THE KIRCHHOFF-HELMHOLTZ INTEGRAL FOR ANISOTROPIC ELASTIC MEDIA

Jörg Schleicher*, Martin Tygel*, Bjørn Ursin**, and Norman Bleistein***

* Departamento de Matemática Aplicada,
Universidade Estadual de Campinas, IMECC/UNICAMP,
Caixa Postal 6065, 13081-970 Campinas (SP), Brazil.

** Department of Petroleum Engineering and Applied Geophysics,
The Norwegian University of Science and Technology,
S.P. Andersensv. 15A, N-7491 Trondheim, Norway.

*** Center of Wave Phenomena,
Colorado School of Mines,
Golden, Colorado 80401-1887, USA.

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Abstract

The Kirchhoff-Helmholtz integral is a powerful tool to model the scattered wavefield from a smooth interface in acoustic or isotropic elastic media due to a given incident wavefield and observation points sufficiently far away from the interface. This integral makes use of the Kirchhoff approximation of the unknown scattered wavefield and its normal derivative at the interface in terms of the corresponding quantities of the known incident field. An attractive property of the Kirchhoff-Helmholtz integral is that its asymptotic evaluation recovers the zero-order ray theory approximation of the reflected wavefield at all observation points where that theory is valid. Here, we extend the Kirchhoff-Helmholtz modeling integral to general anisotropic elastic media. It uses the natural extension of the Kirchhoff approximation of the scattered wavefield and its normal derivative for those media. The anisotropic Kirchhoff-Helmholtz integral also asymptotically provides the zero-order ray theory approximation of the reflected response from the interface. In connection with the asymptotic evaluation of the Kirchhoff-Helmholtz integral, we also derive an extension to anisotropic media of a useful decomposition formula of the geometrical spreading of a primary reflection ray.

1. Introduction

The wavefield scattered from a smooth interface can be represented by a surface integral. Both the field and its normal derivative at the interface appear in the integrand. Fundamental representations for the acoustic, isotropic and anisotropic elastic cases can be found in the literature (see, e.g., Aki and Richards [2]; Baker and Copson [3]; Bleistein [4]); Kupradze [5]). These representations can be recast as modeling formulas for reflection from a transparent interface by exploiting the Kirchhoff approximation which expresses the unknown scattered field and its normal derivative in terms of the known incident field (Bleistein [4], Frazer and Sen [6], Tygel et al. [7]). The result is called the Kirchhoff-Helmholtz integral.

We derive the extension of the Kirchhoff-Helmholtz integral to anisotropic elastic media starting with an integral representation for the wavefield at a receiver point (Aki and Richards [2]). The incoming wavefield and the Green's function from the receiver point to the interface are both replaced by the geometric ray approximation (GRA). The GRA Green's function is often expressed as a function of phase velocities and the relative geometrical-spreading factor that may be computed from mixed second-order traveltime derivatives with respect to the phase-front coordinates that are normal to the phase-velocity vectors. Here we prefer to work with a GRA that is expressed by the group velocities and a relative geometrical-spreading factor that is expressed by the mixed second-order traveltime derivatives with respect to the ray coordinates that are normal to the group-velocity vectors. The relationship between the geometrical-spreading factors is shown in Appendix A.

The Kirchhoff approximation for anisotropic media is done in the same way as for isotropic media. Both the outgoing field and its derivative at the interface are approximated from the specularly reflected wavefield from the source. This approximation has also been indicated by de Hoop and Bleistein [8]. To verify the resulting anisotropic Kirchhoff-Helmholtz integral, we show that the stationary-phase evaluation is the GRA for the reflected wavefield. Necessary decomposition formulas for the geometrical-spreading factors for the reflected wave are given in Appendix B. These are extensions of the isotropic elastic formulas (Hubral et al. [9],[10] and Červený [11]). The GRA of the reflected wavefield is then expressed by the total geometrical spreading and a reflection coefficient which is normalized with respect to the vertical energy flux. This formula also satisfies reciprocity.

2. The geometric ray approximation

Wave propagation in an inhomogeneous anisotropic elastic solid is governed by the elastodynamic equations (Aki and Richards [2]). In the frequency domain, these are

$$\omega^2 \rho U_i + (c_{ijkl} U_{k,l})_{,j} = 0 \ , \quad i,j,k = 1,2,3. \tag{1}$$

Here, ω is the frequency, $U_i = U_i(\boldsymbol{x}, \omega)$ is the i-th component of the displacement vector $\boldsymbol{U}(\boldsymbol{x}.\omega)$, $\rho = \rho(\boldsymbol{x})$ is the density, and $c_{ijkl} = c_{ijkl}(\boldsymbol{x})$ are the elastic parameters of the medium at the point $\boldsymbol{x} = (x_1, x_2, x_3)$. The elastic parameters satisfy the symmetry relations $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. In equation (1), the notation ", j" stands for $\partial/\partial x_j$, and a repeated index implies summation with respect to that index.

The Green's function, $G_{in}(\boldsymbol{x}, \omega; \boldsymbol{x}^s)$, satisfies the equation

$$\omega^2 \rho G_{in} + (c_{ijkl} G_{kn,l})_{,j} = - \, \delta_{in} \delta(\boldsymbol{x} - \boldsymbol{x}^s) \; . \tag{2}$$

For homogeneous boundary conditions, it also satisfies the reciprocity relation (Aki and Richards [2], equation (2.39))

$$G_{ij}(\boldsymbol{x}, \omega; \boldsymbol{x}^s) = G_{ji}(\boldsymbol{x}^s, \omega; \boldsymbol{x}) \; . \tag{3}$$

For a specific ray connecting a source point $\boldsymbol{x}^s$ to a scattering point $\boldsymbol{x}$, the GRA Green's function is (Červený [11], Chapman and Coates [12])

$$G_{ij}(\boldsymbol{x}, \omega; \boldsymbol{x}^s) = h_i^s(\boldsymbol{x}) \; A(\boldsymbol{x}, \boldsymbol{x}^s) \; e^{i\omega T(\boldsymbol{x}, \boldsymbol{x}^s)} \; h_j(\boldsymbol{x}^s) \; , \tag{4}$$

where $\boldsymbol{h}(\boldsymbol{x}^s)$ and $\boldsymbol{h}^s(\boldsymbol{x})$ are the unit polarization vectors in the ray direction at the source $\boldsymbol{x}^s$ and at the point $\boldsymbol{x}$, respectively. $T(\boldsymbol{x}, \boldsymbol{x}^s)$ is the traveltime along the ray from $\boldsymbol{x}$ to $\boldsymbol{x}^s$, and $A(\boldsymbol{x}, \boldsymbol{x}^s)$ is a complex amplitude function taking into account possible caustics and phase-shift at the source. It is given by

$$A(\boldsymbol{x}, \boldsymbol{x}^s) = \frac{e^{-i\frac{\pi}{2}\,\mathrm{sgn}(\omega)\,\kappa(\boldsymbol{x}, \boldsymbol{x}^s)}}{4\pi \, [\rho(\boldsymbol{x}) \, v^s(\boldsymbol{x}) \, \rho(\boldsymbol{x}^s) \, v(\boldsymbol{x}^s)]^{1/2} \, |\det \boldsymbol{Q}_2(\boldsymbol{x}, \boldsymbol{x}^s)|^{1/2}} \; . \tag{5}$$

Here, $\rho(\boldsymbol{x}^s)$ and $\rho(\boldsymbol{x}^r)$ are the densities and $v(\boldsymbol{x}^s)$ and $v^s(\boldsymbol{x})$ the phase velocities in the ray direction at the source $\boldsymbol{x}^s$ and at the point $\boldsymbol{x}$, respectively; $|\det \boldsymbol{Q}_2(\boldsymbol{x}, \boldsymbol{x}^s)|^{1/2}$ denotes the relative geometrical spreading factor computed with respect to local wavefront coordinates. The inverse of the matrix $\boldsymbol{Q}_2$ is given by

$$Q_{2\,ij}^{-1} = - \, \frac{\partial^2 T(\boldsymbol{x}, \boldsymbol{x}^s)}{\partial q_i^s \partial q_j} \; , \quad i, j = 1, 2. \tag{6}$$

The local coordinates q_i^s and q_j are in the wavefront plane (normal to the phase velocity) at the points $\boldsymbol{x}^s$ and $\boldsymbol{x}$, respectively. Finally, $\kappa(\boldsymbol{x}, \boldsymbol{x}^s)$ is the KMAH index for the ray that connects the source $\boldsymbol{x}^s$ to the point $\boldsymbol{x}$.

We recognize that our notations for the phase velocities $v^s(\boldsymbol{x})$ and $v(\boldsymbol{x}^s)$ require some explanation. These are both velocities at the end points of the specific ray that connects $\boldsymbol{x}^s$ to $\boldsymbol{x}$. Due to anisotropy, these velocities depend on the ray direction, as well as position. The additional superscript in $v^s(\boldsymbol{x})$ is used to distinguish it from $v^r(\boldsymbol{x})$, the phase velocity at $\boldsymbol{x}$ for the ray coming from a receiver at $\boldsymbol{x}^r$ (see Figure 1). Our notation for the polarization vectors $h_i^s(\boldsymbol{x})$ and $h_i(\boldsymbol{x}^s)$ follows the same pattern. Since the magnitude of the phase velocity does not change when the ray direction is reversed and, moreover, all the other quantities in the GRA Green's function (4) remain the same when $\boldsymbol{x}^s$ and $\boldsymbol{x}$ are interchanged, it follows that the GRA Green's function also satisfies the reciprocity relation (3).

In evaluating the stationary value of the Kirchhoff-Helmholtz integral to be defined later, it turns out to be more convenient to work with the relative geometrical-spreading factor $|\det \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{x}^s)|^{1/2}$ defined with respect to local ray coordinates. Here,

$$Y_{ij}^{-1} = -\frac{\partial^2 T(\boldsymbol{x}, \boldsymbol{x}^s)}{\partial g_i^s \partial g_j}, \quad i, j = 1, 2, \tag{7}$$

where g_i^s and g_j are local coordinates in the planes normal to the ray (and normal to the group velocity), respectively.

Using the results of Appendix A, we have that

$$V^s(\boldsymbol{x})\, V(\boldsymbol{x}^s)\, |\det \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{x}^s)| = v^s(\boldsymbol{x})\, v(\boldsymbol{x}^s)\, |\det \boldsymbol{Q}_2(\boldsymbol{x}, \boldsymbol{x}^s)|\;, \tag{8}$$

where $V(\boldsymbol{x}^s)$ and $V^s(\boldsymbol{x})$ denote the group velocities in the ray direction at the source $\boldsymbol{x}^s$ and at the point $\boldsymbol{x}$, respectively. When equation (8) is used in equation (4), the amplitude expression becomes

$$A(\boldsymbol{x}, \boldsymbol{x}^s) = \frac{e^{-i\frac{\pi}{2}\,\mathrm{sgn}(\omega)\,\kappa(\boldsymbol{x}, \boldsymbol{x}^s)}}{4\pi\, [\rho(\boldsymbol{x})\, V^s(\boldsymbol{x})\, \rho(\boldsymbol{x}^s)\, V(\boldsymbol{x}^s)]^{1/2}\, |\det \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{x}^s)|^{1/2}}\;. \tag{9}$$

Note again that the amplitude formula satisfies reciprocity.

For our purposes, it suffices to use the "free-space" Green's function in what follows. This is the Green's function in the absence of the reflecting surface. This is a standard mathematical device. Clearly, the creation of this wavefield requires an appropriately smooth continuation of the medium parameters of the upper medium to all space. However, since we will only use the ray theoretic description of the wavefield above the reflector, there is no need to describe this continuation in any detail. We need only assume that the extension of the medium parameters below the reflector do not introduce "turned energy" from below the reflector back into the upper medium. As can be seen from the discussion above, the ray theoretic free space Green's function above the reflector is then derived using only medium parameter values above the reflector.

3. The Kirchhoff-Helmholtz integral

We will consider a wavefield that is generated by a source at $\boldsymbol{x}^s$, reflected at the surface Σ and recorded at $\boldsymbol{x}^r$, as shown in Figure 1. The total field, $\boldsymbol{U}^{\mathrm{tot}}$, above the reflector consists of a direct arrival (the *incident* wavefield), $\boldsymbol{U}^{\mathrm{inc}}$, that is unchanged by the presence of the reflector and the response to the reflector, $\boldsymbol{U}$. It is the latter wavefield for which we will derive a Kirchhoff-Helmholtz integral representation. It is dominated by reflection and therefore, for brevity, we will refer to it as the reflected wavefield. The incident wavefield is analogous to the free-space Green's function, introduced above, but now it is the free-space wave response to the given source. Then

$$\boldsymbol{U}^{\mathrm{tot}}(\boldsymbol{x}) = \boldsymbol{U}^{\mathrm{inc}}(\boldsymbol{x}) + \boldsymbol{U}(\boldsymbol{x}). \tag{10}$$

The dependence on ω will no longer be shown explicitly. The reflected displacement field at the receiver $\boldsymbol{U}(\boldsymbol{x}^r)$ can be expressed as a surface integral involving the outgoing displacement field

$\boldsymbol{U}(\boldsymbol{x})$ and its derivative $\boldsymbol{\nabla U}(\boldsymbol{x})$ at the surface. This result can be derived by starting from a representation theorem that is given in Aki and Richards [2], equation (2.41). Their result is valid for the total field, $\boldsymbol{U}^{\text{tot}}(\boldsymbol{x})$, in bounded media; here, we are considering problems in unbounded media and we want a representation only for $\boldsymbol{U}(\boldsymbol{x})$. In Appendix C, we briefly describe the adaptation of their result to the one we seek and only present the final result here.

In the absence of body forces, and with a Green's function that satisfies the reciprocity relation (3), this representation is

$$\begin{aligned} U_m(\boldsymbol{x}^r) &= \int_\Sigma \Big\{ G_{im}(\boldsymbol{x},\boldsymbol{x}^r)\, c_{ijkl}(\boldsymbol{x})\, U_{k,l}(\boldsymbol{x}) \\ &- G_{km,l}(\boldsymbol{x},\boldsymbol{x}^r)\, c_{ijkl}(\boldsymbol{x})\, U_i(\boldsymbol{x}) \Big\}\, n_j\, d\sigma\,. \end{aligned} \tag{11}$$

Here, $\boldsymbol{n}$ is the normal to the reflector surface Σ pointing outwards (that is, downwards) in Figure 1.

To derive the Kirchhoff-Helmholtz integral, we shall make the following approximations:

(i) The free-space Green's function $G_{ij}(\boldsymbol{x},\boldsymbol{x}^r)$ for the receiver ray segment is replaced by the GRA corresponding to equation (4) with the amplitude function $A(\boldsymbol{x},\boldsymbol{x}^r)$ in the form of equation (9).

(ii) The spatial derivatives of the Green's function are approximated by neglecting variations in the amplitude function, so that

$$\begin{aligned} G_{ij,k}(\boldsymbol{x},\boldsymbol{x}^r) &\approx i\omega\, T_{,k}\, G_{ij}(\boldsymbol{x},\boldsymbol{x}^r) \\ &= i\omega\, p_k^r\, G_{ij}(\boldsymbol{x},\boldsymbol{x}^r)\,, \end{aligned} \tag{12}$$

where $p_k^r = p_k^r(\boldsymbol{x}) = T_{,k}(\boldsymbol{x},\boldsymbol{x}^r)$ is the k-th component of the slowness vector $\boldsymbol{p}^r(\boldsymbol{x})$ at the point $\boldsymbol{x}$ for the ray coming from the receiver at $\boldsymbol{x}^r$. This approximation is compatible with GRA in (i) as both have errors $O(\omega^{-1})$ when compared to the terms that are retained.

(iii) The incident wavefield at the surface Σ from an arbitrary source at $\boldsymbol{x}^s$ is given by a GRA in the form

$$U_i^{\text{inc}}(\boldsymbol{x}) = h_i^s(\boldsymbol{x})\; A^{\text{inc}}(\boldsymbol{x},\boldsymbol{x}^s)\; e^{i\omega\, T(\boldsymbol{x},\boldsymbol{x}^s)}\,, \tag{13}$$

with the amplitude $A^{\text{inc}}(\boldsymbol{x},\boldsymbol{x}^s)$ and the traveltime $T(\boldsymbol{x},\boldsymbol{x}^s)$.

(iv) The outgoing reflection wavefield $U_i(\boldsymbol{x})$ at the surface is now replaced by a Kirchhoff-type approximation for anisotropic media, namely a sum of GRA wavemodes (one qP and two qS waves), each of them of the form

$$U_i^{\text{refl}}(\boldsymbol{x}) = h_i^{\text{spec}}(\boldsymbol{x})\; R^A(\boldsymbol{x},\boldsymbol{p}^s)\; A^{\text{inc}}(\boldsymbol{x},\boldsymbol{x}^s)\; e^{i\omega\, T(\boldsymbol{x},\boldsymbol{x}^s)}\,, \tag{14}$$

where $h_i^{\text{spec}}(\boldsymbol{x})$ is the polarization vector corresponding to a specular reflected wave of proper mode at the point $\boldsymbol{x}$ on Σ, due to the incident wave and $R^A(\boldsymbol{x},\boldsymbol{p}^s)$ is the amplitude-normalized plane-wave reflection coefficient for our choice of incoming and outgoing wave mode. Also, $\boldsymbol{p}^s(\boldsymbol{x})$ is the slowness vector at the point $\boldsymbol{x}$ for the ray coming from the source.

(v) The spatial derivatives of the outgoing wavefield at the surface will be given by the analogous anisotropic Kirchhoff approximation, namely the corresponding sum of GRA wavemodes

$$U_{i,k}^{\rm refl}(\boldsymbol{x}) = i\omega\, p_k^{\rm spec}\; U_i^{\rm refl}(\boldsymbol{x}) \;, \tag{15}$$

where $\boldsymbol{p}^{\rm spec}(\boldsymbol{x})$ is the slowness vector of the specular reflected wave, that is, it is related to $\boldsymbol{p}^s$ by Snell's law for a plane wave incident on a plane reflector.

Due to linearity, when all these approximations are used in the representation integral in equation (11), the result is a sum of integrals, one for each wavemode in the reflected field. Each of these integrals has the form

$$\begin{aligned} U_m^{KH}(\boldsymbol{x}^r) &= i\omega \int_\Sigma h_m(\boldsymbol{x}^r)\; A(\boldsymbol{x},\boldsymbol{x}^r)\; A^{\rm inc}(\boldsymbol{x},\boldsymbol{x}^s)\; c_{ijkl}(\boldsymbol{x})\; n_j(\boldsymbol{x}) \\ &\times \left\{ h_i^r(\boldsymbol{x})\, p_l^{\rm spec}(\boldsymbol{x})\, h_k^{\rm spec}(\boldsymbol{x}) - h_k^r(\boldsymbol{x})\, p_l^r(\boldsymbol{x})\, h_i^{\rm spec}(\boldsymbol{x}) \right\} \\ &\times e^{i\omega[T(\boldsymbol{x},\boldsymbol{x}^r)+T(\boldsymbol{x},\boldsymbol{x}^s)]}\; R^A(\boldsymbol{x},\boldsymbol{p}^s(\boldsymbol{x}))\; d\sigma \;. \end{aligned} \tag{16}$$

The above-obtained Kirchhoff-Helmholtz modeling integral is of particular interest when the incident wavefield of equation (13), $U_i^{\rm inc}(\boldsymbol{x})$, is replaced by the GRA Green's function of equation (4), $G_{in}(\boldsymbol{x},\omega;\boldsymbol{x}^s)$. Note that the GRA Green's function also consists of three wavemodes (one qP and two qS). The explicit expression of the resulting representation integral due to each of these modes is readily found to be

$$\begin{aligned} G_{mn}^{KH}(\boldsymbol{x}^r,\boldsymbol{x}^s) &= i\omega \int_\Sigma h_m(\boldsymbol{x}^r)\; A(\boldsymbol{x},\boldsymbol{x}^r)\; A(\boldsymbol{x},\boldsymbol{x}^s)\; c_{ijkl}(\boldsymbol{x})\; n_j(\boldsymbol{x}) \\ &\times \left\{ h_i^r(\boldsymbol{x})\, p_l^{\rm spec}(\boldsymbol{x})\, h_k^{\rm spec}(\boldsymbol{x}) - h_k^r(\boldsymbol{x})\, p_l^r(\boldsymbol{x})\, h_i^{\rm spec}(\boldsymbol{x}) \right\} \\ &\times e^{i\omega[T(\boldsymbol{x},\boldsymbol{x}^r)+T(\boldsymbol{x},\boldsymbol{x}^s)]}\; R^A(\boldsymbol{x},\boldsymbol{p}^s(\boldsymbol{x}))\; h_n(\boldsymbol{x}^s)\; d\sigma \;. \end{aligned} \tag{17}$$

The Kirchhoff-Helmholtz integral does not satisfy reciprocity, in contrast to the Born-Kirchhoff scattering integral derived in Ursin and Tygel [13]. Note that the geometrical-spreading decomposition formula given in that reference is only valid for isotropic media. In anisotropic media, the results from the present Appendix B should be used, instead. However, the surface scattering integral is correctly stated there.

4. The stationary-phase approximation

We want to compute the stationary values of the surface scattering integral of the type

$$I = i\omega \int_\Sigma b(\boldsymbol{x})\; e^{i\omega T(\boldsymbol{x})}\; d\boldsymbol{\sigma} \;, \tag{18}$$

to leading order in the high-frequency ω.

The stationary points, $\tilde{\boldsymbol{x}}$, are those points for which

$$\frac{\partial T}{\partial \sigma_j} = \frac{\partial T}{\partial x_k} \frac{\partial x_k}{\partial \sigma_j} = \boldsymbol{\nabla} T \cdot \boldsymbol{t}_j \ = 0 \ , \quad i,j,= 1,2 \ , \tag{19}$$

where $\boldsymbol{t}_j$ are the surface tangents. This condition is equivalent to Snell's law. For simplicity, we assume that there is only one stationary point $\tilde{\boldsymbol{x}}$ and also that this is a regular stationary point. This means that the second-derivative matrix,

$$H_{ij} = \frac{\partial^2 T}{\partial \sigma_i \partial \sigma_j} = \frac{\partial^2 T}{\partial x_n \partial x_k} \frac{\partial x_n}{\partial \sigma_i} \frac{\partial x_k}{\partial \sigma_j} \ , \qquad i,j = 1,2 \ , \tag{20}$$

evaluated at $\tilde{\boldsymbol{x}}$ is non-singular, $\det \boldsymbol{H} \neq 0$. Then the stationary value of the integral is (Bleistein [4], equation (2.8.23))

$$\tilde{I} = i\omega \left(\frac{2\pi}{|\omega|} \right) \ |\det \boldsymbol{H}|^{-1/2} \ e^{i \frac{\pi}{4} \operatorname{sgn}(\omega) \operatorname{Sgn}(\boldsymbol{H})} \ b(\tilde{\boldsymbol{x}}) \ e^{i\omega T(\tilde{\boldsymbol{x}})} \ , \tag{21}$$

where $\tilde{\boldsymbol{x}} = \boldsymbol{x}(\tilde{\boldsymbol{\sigma}})$ is the stationary point and $\operatorname{Sgn}(\boldsymbol{H})$ is the signature of the matrix $\boldsymbol{H}$, that is, the difference between the number of its positive eigenvalues and the number of its negative eigenvalues.

The stationary point $\tilde{\boldsymbol{x}}$ is a point of specular reflection, so that $\boldsymbol{h}^{\mathrm{spec}}(\tilde{\boldsymbol{x}}) = \boldsymbol{h}^r(\tilde{\boldsymbol{x}})$ and $\boldsymbol{p}^{\mathrm{spec}}(\tilde{\boldsymbol{x}}) = -\boldsymbol{p}^r(\tilde{\boldsymbol{x}})$ because $\boldsymbol{p}^r(\tilde{\boldsymbol{x}})$ is the slowness for the ray going from $\boldsymbol{x}^r$ to $\boldsymbol{x}$, so that it is pointing downwards at the interface.

A stationary-phase evaluation of the integral (18) yields

$$\begin{aligned} G_{mn}^{KH}(\boldsymbol{x}^r, \boldsymbol{x}^s) \ &\approx \ 4\pi \ h_m(\boldsymbol{x}^r) \ |\det \boldsymbol{H}|^{-1/2} \ e^{-i\frac{\pi}{2} \operatorname{sgn}(\omega) \, [1 - \frac{1}{2} \operatorname{Sgn}(\boldsymbol{H})]} \\ &\times \ c_{ijkl}(\tilde{\boldsymbol{x}}) \, h_l^r(\tilde{\boldsymbol{x}}) \, h_k^r(\tilde{\boldsymbol{x}}) \, p_l(\tilde{\boldsymbol{x}}) \, n_j(\tilde{\boldsymbol{x}}) \, R^A(\tilde{\boldsymbol{x}}, \boldsymbol{p}^s(\tilde{\boldsymbol{x}})) \\ &\times \ A(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \, A(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \, e^{i\omega[T(\boldsymbol{x}^r, \boldsymbol{x}) + T(\boldsymbol{x}, \boldsymbol{x}^s)]} \, h_n(\boldsymbol{x}^s) \end{aligned} \tag{22}$$

for the reflected field. The matrix $\boldsymbol{H}$ is called the Fresnel matrix corresponding to the reflection ray (Hubral et al. [10]). Here we note that (Červený [11])

$$c_{ijkl}(\tilde{\boldsymbol{x}}) \, h_i^r(\tilde{\boldsymbol{x}}) \, h_k^r(\tilde{\boldsymbol{x}}) \, p_i^r(\tilde{\boldsymbol{x}}) \, n_j(\tilde{\boldsymbol{x}}) = \rho(\tilde{\boldsymbol{x}}) \, V_j^r(\tilde{\boldsymbol{x}}) \, n_j(\tilde{\boldsymbol{x}}) = \rho(\tilde{\boldsymbol{x}}) \, V^r(\tilde{\boldsymbol{x}}) \ \cos\alpha^r \ , \tag{23}$$

where α^r is the angle between the surface normal and the ray coming from the receiver; see Figure 1. We now use the energy-flux normalized reflection coefficient

$$R(\tilde{\boldsymbol{x}}) = R^A(\tilde{\boldsymbol{x}}, \boldsymbol{p}^s(\tilde{\boldsymbol{x}})) \ \left[\frac{V^r \ \cos\alpha^r}{V^s \ \cos\alpha^s} \right]^{1/2} \ . \tag{24}$$

Combining equations (22) to (24), and taking into account the amplitude formula in equation (9), we obtain that the stationary value of the Kirchhoff-Helmholtz integral is equal to the GRA for the reflected wavefield

$$G_{mn}^{KH}(\boldsymbol{x}^r, \boldsymbol{x}^s) \approx G_{mn}^{R}(\boldsymbol{x}^r, \boldsymbol{x}^s) = h_m(\boldsymbol{x}^r) \ A(\boldsymbol{x}^r, \boldsymbol{x}^s) \ e^{i\omega \, T(\boldsymbol{x}^r, \boldsymbol{x}^s)} \ R(\tilde{\boldsymbol{x}}) \ h_n(\boldsymbol{x}^s) \ . \tag{25}$$

Here, $R(\tilde{\boldsymbol{x}})$ is given in equation (24), the total traveltime is

$$T(\boldsymbol{x}^r, \boldsymbol{x}^s) = T(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) + T(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \ , \tag{26}$$

and the total amplitude is given as in equation (9), with

$$\kappa(\boldsymbol{x}^r, \boldsymbol{x}^s) = \kappa(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) + \kappa(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) + 1 - \mathrm{Sgn}(\boldsymbol{H})/2 \ . \tag{27}$$

Finally,

$$|\det \boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s)| = \frac{|\det \boldsymbol{H} \ \det \boldsymbol{Y}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \ \det \boldsymbol{Y}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)|}{\cos\alpha^r \ \cos\alpha^s} \ , \tag{28}$$

where equation (B-16) has been used.

5. Conclusions

We have extended the Kirchhoff-Helmholtz integral to general anisotropic media. The upgoing scattered field at the interface was replaced by the specularly reflected field, as approximated by the GRA. Within the validity of the GRA, the new integral formula can be used to compute multiply reflected and converted waves in anisotropic media. This also includes a possible wave-mode conversion at the interface. The present approach provides a "single-event" approximation that enables us to determine one specifically chosen reflection without having to calculate all other events that might be considered noise in the actual problem. The complete wavefield at the receiver is, then, the superposition of all possible events that can be calculated independently (but simultaneously, if so desired) by the corresponding Kirchhoff-Helmholtz integrals.

We have also extended the decomposition formula for the relative geometrical spreading factor from isotropic to anisotropic elastic media. This generalization has been done independently, based only on ray-theoretical arguments. The resulting decomposition formula provides the means to calculate the geometrical spreading of a primary reflected ray in terms of the spreading factors of the incident and reflected ray segments and a third factor that accounts for the influence of the interface.

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Appendix A

Relationship between the relative geometrical-spreading matrices

In this appendix we derive expressions for the relative geometrical spreading given through the matrices $\boldsymbol{Q}_2$ and $\boldsymbol{Y}$ defined in the main text. We start with the general expression

$$\frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial x_i^s \partial x_j^r} = \frac{\partial g_n^s}{\partial x_i^s} \frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial g_n^s \partial g_m^r} \frac{\partial g_m^r}{\partial x_j^r} \tag{A-1}$$

Here, the indices i and j are fixed and vary from 1 to 3. A summation from 1 to 3 is understood for the indices n and m. The orthogonal ray coordinates systems g_n^s and g_m^r are chosen such that g_3^s and g_3^r are in the direction of the ray (and the group velocity) at $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$, respectively. We use the notation of (26) for the total traveltime from $\boldsymbol{x}^s$ to the reflector to $\boldsymbol{x}^r$. (The dependence of the traveltime on the reflector point is unimportant in this discussion.) We have that

$$\frac{\partial T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial g_i^s} = p_i^{(g)}(\boldsymbol{x}^s) \ , \tag{A-2}$$

where $p_i^{(g)}(\boldsymbol{x}^s)$ denote the components of the slowness vector at $\boldsymbol{x}^s$, expressed in the g_i^s-coordinate system. Observe that as the receiver position is changed in the ray direction, that is, as g_3^r varies along the ray, this gradient does not vary. Thus, taking the derivative with respect to g_3^r—that is, taking a derivative along the ray at $\boldsymbol{x}^r$—does not change this slowness. This means that

$$\frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial g_3^r \partial g_i^s} = 0 \ . \tag{A-3}$$

Similarly, we have that

$$\frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial g_3^s \partial g_i^r} = 0 \ . \tag{A-4}$$

This means that the summations over the indices n and m in equation (A-1) need to be taken from 1 to 2 only. Next we define the 2×2 matrix $\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s)$ with components

$$B_{ij}(\boldsymbol{x}^r, \boldsymbol{x}^s) = - \left[\frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial x_i^s \partial x_j^r} \right]^{-1} \ , \quad i, j = 1, 2. \tag{A-5}$$

Then the upper left 2×2 part in equation (A-1) may be written

$$\boldsymbol{B}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{\Lambda}^T(\boldsymbol{x}^s)\, \boldsymbol{Y}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s)\, \boldsymbol{\Lambda}(\boldsymbol{x}^r) \ , \tag{A-6}$$

where $\boldsymbol{\Lambda}(\boldsymbol{x}^s)$ is the upper left 2×2 sub-matrix of the full 3×3 transformation matrix $(\partial g_m^s / \partial x_i)$. This is a general 3-D rotation matrix that can be decomposed into three elementary rotations, being one around the 3-axis, a second one around the resulting 2-axis, and a third one around the new 3-axis. Therefore, their upper left 2×2 submatrices can be decomposed into three elementary matrices, being two rotation matrices and a projection matrix, namely

$$\mathbf{\Lambda}(\boldsymbol{x}^s) = \begin{pmatrix} \cos\gamma(\boldsymbol{x}^s) & \sin\gamma(\boldsymbol{x}^s) \\ -\sin\gamma(\boldsymbol{x}^s) & \cos\gamma(\boldsymbol{x}^s) \end{pmatrix} \begin{pmatrix} \cos\alpha(\boldsymbol{x}^s) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi(\boldsymbol{x}^s) & \sin\phi(\boldsymbol{x}^s) \\ -\sin\phi(\boldsymbol{x}^s) & \cos\phi(\boldsymbol{x}^s) \end{pmatrix} . \quad \text{(A-7)}$$

Here, $\phi(\boldsymbol{x}^s)$ and $\gamma(\boldsymbol{x}^s)$ denote the in-plane rotation angles around the old and new 3-axes, respectively, and $\alpha(\boldsymbol{x}^s)$ denotes the angle between the group velocity vector and the x_3^s-axis. The fact that the form (A-7) of the 2×2 matrix $\mathbf{\Lambda}(\boldsymbol{x}^s)$ is exact can be verified by computing the upper left 2×2 matrix of the full 3×3 transformation matrix in equation (A-1) following the previous description. The matrix $\mathbf{\Lambda}(\boldsymbol{x}^r)$ is similarly defined. Then it follows directly that

$$\det \boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \cos\alpha(\boldsymbol{x}^r) \cos\alpha(\boldsymbol{x}^s) \ \det \boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) \ . \quad \text{(A-8)}$$

Next we choose the $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$ coordinate systems to be equal to the phase-front coordinate $\boldsymbol{q}^s$ and $\boldsymbol{q}^r$, respectively. Then, equation (A-1) can be written as in equation (A-6), so that

$$\boldsymbol{Q}_2^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \mathbf{\Gamma}^T(\boldsymbol{x}^s) \, \boldsymbol{Y}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) \, \mathbf{\Gamma}(\boldsymbol{x}^r) \ , \quad \text{(A-9)}$$

where the 2×2 transformation matrix now is defined by

$$\mathbf{\Gamma}(\boldsymbol{x}^s) = \begin{pmatrix} \cos\lambda(\boldsymbol{x}^s) & \sin\lambda(\boldsymbol{x}^s) \\ -\sin\lambda(\boldsymbol{x}^s) & \cos\lambda(\boldsymbol{x}^s) \end{pmatrix} \begin{pmatrix} \cos\chi(\boldsymbol{x}^s) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\mu(\boldsymbol{x}^s) & \sin\mu(\boldsymbol{x}^s) \\ -\sin\mu(\boldsymbol{x}^s) & \cos\mu(\boldsymbol{x}^s) \end{pmatrix} . \quad \text{(A-10)}$$

Here, $\lambda(\boldsymbol{x}^s)$ and $\mu(\boldsymbol{x}^s)$ are rotation angles defined in a similar way as $\gamma(\boldsymbol{x}^s)$ and $\phi(\boldsymbol{x}^s)$ defined previously, and $\chi(\boldsymbol{x}^s)$ is the angle between the phase velocity and group velocity vectors at $\boldsymbol{x}^s$. The matrix $\mathbf{\Gamma}(\boldsymbol{x}^r)$ is similarly defined.

We have that (Červený [11])

$$\cos\chi = \frac{v}{V} \ , \quad \text{(A-11)}$$

so that equation (A-9) directly yields

$$V(\boldsymbol{x}^r) \, V(\boldsymbol{x}^s) \ \det \boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s) = v(\boldsymbol{x}^r) \, v(\boldsymbol{x}^s) \ \det \boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s) \ . \quad \text{(A-12)}$$

A direct relationship between the matrices $\boldsymbol{B}$ and $\boldsymbol{Q}_2$ can be established by combining equations (A-6) and (A-9). It reads

$$\boldsymbol{Q}_2^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \mathbf{\Gamma}^T(\boldsymbol{x}^s) \, \mathbf{\Lambda}^{-T}(\boldsymbol{x}^s) \, \boldsymbol{B}^{-1}(\boldsymbol{x}^r, \boldsymbol{x}^s) \, \mathbf{\Lambda}^{-1}(\boldsymbol{x}^r) \, \mathbf{\Gamma}(\boldsymbol{x}^r) \ . \quad \text{(A-13)}$$

It is instructive to note in the above derivation the crucial role played by the matrix $\boldsymbol{Y}$ (in the $\boldsymbol{g}$-system) to establish the relationship (A-13) between the $\boldsymbol{B}$ matrix (in the $\boldsymbol{x}$-system) and the $\boldsymbol{Q}_2$ matrix (in the $\boldsymbol{q}$-system). A question that naturally arises is why is the intermediary $\boldsymbol{Y}$

matrix actually needed. A mathematical argument is that the invariance properties (A-3) and (A-4) are only valid in the $\boldsymbol{g}$ system, so that no one-step 2×2 matrix transformation is possible between quantities in the $\boldsymbol{x}$ and $\boldsymbol{q}$ coordinate systems. Physically, equations (A-3) and (A-4) express the fact that any dislocation of one end point of the ray in direction other than that of the group velocity also affects the direction of the slowness vector at its other end point.

Appendix B

Decomposition of the relative geometrical spreading

In this appendix, we derive the geometrical-spreading decomposition formula (28) in terms of the second-order mixed derivatives of the traveltime. This formula is crucial to the verification that the asymptotic evaluation of the Kirchhoff-Helmholtz integral provides the GRA for the reflected wave.

First, we derive a decomposition formula for the second-order mixed derivatives of traveltime for general Cartesian coordinate systems. Upon suitable specification of the coordinate systems, this formula will provide the relationships for the matrices $\boldsymbol{Q}_2$ and $\boldsymbol{Y}$ that are needed to derive the expressions for the geometrical-spreading used in the main text.

We consider fixed source and receiver pair, as well as a given smooth reflector Σ, as shown in Figure 1. Points in the vicinity of the source will be represented by $\boldsymbol{x}^s$ in a fixed, 3-D Cartesian system. Analogously, points in the vicinity of the receiver will be represented by $\boldsymbol{x}^r$ in a second, also fixed, 3-D Cartesian system. We assume that the given source-receiver pair determines a unique reflection point on the reflector. Points on the reflector surface Σ in the vicinity of the reflection point, will be represented by $\boldsymbol{\sigma}$ in a 2-D curvilinear coordinate system.

We can express the diffraction traveltime as the sum of the traveltimes,

$$T^D(\boldsymbol{x}^r,\boldsymbol{\sigma},\boldsymbol{x}^s) = T(\boldsymbol{x}^r,\boldsymbol{\sigma}) + T(\boldsymbol{\sigma},\boldsymbol{x}^s)\ , \tag{B-1}$$

along the ray segments that connect $\boldsymbol{x}^s$ to $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$ to $\boldsymbol{x}^r$, respectively. The reflection traveltime that corresponds to a source at $\boldsymbol{x}^s$ and a receiver at $\boldsymbol{x}^r$ will be expressed as

$$T^R(\boldsymbol{x}^r,\boldsymbol{x}^s) = T^D(\boldsymbol{x}^r,\tilde{\boldsymbol{\sigma}},\boldsymbol{x}^s)\ , \tag{B-2}$$

where the stationary point $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{x}^r,\boldsymbol{x}^s)$ is such that

$$\left.\frac{\partial T^D(\boldsymbol{x}^r,\boldsymbol{\sigma},\boldsymbol{x}^s)}{\partial\sigma_i}\right|_{\tilde{\boldsymbol{\sigma}}} = 0\ , \quad i = 1,2\ . \tag{B-3}$$

It is our aim to show that the second-order mixed derivatives of the reflection traveltime satisfies an important decomposition formula. We start by differentiating the reflection traveltime $T^R(\boldsymbol{x}^r,\boldsymbol{x}^s)$ with respect to x^r_j. We have

$$\frac{\partial T^R}{\partial x^r_j} = \frac{\partial}{\partial x^r_j}[T^D(\boldsymbol{x}^r,\tilde{\boldsymbol{\sigma}}(\boldsymbol{x}^r,\boldsymbol{x}^s),\boldsymbol{x}^s)] = \left.\frac{\partial T^D}{\partial x^r_j}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial T^D}{\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial\tilde{\sigma}_k}{\partial x^r_j}\ . \tag{B-4}$$

We next differentiate the above expression with respect to $\boldsymbol{x}^s_i$ to obtain

$$\frac{\partial^2 T^R}{\partial x^s_i\partial x^r_j} = \frac{\partial}{\partial x^s_i}\left\{\left.\frac{\partial T^D}{\partial x^r_j}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial T^D}{\partial\sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}}\frac{\partial\tilde{\sigma}_k}{\partial x^r_j}\right\}$$

$$
\begin{aligned}
&= \left\{ \left.\frac{\partial^2 T^D}{\partial x_i^s \partial x_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} + \frac{\partial \tilde{\sigma}_l}{\partial x_i^s} \left.\frac{\partial^2 T^D}{\partial \sigma_l \partial x_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} \right\} + \left\{ \left.\frac{\partial^2 T^D}{\partial x_i^s \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_k}{\partial x_j^r} \right. \\
&+ \left. \frac{\partial \tilde{\sigma}_l}{\partial x_i^s} \left.\frac{\partial^2 T^D}{\partial \sigma_l \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_k}{\partial x_j^r} + \left.\frac{\partial T^D}{\partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial^2 \tilde{\sigma}_k}{\partial x_i^s \partial x_j^r} \right\} . \qquad \text{(B-5)}
\end{aligned}
$$

We now observe, by the very definition of T^D as a traveltime sum along the incoming and outcoming ray segments, the properties

$$
\begin{aligned}
\frac{\partial^2 T^D(\boldsymbol{x}^s, \tilde{\boldsymbol{\sigma}}, \boldsymbol{x}^r)}{\partial x_i^s \partial \sigma_k} &= \frac{\partial^2 T(\boldsymbol{x}^s, \tilde{\boldsymbol{\sigma}})}{\partial x_i^s \partial \sigma_k} , \\
\frac{\partial^2 T^D(\boldsymbol{x}^s, \tilde{\boldsymbol{\sigma}}, \boldsymbol{x}^r)}{\partial x_j^r \partial \sigma_k} &= \frac{\partial^2 T(\boldsymbol{x}^r, \tilde{\boldsymbol{\sigma}})}{\partial x_i^s \partial \sigma_k} , \qquad \text{(B-6)} \\
\frac{\partial^2 T^D(\boldsymbol{x}^s, \tilde{\boldsymbol{\sigma}}, \boldsymbol{x}^r)}{\partial x_i^s \partial x_j^r} &= 0 .
\end{aligned}
$$

After the use of the stationary condition (B-3), our expression for the traveltime second derivative becomes

$$
\begin{aligned}
\frac{\partial^2 T^R(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial x_i^s \partial x_j^r} &= \frac{\partial \tilde{\sigma}_l}{\partial x_i^s} \left.\frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{\sigma})}{\partial \sigma_l \partial x_j^r}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial^2 T(\boldsymbol{x}^s, \boldsymbol{\sigma})}{\partial x_i^s \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_k}{\partial x_j^r} \\
&+ \frac{\partial \tilde{\sigma}_l}{\partial x_i^s} \left.\frac{\partial^2 T^D}{\partial \sigma_l \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_k}{\partial x_j^r} . \qquad \text{(B-7)}
\end{aligned}
$$

In the above expression, the unknown quantities $\partial\tilde{\sigma}_i / \partial x_j^{s,r}$ can be determined by differentiating the stationary condition (B-3) with respect to the source/receiver coordinates $x_i^{s,r}$. Since the stationary condition holds independently of $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$, we find

$$
\frac{\partial}{\partial x_i^{s,r}} \left[\left.\frac{\partial T^D}{\partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \right] = \left.\frac{\partial^2 T^D}{\partial x_i^{s,r} \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} + \left.\frac{\partial^2 T^D}{\partial \sigma_l \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_l}{\partial x_i^{s,r}} = 0 . \qquad \text{(B-8)}
$$

We now use the properties (B-6) to obtain

$$
\begin{aligned}
\left.\frac{\partial^2 T^D}{\partial \sigma_l \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_l}{\partial x_i^s} + \left.\frac{\partial^2 T(\boldsymbol{x}^s, \tilde{\boldsymbol{\sigma}})}{\partial x_i^s \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} &= 0, \\
& \qquad \text{(B-9)} \\
\left.\frac{\partial^2 T^D}{\partial \sigma_l \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} \frac{\partial \tilde{\sigma}_l}{\partial x_i^r} + \left.\frac{\partial^2 T(\boldsymbol{x}^r, \tilde{\boldsymbol{\sigma}})}{\partial x_i^r \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} &= 0.
\end{aligned}
$$

Recasting the above equations in matrix form, leads to the alternative system

$$
\begin{aligned}
\frac{\partial \tilde{\sigma}_l}{\partial x_i^s} &= - \left.\frac{\partial^2 T(\boldsymbol{x}^s, \boldsymbol{\sigma})}{\partial x_i^s \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} D_{kl}, \\
& \qquad \text{(B-10)} \\
\frac{\partial \tilde{\sigma}_l}{\partial x_i^s} &= - D_{lk} \left.\frac{\partial^2 T(\boldsymbol{x}^s, \boldsymbol{\sigma})}{\partial x_i^s \partial \sigma_k}\right|_{\tilde{\boldsymbol{\sigma}}} ,
\end{aligned}
$$

where D_{kl} denote the components of the symmetric matrix

$$\boldsymbol{D} = \boldsymbol{H}^{-1} = \left(\frac{\partial^2 T^D(\boldsymbol{x}^r, \boldsymbol{\sigma}, \boldsymbol{x}^s)}{\partial\sigma_i \partial\sigma_j} \bigg|_{\tilde{\boldsymbol{\sigma}}} \right)^{-1} . \tag{B-11}$$

This is the inverse of the so-called Fresnel matrix $\boldsymbol{H}$ corresponding to the source-receiver points $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$ and the reflection point $\tilde{\boldsymbol{\sigma}}$.

Substituting these expressions into equation (B-7) and using the definition of $\boldsymbol{D}$, this gives the decomposition formula

$$\frac{\partial^2 T^R(\boldsymbol{x}^r, \boldsymbol{x}^s)}{\partial x_i^s \partial x_j^r} = - \frac{\partial^2 T(\boldsymbol{\sigma}, \boldsymbol{x}^s)}{\partial x_i^s \partial\sigma_j} \bigg|_{\tilde{\boldsymbol{\sigma}}} D_{kl} \frac{\partial^2 T(\boldsymbol{x}^r, \boldsymbol{\sigma})}{\partial\sigma_l \partial x_j^r} \bigg|_{\tilde{\boldsymbol{\sigma}}} . \tag{B-12}$$

Using the matrix $\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s)$ defined by equation (A-5), this can now be written

$$\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{B}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; \boldsymbol{H}(\tilde{\boldsymbol{x}}) \; \boldsymbol{B}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) . \tag{B-13}$$

¿From equation (A-6), we have that

$$\boldsymbol{B}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{\Lambda}^{-1}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; \boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s)) \; \boldsymbol{\Lambda}^{-T}(\boldsymbol{x}^s) . \tag{B-14}$$

When this expression and similar expressions for $\boldsymbol{B}(\boldsymbol{x}^s, \tilde{\boldsymbol{x}})$ and $\boldsymbol{B}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)$ are used in equation (B-13), we obtain

$$\boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{Y}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; (\boldsymbol{\Lambda}^r)^{-T}(\tilde{\boldsymbol{x}}) \; \boldsymbol{H}(\tilde{\boldsymbol{x}}) \; (\boldsymbol{\Lambda}^s)^{-1}(\tilde{\boldsymbol{x}}) \boldsymbol{Y}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) . \tag{B-15}$$

Here, $\boldsymbol{\Lambda}^s(\tilde{\boldsymbol{x}})$ and $\boldsymbol{\Lambda}^r(\tilde{\boldsymbol{x}})$ are the transformation matrices at $\tilde{\boldsymbol{x}}$ corresponding to the ray from the source and receiver, respectively.

¿From the above equation, it follows that

$$\det \boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \frac{\det \boldsymbol{Y}(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; \det \boldsymbol{H}(\tilde{\boldsymbol{x}}) \det \boldsymbol{Y}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)}{\cos\alpha^r \; \cos\alpha^s} . \tag{B-16}$$

Equation (A-9) can be recast into the form

$$\boldsymbol{Y}(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{\Gamma}(\boldsymbol{x}^r) \boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s) \boldsymbol{\Gamma}^T(\boldsymbol{x}^s) , \tag{B-17}$$

so that equation (B-15) leads to the decomposition formula,

$$\boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s) = \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; (\boldsymbol{\Gamma}^r)^T(\tilde{\boldsymbol{x}}) \; (\boldsymbol{\Lambda}^r)^{-T}(\tilde{\boldsymbol{x}}) \; \boldsymbol{H}(\tilde{\boldsymbol{x}}) \; (\boldsymbol{\Lambda}^s)^{-1}(\tilde{\boldsymbol{x}}) \; \boldsymbol{\Gamma}^s(\tilde{\boldsymbol{x}}) \; \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) . \tag{B-18}$$

Here, $\boldsymbol{\Gamma}^s(\tilde{\boldsymbol{x}})$ and $\boldsymbol{\Gamma}^r(\tilde{\boldsymbol{x}})$ are transformation matrices for the ray coming from the source to the receiver, respectively. Finally, we obtain

$$\det \boldsymbol{Q}_2(\boldsymbol{x}^r, \boldsymbol{x}^s) = \det \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}) \; \det \boldsymbol{H}(\tilde{\boldsymbol{x}}) \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \; \frac{\cos\chi^r \; \cos\chi^s}{\cos\alpha^r \; \cos\alpha^s} \tag{B-19}$$

as the decomposition formula for the relative geometrical-spreading factor in anisotropic media.

Appendix C

Derivation of Equation (11)

In this appendix, we discuss the derivation of equation (11) for the upward scattered wave. First, let us consider the total wavefield above the reflector, $\boldsymbol{U}^{\mathrm{tot}}(\boldsymbol{x})$. Then, Aki and Richards [2], equation (2.41) states that

$$\begin{aligned} U_m^{\mathrm{tot}}(\boldsymbol{x}^r) &= \int_S \Big\{ G_{im}(\boldsymbol{x}, \boldsymbol{x}^r)\, c_{ijkl}(\boldsymbol{x})\, U_{k,l}^{\mathrm{tot}}(\boldsymbol{x}) \\ &- G_{km,l}(\boldsymbol{x}, \boldsymbol{x}^r)\, c_{ijkl}(\boldsymbol{x})\, U_i^{\mathrm{tot}}(\boldsymbol{x}) \Big\}\, n_j \, d\sigma \,. \end{aligned} \qquad \text{(C-1)}$$

Here, S is a surface confined to the upper domain, possibly including a portion of the reflector, Σ, but also including the points, $\boldsymbol{x}^s$ and $\boldsymbol{x}^r$. For our purposes, we introduce a sphere of radius R, centered at a point on Σ. We then take S to consist of the portion of the sphere above the reflector, denoted by S_R and the portion of Σ interior to the sphere, denoted by Σ_R. See Figure 2.

Our first objective is to allow the radius of the sphere approach infinity and argue away the integral over the spherical portion of S; then, only the integral over Σ will remain. For isotropic elastic media the appropriate generalization of the Sommerfeld radiation conditions to assure this result is readily available in the literature. See, for example, Achenbach, et al, [1] or Kupradze [5]. Discussion of the appropriate extension of these radiation conditions to anisotropic media are less accessible. However, there is an alternative method for assuring that the integral over this sphere vanishes with increasing radius. Recall that the underlying problem here is an initial value problem in the time domain, a so-called *causal* problem. For such problems and our sign convention in the phase, the Fourier transform is initially defined in some upper-half complex-valued ω-plane, above all singularities of the transformed wavefield. Further, the solution must decay to zero as $|\omega| \to \infty$ in that upper half plane. Note that solutions with phase factor, $\exp\{i\omega T(\boldsymbol{x}, \boldsymbol{x}^s)\}$, have this property, while solutions with phase factor, $\exp\{-i\omega T(\boldsymbol{x}, \boldsymbol{x}^s)\}$, do not. Thus, this property distinguishes between incoming and outgoing wave types and identifies the acceptable fields at infinity. Further, the solutions decay exponentially in the upper half plane. We can think of this condition as a *causality condition.* Typically, these solutions have singularities on the $\Re\{\omega\}$ axis. Thus, solutions on that line should be obtained by analytic continuation from above. It is only when we insist on evaluating solutions solely for real values of ω that we need to resort to radiation conditions to distinguish between incoming (unacceptable) and outgoing (acceptable) solutions at infinity.

With this in mind, then, we allow the radius R of S_R approach infinity. In that limit, the exponential decay of the wavefields assures that the integral over S_R approaches zero and the surface Σ_R approaches Σ. Thus, we can replace S by Σ in (C-1).

Next, we must address the question of concluding the same representation result for the upward scattered wave, $\boldsymbol{U}(\boldsymbol{x})$.

Both $\boldsymbol{U}^{\mathrm{inc}}(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$ also satisfy the appropriate radiation conditions or causality condition for observation points above the reflector. It is then fairly straightforward to show that $\boldsymbol{U}^{\mathrm{inc}}(\boldsymbol{x}^r)$ is given by the right side of (11) if the wavefield used in the integrand is $\boldsymbol{U}^{\mathrm{inc}}(\boldsymbol{x})$. When we subtract that identity from the representation for the total wavefield, the result is (11) for an upward scattered wavefield, alone.

The wavefield, $\boldsymbol{U}^{\mathrm{inc}}(\boldsymbol{x}^r)$, contains no response to the reflector, since it is derived in the absence of the reflector. Thus, all responses to the presence of the reflector are contained in $\boldsymbol{U}(\boldsymbol{x})$. What is important to us, here, is that it contains all of the reflected wavefields, unconverted and converted, in response to the point source on the right side of (2).

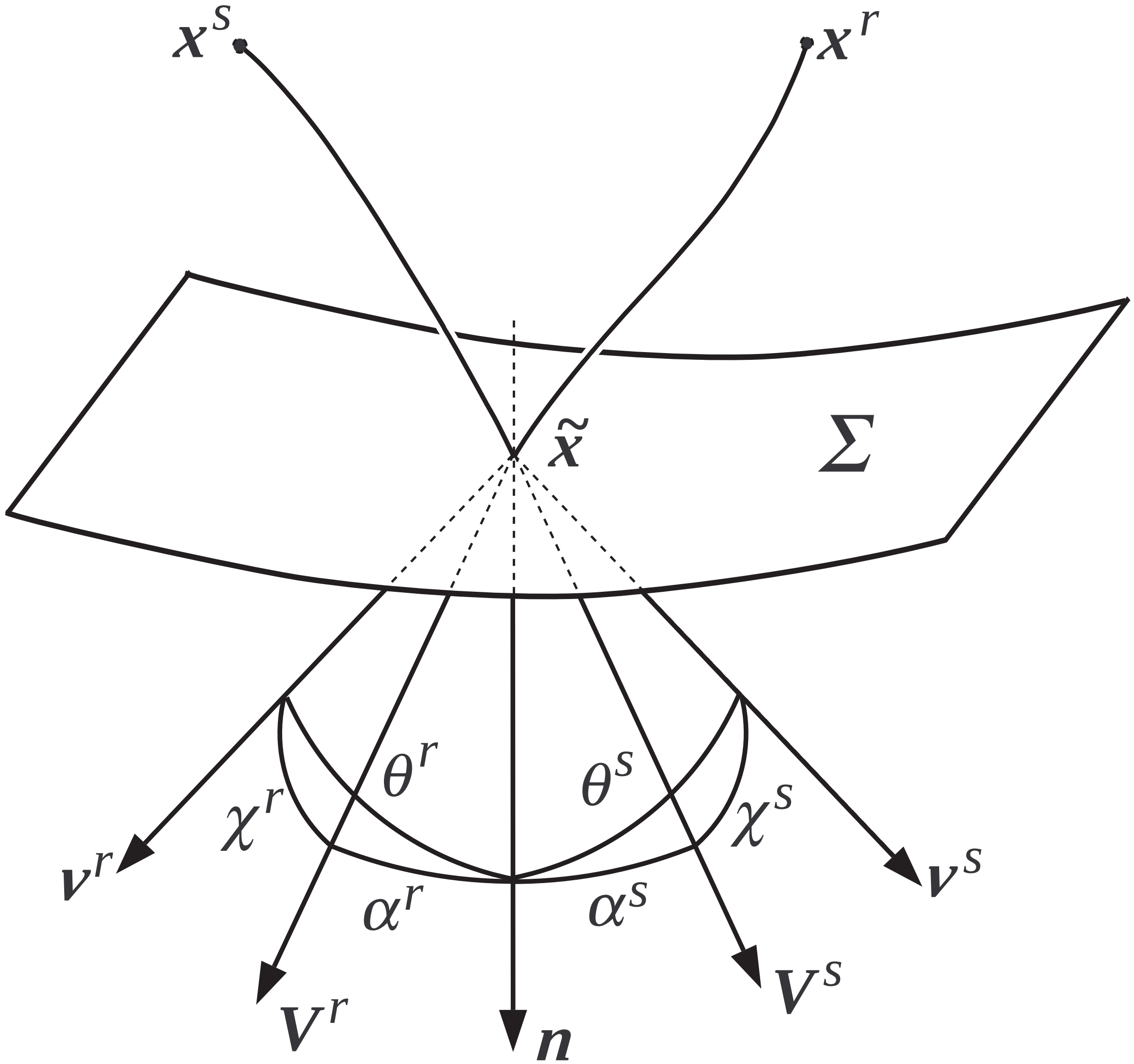

Figure 1: Geometry of the reflection point

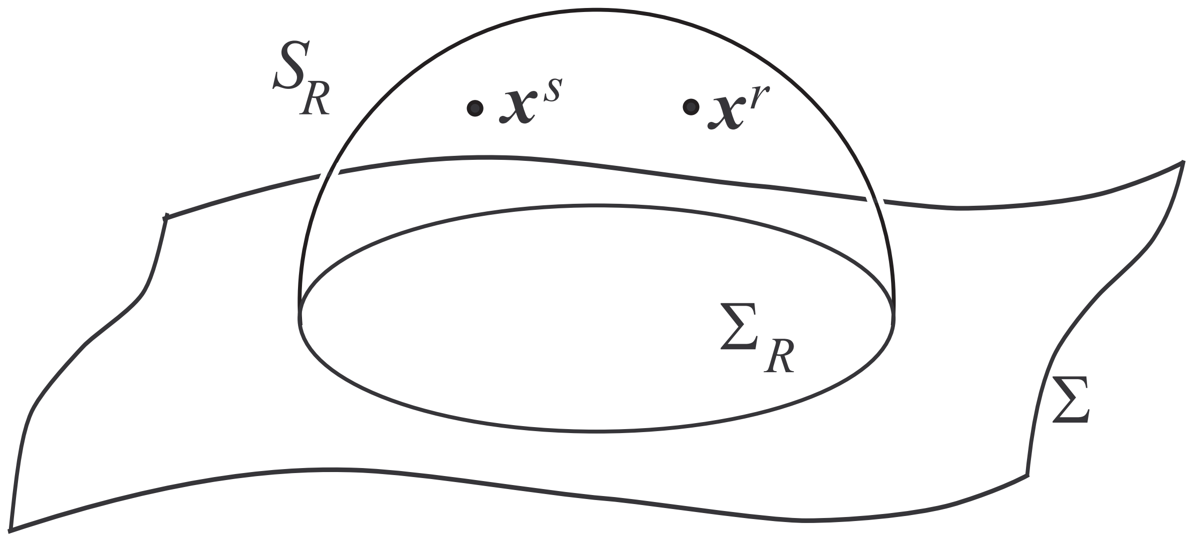

Figure 2: The integration domain for equation (C-1)