabla\eta^m) & \leq & k_1\|\nabla\eta^m\|\|\nabla S\|. \end{array}$$

Observe that

$$\begin{array}{rcl} (\nu(\eta^m + S)\nabla\mathbf{v}^m, \nabla\mathbf{v}^m) & \geq & \nu_0\|\nabla\mathbf{v}^m\|^2, \\ (k(\eta^m + S)\nabla\eta^m, \nabla\eta^m) & \geq & k_0\|\nabla\eta^m\|^2, \end{array}$$

the estimates and equalities imply

$$\begin{array}{rcl} \frac{d}{dt}\|\mathbf{v}^m\|^2 + \nu_0\|\nabla\mathbf{v}^m\|^2 & \leq & 3\alpha\|\mathbf{g}\|_{\frac{3}{2}}\|\eta^m\|_6\|\mathbf{v}^m\|_6 + 3\alpha\|\mathbf{g}\|\|S\|_3\|\mathbf{v}^m\|_6, \\ \frac{d}{dt}\|\eta^m\|^2 + k_0\|\nabla\eta^m\|^2 & \leq & 3\|\mathbf{v}^m\|_6\|\nabla\eta^m\|\|S\|_3 + 3\|S_t\|_{\frac{6}{5}}\|\eta^m\|_6 + k_1\|\nabla\eta^m\|\|\nabla S\|. \end{array}$$

The Ladyzhenskaya's inequality implies

$$\begin{array}{rl} & \frac{d}{dt}(\|\mathbf{v}^m\|^2 + \|\eta^m\|^2) + \nu_0\|\nabla\mathbf{v}^m\|^2 + k_0\|\nabla\eta^m\|^2 \\ \leq & \frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}(\frac{k_0}{2}\|\nabla\eta^m\|^2 + \frac{\nu_0}{2}\|\nabla\mathbf{v}^m\|^2) + \frac{9\alpha^2 C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2 \\ & +\frac{\nu_0}{2}\|\nabla\mathbf{v}^m\|^2 + \frac{9C_L^2}{k_0}\|S\|_3^2\|\nabla\mathbf{v}^m\|^2 + \frac{3k_0}{4}\|\nabla\eta^m\|^2 \\ & +\frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}} + \frac{k_1^2}{k_0}\|\nabla S\|^2. \end{array}$$

Thus,

$$\frac{d}{dt}(\|\mathbf{v}^m\|^2+\|\eta^m\|^2)+\frac{\nu_0}{2}(1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}-\frac{9C_L^2}{k_0}\|S\|_3^2)\|\nabla\mathbf{v}^m\|^2+\frac{k_0}{2}(1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}})\|\nabla\eta^m\|^2$$
$$\leq \frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}}+\frac{k_1^2}{k_0}\|\nabla S\|^2+\frac{9\alpha^2C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2.$$

We put $\gamma=1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}$ and $f(t)=\frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}}+\frac{k_1^2}{k_0}\|\nabla S\|^2+\frac{9\alpha^2C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2$. This proves Lemma 8.

Proof of the Lemma 7

Now, we prove the existence of the solution $(\mathbf{v}^m,\eta^m)$ for (P_m). Let m be arbitrarily fixed. Let $\{e^i(x)\}_{i=1}^{\infty}\subseteq C_{0,\sigma}^{\infty}(\Omega_m)$ (respec. $\{\phi^i(x)\}_{i=1}^{\infty}\subseteq C_0^{\infty}(\Omega_m)$) be a sequence of functions orthonormal in $L^2(\Omega_m)$ and total in $J(\Omega_m)$ (respec. $H_0^1(\Omega_m)$). As k^{-th} approximate solution of (P_m), we choose the functions

$$\mathbf{v}^k(t,x)=\sum_{j=1}^{k}c_{kj}(t)e^j(x)\,,\quad \eta^k(t,x)=\sum_{j=1}^{k}d_{kj}(t)\phi^j(x)\,.$$

which satisfy the equations

$$\begin{aligned}(\mathbf{v}_t^k,\varphi^j)+(\nu(\eta^k+S)\nabla\mathbf{v}^k,\nabla\varphi^j)+B(\mathbf{v}^k,\mathbf{v}^k,\varphi^j)-\alpha(\eta^k\mathbf{g},\varphi^j)-\alpha(S\mathbf{g},\varphi^j) &= 0\\ (\eta_t^k,\phi^j)+(\kappa(\eta^k+S)\nabla\eta^k,\nabla\phi^j)+b(\mathbf{v}^k,\eta^k,\phi^j)+(\kappa(\eta^k+S)\nabla S,\nabla\phi^j)+b(\mathbf{v}^k,S,\phi^k) &= 0\,,\end{aligned}\tag{7}$$

for $1\leq j\leq k$.

Note that the solutions $(\mathbf{v}^k,\eta^k)$ must satisfy the estimate (7). Thus, we have

$$\frac{d}{dt}(\|\mathbf{v}^k\|^2+\|\eta^k\|^2)+M(\|\nabla\mathbf{v}^k\|^2+\|\nabla\eta^k\|^2)\leq f(t),$$

where

$$M=\min\{\frac{\nu_0}{2}(\gamma-\frac{9C_L^2}{k_0}\|S\|_3^2),\frac{k_0}{2}\gamma).$$

Let d_m be the diameter of Ω_m. Making use of Poincaré inequality, we obtain

$$\frac{d}{dt}(\|\mathbf{v}^k\|^2+\|\eta^k\|^2)+\lambda_m(\|\mathbf{v}^k\|^2+\|\eta^k\|^2)\leq f(t)$$

where $\lambda_m=\frac{2M}{d_m^2}$. Or equivalently,

$$\frac{d}{dt}e^{\lambda_m t}(\|\mathbf{v}^k\|^2+\|\eta^k\|^2)\leq e^{\lambda_m t}f(t).$$

Integrating from 0 to T, we get

$$e^{\lambda_m T}(\|\mathbf{v}^k(T)\|^2 + \|\eta^k(T)\|^2) \leq \|\mathbf{v}^k(0)\|^2 + \|\eta^k(0)\|^2 + \int_0^T e^{\lambda_m t} f(t)dt.$$

We denote by $z^k(t)$ the vector $(\mathbf{v}^k, \eta^k)$ and $\|z^k(t)\|^2 = \|\mathbf{v}^k(t)\|^2 + \|\eta^k(t)\|^2$. With this notation, the above inequality is rewritten as

$$e^{\lambda_m T}\|z^k(T)\|^2 \leq \|z^k(0)\|^2 + \int_0^T e^{\lambda_m t} f(t)dt.$$

Now, let us define the mapping $L^k : [0, T] \to I\!R^{2k}$ as

$$L^k(t) = (c_{1k}(t), ..., c_{kk}(t), d_{1k}(t), ..., d_{kk}(t))$$

where $c_{ik}(t), d_{ik}(t)$, $i = 1, ..., k$ are respectively the coefficient of the expansion of $\mathbf{v}^k(t)$ and $\eta^k(t)$, as defined before.

Keep on mind that

$$\|L^k(t)\|_{R^{2k}} = \|z^k(t)\|, \tag{8}$$

since we have chosen the basis $\{e^i(x)\}_{i=1}^{\infty}$ and $\{\phi^i(x)\}_{i=1}^{\infty}$ to be orthonormal in $(L^2(\Omega_m))^n$.

Now, we define the mapping $\Phi^k : I\!R^{2k} \to I\!R^{2k}$ as follows: given $L_0 \in I\!R^{2k}$ and define $\Phi^k(L_0) = L^k(T)$, where $L^k(t)$ corresponds to the solution of problem (P1) with initial value corresponding to L_0. It is easy to see that Φ^k is continuous. We want to prove that Φ^k has a fixed point. As a consequence of fixed point theorem of Brouwer, it is enough to prove that for any $\lambda \in [0, 1]$, a possible solution of the equation

$$L_0^k(\lambda) = \lambda\Phi^k(L_0^k(\lambda)) \tag{9}$$

is bounded independent by λ.

Since $L_0^k(0) = 0$, by (3.11), it is enough to prove this fact for $\lambda \in (0, 1]$. In this case, (3.11) is equivalent to $\Phi^k(L_0^k(\lambda)) = L_0^k(\lambda)/\lambda$. By definition of Φ^k and condition (3.10), inequality (3.9) implies that

$$e^{\lambda_m T}\|L_0^k(\lambda)/\lambda\|^2_{I\!R^{2k}} \leq \|L_0^k(\lambda)\|^2_{I\!R^{2k}} + \int_0^T e^{\lambda_m t} f(t)dt,$$

which yields

$$\|L_0^k(\lambda)\|^2_{I\!R^{2k}} \leq \frac{\int_0^T e^{\lambda_m t}(f(t)dt}{e^{\lambda_m T} - 1} = N, \tag{10}$$

since $\lambda \in (0,1]$. This bound is independent of $\lambda \in [0,1]$ and, therefore, Φ^k has a fixed point $L_0^k(1)$ satisfying the same bound as (3.12).

This corresponds to the existence of a solution $\mathbf{v}^k(t), \eta^k(t))$ of (P$_1$) satisfying $\mathbf{v}^k(0) = \mathbf{v}^k(T)$, and $\eta^k(0) = \eta^k(T)$, that is a reproductive approximated solution.

Moreover, $\|\mathbf{v}^k(0)\| + \|\eta^k(0)\|^2 = \|L_0^k(1)\|^2_{I\!R^{2k}} \leq N$, which is also independent of k.

On the other hand, from (3.6) we have

$$\begin{aligned} & \|\mathbf{v}^k(t)\|^2 + \|\eta^k(t)\|^2 + M\int_0^t ((\nabla \mathbf{v}^k, \nabla \mathbf{v}^k) + a(\nabla \eta^k, \nabla \eta^k))ds \\ \leq & \int_0^T f(t)dt + \|\mathbf{v}^k(0)\|^2 + \|\eta^k(0)\|^2 \\ \leq & N(f) + N, \end{aligned}$$

for $k \geq 1$, where $N(f) = \int_0^T f(t)dt$.

Moreover, the sequence $(\mathbf{v}^k, \eta^k)$ is bounded in $L^2(0,T;J(\Omega_m)) \times L^2(0,T;H_0^1(\Omega_m))$ and in $L^\infty_\pi(0,T;H(\Omega_m) \times L^\infty_\pi(0,T;L^2(\Omega_m))$.

Since $J(\Omega_m)$ (respectively $H_0^1(\Omega_m)$) is compactly embedded in $H(\Omega_m)$ (respectively $L^2(\Omega_m)$) we can choose subsequences, which we again denote by $(\mathbf{v}^k, \eta^k)$, and elements $\overline{\mathbf{u}}^m \in L^2(0,T;J(\Omega_m))$, $\overline{\eta}^m \in L^2(0,T;H_0^1(\Omega_m))$ such that

$$\begin{aligned} \mathbf{v}^k &\to \overline{\mathbf{u}}^m \text{ weakly in } L^2(0,T;J(\Omega_m)) \text{ and weakly* in } L^\infty(0,T;H(\Omega_m)), \\ \eta^k &\to \overline{\eta}^m \text{ weakly in } L^2(0,T;H_0^1(\Omega_m)) \text{ and weakly* in } L^\infty(0,T;L^2(\Omega_m)). \end{aligned}$$

Furthermore, by using the Lemma 2 and (3.13) we see that

$$\begin{aligned} \mathbf{v}^k &\to \overline{\mathbf{u}}^m \text{ strongly in } L^2(0,T;H(\Omega_m)), \\ \eta^k &\to \overline{\eta}^m \text{ strongly in } L^2(0,T;L^2(\Omega_m)). \end{aligned}$$

Now, it is enough to take the limit $k \to \infty$ in (P_m). Therefore, $(\overline{\mathbf{u}}^m, \overline{\eta}^m)$ is a required weak solution to problem (P_m).

Lemma 7 *Let $(\overline{\mathbf{u}}^m, \overline{\eta}^m)$ be a weak solution for (P_m) obtained in Lemma 7. Put*

$$\mathbf{u}^m(t,x) = \begin{cases} \overline{\mathbf{u}}^m(t,x) & \text{if } x \in \Omega_m, \\ 0 & \text{if } x \in \Omega \setminus \Omega_m, \end{cases}$$

$$\varphi^m(t,x) = \begin{cases} \overline{\eta}^m(t,x) & \text{if } x \in \Omega_m, \\ 0 & \text{if } x \in \Omega \setminus \Omega_m. \end{cases}$$

Then it follows

$$\mathbf{u}^m \in L^2(0,T;J(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega)),$$

$$\varphi^m \in L^2(0,T;H^1_0(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega))$$

and,

$$\int_0^T \|\nabla \mathbf{u}^m\|^2 \leq \ell_1, \quad \int_0^T \|\nabla \varphi^m\|^2 \leq \ell_2,$$

$$\int_0^T \|\mathbf{u}^m\|^2_{L^6(\Omega)} \leq \ell_1, \quad \int_0^T \|\varphi^m\|^2_{L^6(\Omega)} \leq \ell_2,$$

where ℓ_1, ℓ_2 are taken uniformly in m.

Proof. From (3.6), we have, integrating in $[0,T]$

$$M \int_0^T (\|\nabla \mathbf{v}^k(t)\|^2 + \|\nabla \eta^k(t)\|^2) dt \leq N(f), \tag{11}$$

since $\mathbf{v}^k(t)$, $\eta^k(t)$ are reproductive with period T. Consequently, if we take $k \to \infty$ in (3.13),then we obtain by the lower semicontinuity of the norm with respect to the weak convergence

$$M \int_0^T (\|\nabla \overline{\mathbf{u}}^m(t)\|^2 + \|\nabla \overline{\eta}^m(t)\|^2) dt \leq N(f). \tag{12}$$

On the other hand, the equality $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ in $L^2(\Omega_m)$ implies $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ for a.e. $x \in \Omega_m$ and by using the Lemma 2 we obtain $\overline{\mathbf{u}}^m(t) \in L^6(\Omega_m)$, therefore we find $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ as elements of $L^6(\Omega_m)$. Thus, we obtain $\overline{\mathbf{u}}^m \in L^2_\pi(0,T;L^6(\Omega_m))$. Analogously, we show that $\overline{\eta}^m \in L^2_\pi(0,T;L^6(\Omega_m))$.

¿From this and (3.15) , it follows that for all $m \geq 1$,

$$\overline{\mathbf{u}}^m \in L^2(0,T;J(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega)),$$

$$\overline{\eta}^m \in L^2(0,T;H_0^1(\Omega)) \cap L_\pi^2(0,T;L^6(\Omega)),$$

and

$$\begin{aligned} & \frac{1}{C_L}\int_0^T (\|\overline{\mathbf{u}}^m(t)\|_{L^6(\Omega)}^2 + \|\overline{\eta}^m(t)\|_{L^6(\Omega)}^2)dt \\ \leq & \int_0^T (\|\nabla\overline{\mathbf{u}}^m(t)\|^2 + \|\nabla\overline{\eta}^m(t)\|^2)dt \\ \leq & \frac{1}{C_0}N(f). \end{aligned} \tag{13}$$

4 Proof of Theorem 5

According to the uniform estimate (3.16), we can choose subsequences $\mathbf{u}^{m'}$ and $\varphi^{m'}$ and

$\mathbf{u} \in L^2(0,T;J(\Omega)) \cap L_\pi^2(0,T;L^6(\Omega))$ and $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L_\pi^2(0,T;L^6(\Omega))$ such that

$$\begin{aligned} \mathbf{u}^{m'} &\to \mathbf{u} \text{ weakly in } L^2(0,T;J(\Omega)) \text{ and weakly in } L_\pi^2(0,T;L^6(\Omega)), \\ \varphi^{m'} &\to \varphi \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \text{ and weakly in } L_\pi^2(0,T;L^6(\Omega)). \end{aligned} \tag{4.1}$$

Now, we claim that there exist subsequences $\mathbf{u}^{m'}$ and $\varphi^{m'}$ such that for any bounded $\Omega' \subset \Omega$

$$\begin{aligned} \mathbf{u}^{m'} &\to \mathbf{u} \text{ strongly in } L^2(0,T;L^2(\Omega')), \\ \varphi^{m'} &\to \varphi \text{ strongly in } L^2(0,T;L^2(\Omega')). \end{aligned}$$

We put $K_j = \overline{\Omega_j}$, then $\{K_j\}_{j=1}^\infty$ a sequence of compact sets such that $K_1 \subseteq K_2 \subseteq ... \to \Omega$ $(j \to \infty)$. Here, for each K_j we take $\alpha_j(x) \in C_0^\infty(\Omega)$ with the property $0 \leq \alpha \leq 1$, $\alpha_j|_{K_j} \equiv 1$, and supp $\alpha_j \subset \Omega_{j+1}$. We note that $K_j \subset$ supp α_j. Here and from now on, let us denote $\|.\|_{\Omega_j} \equiv \|.\|_{L^2(\Omega_j)}$ and $d_j =$ diameter of Ω_j. Then we construct the desired $\{\mathbf{u}^{m'}\}$ as follows. First we consider a sequence $\{\alpha_j(x)\mathbf{u}^m(x)\}_{m=1}^\infty$; this is a uniformly bounded sequence of $L^2(0,T;H_0^1(\Omega_2))$. Indeed, noting that $\mathbf{u}^m(\Gamma)=0$ and using Poincaré's inequality on Ω_2, we see that $\|\alpha_1\mathbf{u}^m\|_{\Omega_2} \le \|\mathbf{u}^m\|_{\Omega_2} \le \frac{d_2}{2}\|\nabla\mathbf{u}^m\|_{\Omega_2}$. Hence we have by (3.16)

$$\begin{aligned}\int_0^T \|\alpha_1\mathbf{u}^m(t)\|_{\Omega_2}^2 dt &\le \frac{d_2^2}{2}\int_0^T \|\nabla\mathbf{u}^m(t)\|^2 dt \\ &\le \frac{d_2^2}{2C_0}N(f).\end{aligned}$$

Moreover, $\|\nabla(\alpha_1\mathbf{u}^m)\|_{\Omega_2} \le \|(\nabla\alpha_1)\mathbf{u}^m\|_{\Omega_2}+\|\alpha_1(\nabla\mathbf{u}^m)\|_{\Omega_2} \le (\frac{d_2}{2}\|\nabla\alpha_1\|_{L^\infty(\Omega_2)}+ \|\alpha_1\|_{L^\infty(\Omega_2)})\|\nabla\mathbf{u}^m\|_{\Omega_2}$.

Therefore, we have

$$\begin{aligned}&\int_0^T \|\nabla(\alpha_1\mathbf{u}^m)(t)\|_{\Omega_2}^2 dt \\ \le\ & (\frac{d_2}{\sqrt{2}}\|\nabla\alpha_1\|_{L^\infty(\Omega_2)} + \|\alpha_1\|_{L^\infty(\Omega_2)})^2\frac{d_2^2}{2C_0}N(\mathbf{f})T.\end{aligned}$$

These estimates imply that $\{\alpha_1\mathbf{u}^m\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega_2))$. Consequently, there exists a subsequence $\{\alpha_1\mathbf{u}^{1p}\}_{p=1}^\infty$ which converges weakly in $L^2(0,T;H_0^1(\Omega_2))$. Furthermore, according to Lemma 2, we get

$$\begin{aligned}\int_0^T \|\alpha_1\mathbf{u}^{1p}-\alpha_1\mathbf{u}^{1q}\|_{\Omega_2}^2 dt &\le \sum_{n=1}^{l_\varepsilon}\int_0^T (\alpha_1\mathbf{u}^{1p}-\alpha_1\mathbf{u}^{1q},e^n)_{\Omega_2}^2 + \varepsilon\int_0^T \|\alpha_1\mathbf{u}^{1p}-\alpha_1\mathbf{u}^{1q}\|_{W^{1,2}(\Omega_2)}^2 dt \\ &\le \sum_{n=1}^{l_\varepsilon}\int_0^T (\alpha_1\mathbf{u}^{1p}-\alpha_1\mathbf{u}^{1q},e^n)_{\Omega_2}^2 + 4\varepsilon C_{\alpha_1}N(\mathbf{f})\end{aligned}$$

where C_{α_1} depends on $.\|\alpha_1\|_\infty, \|\nabla\alpha_1\|_\infty$ and is independent of p and q. Consequently, if $p,q\to\infty$, we have in ( ), since ε is arbitrary in ( ), the sequence $\{\alpha_1\mathbf{u}^{1p}\}_{p=1}^\infty$ converges strongly in $L^2(0,T;L^2(\Omega_2))$. This implies that $\{\mathbf{u}^{1p}\}_{p=1}^\infty$ converges strongly in $L^2(0,T;L^2(K_1))$. Using the same reasoning as before, we obtain $\{\mathbf{u}^{jp}\}_{p=1}^\infty$ $(j=1,2,...)$. We choose diagonal components and denote them by $\{\mathbf{u}^{m'}\}_{m'=1}^\infty$, then it converges on all K_j in $L^2(0,T;L^2(K_j))$ sense. The proof for $\{\varphi^{m'}\}_{m'=1}^\infty$,can be done in a similar way.

Once we obtain these convergence and limit results, we can show that $(\mathbf{u}, \varphi)$ is the desired reproductive weak solution for (P1) and conditions (1) through (4). Indeed, let $(\mathbf{v}, \psi)$ be any arbitrary test function. Then we find a bounded domain Ω' and k_0 such that supp $\mathbf{v}$, supp $\psi \subseteq \Omega' \subseteq \Omega_{k_0} \subseteq \Omega_k$, for all $k \geq k_0$. Moreover, by Lemma 2.1 and (3.16)

$$\begin{aligned}
& \int_0^T (\mathbf{u}^k \cdot \nabla \mathbf{v}, \mathbf{u}^k) - (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{u}) dt \\
\leq \; & \int_0^T \{3\|\mathbf{u}^k - \mathbf{u}^k\|^2_{L^2(\Omega')} \|\mathbf{u}^k\|_{L^6(\Omega)} \|\nabla \mathbf{v}\|_{L^3(\Omega')} + 3\|\mathbf{u}^k - \mathbf{u}^k\|^2_{L^2(\Omega')} \|\mathbf{u}\|_{L^6(\Omega)} \|\nabla \mathbf{v}\|_{L^3(\Omega')}\} dt \\
\leq \; & 9(\int_0^T \|\mathbf{u}^k - \mathbf{u}\|^2_{L^2(\Omega')} dt)^{1/2} (\int_0^T \|\mathbf{u}^k\|^2_{L^6(\Omega)} dt)^{1/2} \sup \|\nabla \mathbf{v}\|_{L^3(\Omega')} \\
& +9(\int_0^T \|\mathbf{u}^k - \mathbf{u}\|^2_{L^2(\Omega')} dt)^{1/2} (\int_0^T \|\mathbf{u}\|^2_{L^6(\Omega)} dt)^{1/2} \sup \|\nabla \mathbf{v}\|_{L^3(\Omega')}.
\end{aligned}$$

Using convergences (4.1) and the above estimate, we get

$$\int_0^T (\mathbf{u}^k \cdot \nabla \mathbf{v}, \mathbf{u}^k) - (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{u}) dt \to 0,$$

as $k \to \infty$. The other convergences are in the same way established. Thus, $(\mathbf{u},\varphi)$ is a reproductive weak solution for problem (P1) and conditions (1) through (4).

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