Some Results on Centers of Polytopes

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Abstract

The main ingredient for polynomiality in interior point methods is the centering procedure. All interior point algorithms for solving linear programming problems, known to be polynomial, has an explicit or implicit mechanism for finding a center of the linear programming polytope. Therefore, we consider the study of center of polytope as a serious work to be done. In this work, we want to talk about three kinds of center of a polyotpe and its relations among them.

1 Introduction

Since 1984, when Karmarkar [9] proposed his famous projective-scaling algorithm, several researchers developed and implemented new ideias in interior

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point methods for solving linear programming problems.Every time an algorithm was proved to be polynomial-time algorithm, we observed that the algorithm carries out an explicity or implicit centering procedure. This fact, motivated us to study centers of polytopes, trying to obtain centers in a efficient way.

The first center of a polytopes used in optmization was the center of gravity (or centroid) developed by Levin [12] in his method of central sections for minimizing a convex function f over a convex polytope P. The problem with this center is that it is very difficult to compute. Yudin and Nemirovskii proposed a modified version of Levin's method [21] which used ellipsoids instead of a polyhedra. In fact, it was a special case of Shor's algorithm [17] with space dilation in the direction of the subgradient. Shor developed the ellipsoid method [17] in 1977 and in 1979, Khachyan [10] showed that the ellipsoid method is a polynomial-time algorithm for solving linear programming problems.

Later on, Tarasov, Khachyan and Erlich [18] developed the method of inscribed ellipsoid in a polytope as the center of the polytope.

Finally, in 1984, Karmarkar [9] presented his innovative polynomial-time algorithm for solving linear programming problems which also computes centers of polytopes. Although his method uses centers in a different way, since in each iteration, the current point is mapped by a projective transformation such that it becomes the analytic center (see definition in the next section) of the transformed region.

Renegar [13] used the analytic center to develop his polynomial-time algorithm. The analytic center is, no doubt about that, the most used notion of center of poloytope in linear optmization. It is easy to compute but, on the other hand, it can be pushed towards a boundary (not a good feature) of the polytope depending on the spatial positions of the hyperplanes associated with the constraints that define the polytope.

Vaidya [20] uses another notion of center, the volumetric center, which is the center of ellipsoid with largest volume among a certain set of ellipsoids that are contained in P. By using this notion of center, he also proved that a polynomial-time algorithm can be obtained for solving linear programming problems.

From what is presented above, we realize that centers of a polytope play an important role on linear optmization. Therefore, we presented in this paper, three different notions of center of polytopes : Helly center, John

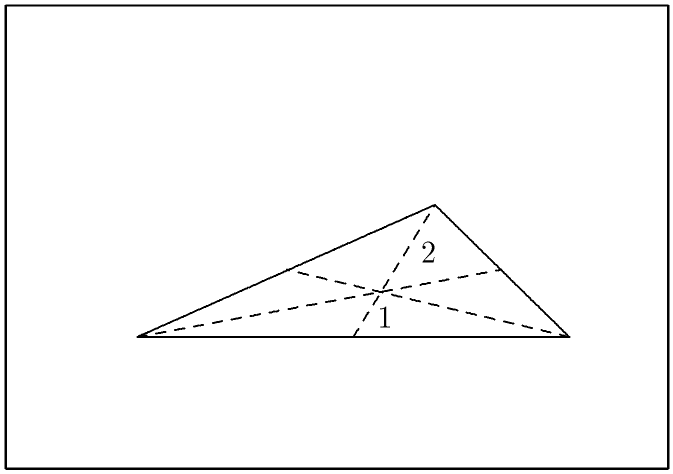

Figure 1: The centroid of a triangle divides medians in the ratio 1:2.

Center and Analytic Center. we described the relations among them by showing that the John center is a Helly center. We also showed that a convex body has infinitely many Helly centers. Finally, we showed how we can solve linear programming problems in polynomial time if we can get any one of these centers in polynomial time.

2 Definitions of Center of a Polytope

There are several notions for the center of a convex set. We begin by describing a few of them. Intuitively speaking, we would expect the center of a convex set to be a point which divides each chord through the point into two equal parts. But clearly this is too much to hope for. For example, consider a triangle in the plane. The line segments drawn from each vertex to the midpoint of the opposite side intersect in the centroid and are divided by the centroid into parts whose lengths are in the ratio 1:2 (See Figure 1). For any other point in the triangle there is a chord which is divided by this point in such a way that one part is more than twice as long as the other. So the centroid is the point which most nearly agrees with our intuitive feeling of what a center should be.

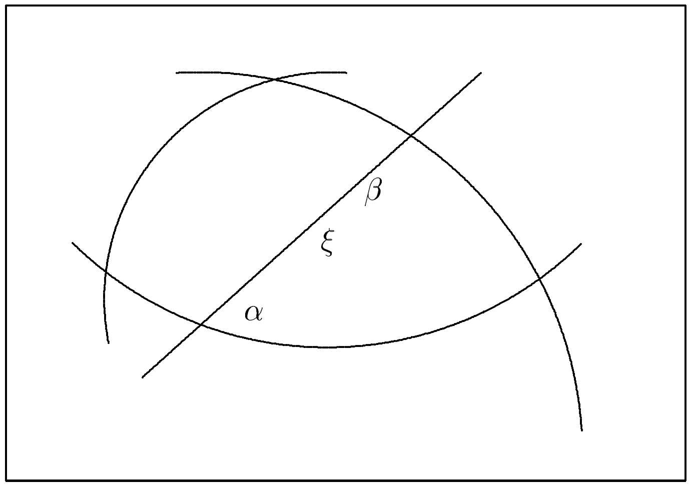

Figure 2: A Helly center in a convex body

2.1 Helly Centers

The above result about centroids for triangles has an analog for convex sets in n dimensions. Let C be a convex body in $\Re^n$ (See Figure 2). It is shown in [11] that there exists a point $\xi \in C$ which divides any chord through ξ into parts of lengths α and β such that

$$\frac{1}{n+1} \leq \frac{\alpha}{\alpha+\beta} \leq \frac{n}{n+1}. \tag{1}$$

We will refer to ξ as a Helly center, or simply an H center, because of its connection with the following theorem due to Eduard Helly [11].

Theorem 2.1 *(Eduard Helly, 1913)*
Suppose K is a family of at least $(n+1)$ convex sets in the $n-$dimensional Euclidean space $\Re^n$ and K is finite or each member of K is compact. Then if each $(n+1)$ members of K have a common point, there is a point common to all members of K.

The next theorem is a consequence of Helly's Theorem, and its proof is a validation of the inequalities in (1).

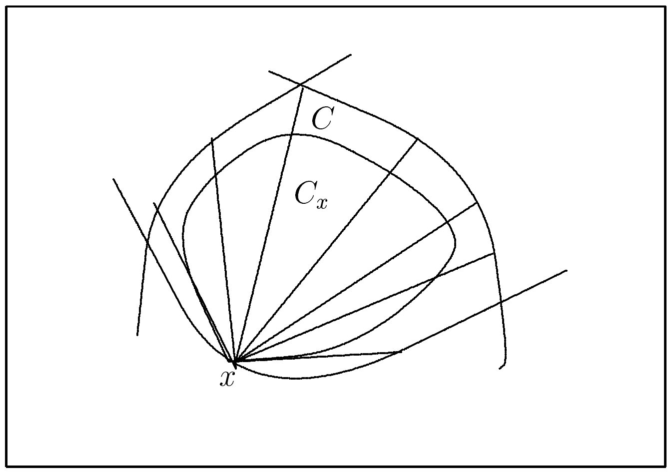

Figure 3: Illustration of the set C_x

Theorem 2.2 *[11]*
If C is a convex body in $\Re^n$, there exist a point $z \in C$ such that for each chord $[u, v]$ of C which passes through z

$$\frac{1}{n+1} \quad \leq \quad \frac{\| z - u \|}{\| v - u \|} \quad \leq \quad \frac{n}{n+1}. \tag{2}$$

The proof given in [11] is instructive so we will repeat it here.

Proof :
If C is a convex body in $\Re^n$, for each point $x \in C$ we define a set $C_x \subset C$ by

$$C_x = x + \frac{n}{n+1}(C - x).$$

We claim that $\cap_{x \in C} C_x \neq \emptyset$, and to prove this it suffices (in view of Helly's Theorem) to show that if $x_0, x_1, \ldots, x_n$ are points of C, then $\cap_{i=0}^{n} C_{x_i}$ includes the point $y = \frac{1}{n+1} \sum_{i=0}^{n} x_i$.

This is evidently correct, since for each j it is true that $y = \frac{1}{n+1} \sum_{i=0}^{n} x_i$ can be rewritten as

$$y \quad = \frac{1}{n+1}(\sum_{i=0}^{n} x_i) - x_j + x_j = x_j + \frac{1}{n+1}(\sum_{i \neq j} x_i - n x_j)$$

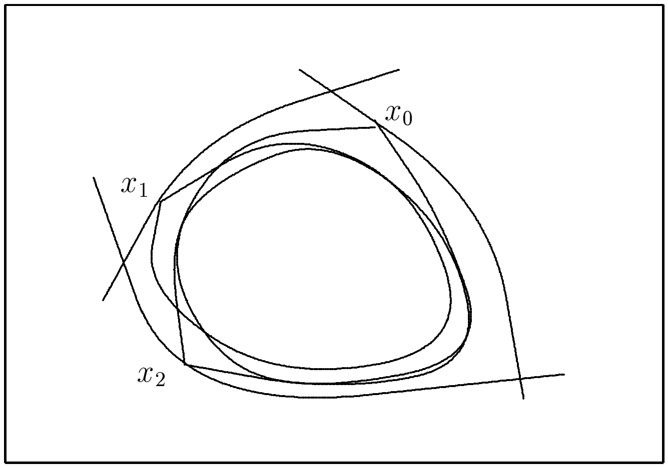

Figure 4: Intersection of all C_{x_i}

$$= x_j + \frac{n}{n+1}([\frac{1}{n}\sum_{i\neq j} x_i] - x_j),$$

which clearly shows that $y \in C_{x_j}$.

Now, consider an arbitrary chord $[u, v]$ passing through a point $z \in \cap_{x\in C} C_x$. Then

$$z \in u + \frac{n}{n+1}([u, v] - u).$$

This implies $z = u + \frac{n}{n+1}\lambda(v - u)$ for some $\lambda \in [0, 1]$. Therefore,

$$\frac{\| z - u \|}{\| v - u \|} \leq \frac{n}{n+1} \tag{3}$$

and this proves the second inequality in (2).

We have shown that

$$\frac{\| z - v \|}{\| v - u \|} \leq \frac{n}{n+1},$$

since this result can be obtained by interchanging u and v in the argument above. Therefore,

$$
\begin{aligned}
1 &= \frac{\| z - v \| + \| z - u \|}{\| v - u \|} \\
&= \frac{\| z - v \|}{\| v - u \|} + \frac{\| z - u \|}{\| v - u \|} \\
&\leq \frac{n}{n+1} + \frac{\| z - u \|}{\| v - u \|}.
\end{aligned}
$$

This implies that

$$
\tfrac{1}{n+1} \leq \frac{\| z - u \|}{\| v - u \|} \tag{4}
$$

as claimed.

□

This completes the proof of the theorem and the validation of the inequalities in (1).

2.2 John Centers

Another notion of a center for a convex body is the center of the circumscribing ellipsoid of least volume. Given a polytope S, let E denote the ellipsoid of least volume containing S. We define the center of E to be a John center of this polytope. Although these centers are not easy to compute they are of some pedagogical value in the theory of interior point methods for linear programming. We refer to them as J centers. This is because of the following result due to John [7].

Theorem 2.3 *(Fritz John, 1947)*
Let S be a polytope in the Euclidean $n-$dimensional space $\Re^n$. Let E denote the ellipsoid of least volume containing S. Then E is unique and if we shrink E by a factor n about its center, we obtain an ellipsoid contained in S.

It turns out that a J center is also an H center. Any algorithm for computing one of these centers gives rise to an algorithm for solving linear programming problems. In the next few pages we will show how this is done for J centers. Afterwards we will indicate how an analogous process can be carried out using H centers. We will also show that most convex sets contain a large subset of H centers, while the J center is unique. Presumably this makes H centers easier to find and consequently more useful for linear programming.

2.2.1 John Centers for LP

Consider the following LP problem,

$$\begin{array}{ll} \text{minimize} & c^t x \\ \text{subject to} & \\ & Ax \le b \\ & x \ge 0, \end{array} \tag{5}$$

where $A = (a_{ij})$ is a $m \times n$ matrix, x and c are $n-$dimensional column vectors, and b is an $m-$dimensional column vector with components $b_1, \ldots, b_m$. We assume that the constraints $Ax \le b, \;\; x \ge 0$ define a **full-dimensional** polytope in $\Re^n$.

Let x^* denote a solution of this problem. The interior point methods for computing a solution x^* begin with a point $x^0 \;\; > \;\; 0$ satisfying $Ax^0 < b$ and generate a sequences $x^0, x^1, \ldots$ satisfying $x^k > 0, Ax^k < b$ and $x^k \longrightarrow x^*$ as $k \longrightarrow \infty$. The simplest of these algorithms is the affine scaling algorithm [1]. Given a point $x^k > 0$ satisfying $Ax^k < b$, this algorithm constructs the ellipsoid

$$\mathrm{E} = \{x | \sum_{i=1}^{m} (\frac{d_i(x)}{d_i(x^k)} - 1)^2 \le 1\},$$

where $d_i(x) = \frac{b_i - \sum_{j=1}^n a_{ij} x_j}{\sqrt{\sum_{j=1}^n a_{ij}^2}}, \;\; i = 1, \ldots, m$, and solves the problem

$$\text{minimize}_{x \in E} \;\; c^t x.$$

The solution of this problem is denoted by x^{k+1}. The sequence $x^0, x^1, \ldots,$ constructed in this way converges to a solution x^* of (5). Throughout this work we will refer to the above ellipsoid as the affine scaling ellipsoid.

Figure 5 illustrates how we can solve linear programming problems in polynomial time using J centers. The method described in the figure is very

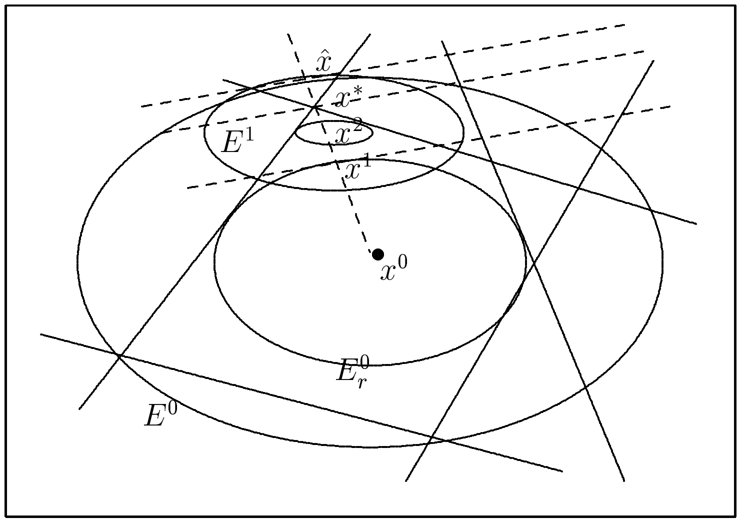

Figure 5: Fritz John's ellipsoids.

much like the affine scaling algorithm, but it uses a different set of ellipsoids. For the linear programming problem (5) the process constructs a sequence $x^0, x^1, \ldots$ converging to a solution x^* as follows.

- Let E^0 denote the ellipsoid of least volume containing the polytope

  $$S^0 \equiv S \equiv \{Ax \leq b, x \geq 0\},$$

  and let x^0 denote the center of E^0.

- Let x^1 be the solution of $min\ \{c^t x\ subject\ to\ x \in E_r^0\}$ where $E_r^0 = \frac{1}{n}E^0$.

- Let E^1 denote the ellipsoid of minimum volume containing the polytope $S^1 \equiv S^0 \cap \{x | c^t x \leq c^t x^1\}$, and let x^2 denote the center of E^1.

- Let x^3 be the solution of $min\ \{c^t x\ subject\ to\ x \in E_r^1\}$ where $E_r^1 = \frac{1}{n}E^1$.

- Repeat this process starting with the polytope $S^0 \bigcap \{x | c^t x \leq c^t x^3\}$.

In order to examine the convergence of this procedure let $\hat{x}$ denote the solution of $\min\{c^t x \text{ subject to } x \in E^0\}$ and let x^* denote the solution of

$\min\{c^t x$ subject to $x \in S\}$. Since $E_r^0 = \frac{1}{n}E^0$ we have

$$\hat{x} - x^0 = n(x^1 - x^0)$$

which implies that

$$c^t\hat{x} - c^t x^0 = n(c^t x^1 - c^t x^0).$$

Since $c^t x^* \geq c^t \hat{x}$ we have

$$c^t x^* - c^t x^0 \geq n(c^t x^1 - c^t x^0). \tag{6}$$

This can be rewritten as

$$c^t x^* - c^t x^0 \geq n\{(c^t x^1 - c^t x^*) - (c^t x^0 - c^t x^*)\}.$$

It follows that

$$c^t x^1 - c^t x^* \leq (1 - \frac{1}{n})(c^t x^0 - c^t x^*).$$

Now, let $S^1 \equiv S^0 \cap \{x | c^t x \leq c^t x^1\}$. By the same reasoning applied above we have

$$c^t x^3 - c^t x^* \leq (1 - \frac{1}{n})(c^t x^2 - c^t x^*). \tag{7}$$

Now, consider $c^t x^2 - c^t x^*$. It is clear that $c^t x^1 \geq$ *max* $\{c^t x$ *subject to* $x \in E_r^1\}$. So

$$c^t x^2 - c^t x^3 \leq c^t x^1 - c^t x^2. \tag{8}$$

Inequality (8) can be rewritten as

$$c^t x^2 - c^t x^* - (c^t x^3 - c^t x^*) \leq c^t x^1 - c^t x^* - (c^t x^2 - c^t x^*). \tag{9}$$

And, substituting (7) in (9), we obtain

$$\begin{aligned} c^t x^2 - c^t x^* &\leq \frac{1}{1+\frac{1}{n}}(c^t x^1 - c^t x^*) \\ &\leq \frac{1-\frac{1}{n}}{1+\frac{1}{n}}(c^t x^0 - c^t x^*) \\ &\leq \exp^{\frac{-2}{n}}(c^t x^0 - c^t x^*). \end{aligned} \tag{10}$$

Combining (7) with (10) we obtain

$$\begin{aligned} c^t x^3 - c^t x^* &\leq (1 - \frac{1}{n}) e^{-\frac{2}{n}} (c^t x^0 - c^t x^*) \\ &< \exp^{-\frac{3}{n}} (c^t x^0 - c^t x^*). \end{aligned} \tag{11}$$

Here we use the facts $1 - \frac{1}{n} \;<\; e^{-\frac{1}{n}}$, $\frac{1-\frac{1}{n}}{1+\frac{1}{n}} \;<\; e^{-\frac{2}{n}}$.

It is clear that if we continue this process we obtain the inequality

$$c^t x^k - c^t x^* \leq \exp^{\frac{-k}{n}} (c^t x^0 - c^t x^*) \tag{12}$$

for all $k \geq 0$.

Let $L = \log_2(c^t x^0 - c^t x^*)$. It is possible to estimate L in terms of the problem data A, b, c. See for example [16]. If we take $k = (p+1)nL$ where p is a positive integer, we obtain from (12)

$$c^t x^k - c^t x^* \;\;\leq\;\; \exp^{-(p+1)L} (c^t x^0 - c^t x^*) \;\;\leq\;\; e^{-pL}.$$

This implies that $c^t x^k$ can be made as close to $c^t x^*$ as desired simply by making k a sufficiently large multiple of nL.

Thus if we had an efficient method for computing J centers for polytopes we would have a polynomial time procedure for solving linear programming problems. However, there is no simple method for computing the J center or even an H center for a polytope. It turns out that each J center is an H center and whereas the J center for a convex set is unique the set of H centers is generally fairly large. Moreover, when the affine scaling algorithm is started at an H center, and an appropriate ellipsoid is used, we obtain the result (12) with n replaced by $n^{\frac{3}{2}}$. This shows that an H center is almost as good as the J center for application to linear programming.

2.2.2 A J Center is an H Center

Theorem 2.4 *Let C be a convex body and let E be the ellipsoid of least volume containing C. Let $\hat{E} = \frac{1}{n} E$ denote the ellipsoid obtained by shrinking E by a factor of n. Let ξ denote the center of E and let $l = [u, v]$ be a line segment through ξ with endpoints in the boundary of C. Then*

$$\frac{1}{n+1} \leq \frac{\| u - \xi \|}{\| u - v \|} \leq \frac{n}{n+1}. \tag{13}$$

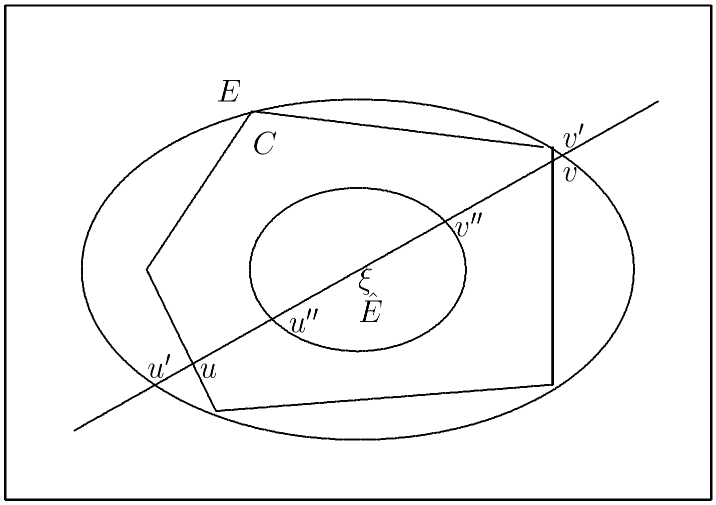

Figure 6: Proving a J center is an H center

Proof :

Let us extend l to a line segment $l' = [u', v']$ with endpoints in the boundary of E. Let u'' and v'' denote the points where l intersects the boundary of $\hat{E}$. See Figure 6. We must show that

$$\frac{1}{n+1} \leq \frac{\| u - \xi \|}{\| u - v \|} \leq \frac{n}{n+1}, \tag{14}$$

and the analogous result with u and v interchanged. Clearly, this is equivalent to (1).

From Figure 6, we see that

$$\begin{aligned} \frac{\|u-\xi\|}{\|u-v\|} = \frac{\|u-\xi\|}{\|u-\xi\|+\|\xi-v\|} &\geq \frac{\| u'' - \xi \|}{\| u'' - \xi \| + \| \xi - v \|} \\ &\geq \frac{\| u'' - \xi \|}{\| u'' - \xi \| + \| \xi - v' \|} \end{aligned}$$

Now, $\| \xi - v' \| = \| \xi - u' \| = n \| u'' - \xi \|$ so

$$\frac{\| u - \xi \|}{\| u - v \|} \geq \frac{\| u'' - \xi \|}{\| u'' - \xi \| + n \| u'' - \xi \|} = \frac{1}{n+1}.$$

This implies that

$$\frac{\| v - \xi \|}{\| u - v \|} \leq \frac{n}{n+1}.$$

By interchanging the roles of u and v in the last two inequalities we obtain (13). □

2.2.3 A Convex Body Has Many H Centers

It turns out that most bounded convex sets have many H centers. Actually, an $n-$dimensional simplex has a unique H center which is of course also the J center. But all other convex bodies in $\Re^n$ have infinitely many H centers. We are going to prove this result for ellipsoids.

Theorem 2.5 *Let E denote the $n-$dimensional ellipsoid defined by*

$$(x-\xi)^t Q (x-\xi) = 1.$$

Let $\hat{E}$ denote the ellipsoid concentric with E and defined by

$$(x-\xi)^t Q (x-\xi) = (\frac{n-1}{n+1})^2.$$

Then all points in $\hat{E}$ are H centers for E.

Proof :

Let $y \in \hat{E}$ be arbitrary. Draw a chord l for E which passes through y. Draw another chord l' through y and ξ. Let α and β denote the lengths of the segments into which y divides l and let α', β' denote the lengths into which y divides l'. Consider the two-dimensional plane containing the lines l and l'. This plane intersects E in a two-dimensional ellipsoid which we will also denote by E. See Figure 7(a). Map this two-dimensional ellipsoid onto the unit circle in $\Re^2$ and label all images in the unit circle as they were labeled in E. We must show that

$$\frac{\alpha}{\alpha+\beta} \geq \frac{1}{n+1}.$$

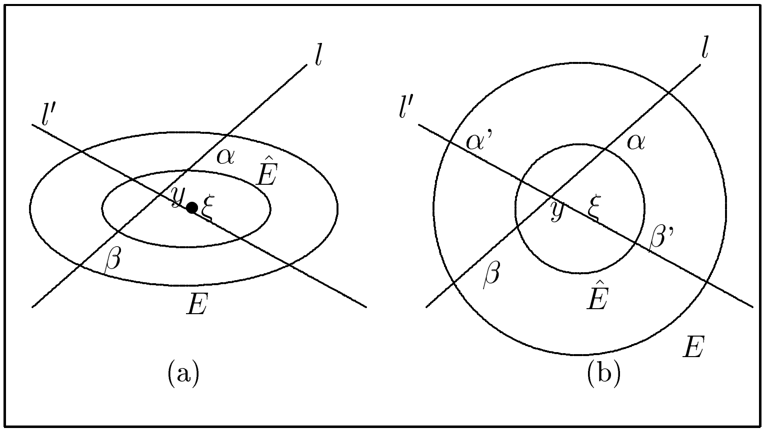

Figure 7: (a) Ellipsoid E (b) Mapping of E and $\hat{E}$ in a unit ball

Refer to Figure 7(b). The radius of $\hat{E}$ is $r = \frac{n-1}{n+1}$. By an elementary theorem in geometry we have $\alpha\beta = \alpha'\beta'$ and clearly $\beta' \geq \beta$. It follows that

$$\begin{aligned} \tfrac{\alpha}{\alpha+\beta} = \tfrac{\alpha\beta}{\alpha\beta+\beta^2} = \tfrac{\alpha'\beta'}{\alpha'\beta'+\beta^2} &\geq \frac{\alpha'\beta'}{\alpha'\beta' + (\beta')^2} \\ &= \frac{\alpha'}{\alpha' + \beta'} = \frac{\alpha'}{2} \\ &\geq \frac{1-r}{2} = \frac{1}{2}(\frac{2}{n+1}) = \frac{1}{n+1}. \end{aligned}$$

By analogy, we have

$$\frac{\beta}{\alpha+\beta} \geq \frac{1}{n+1}.$$

It follows that $\frac{\alpha}{\alpha+\beta} < \frac{n}{n+1}$ and this completes the proof that y is an H center. □

Recently E. R. Barnes [2] proved the following generalization of this theorem.

Theorem 2.6 *Let C be a convex body in $\Re^n$ and let*

$$E \equiv \{x | (x-\xi)^t R(x-\xi) \leq 1\}$$

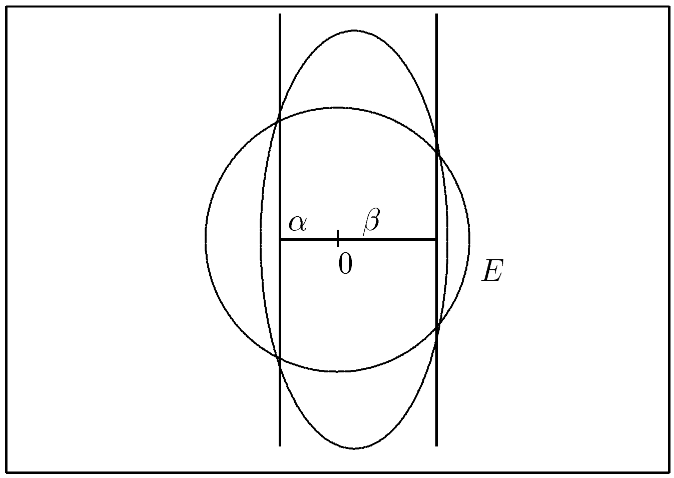

Figure 8: Illustration of Todd's result

denote the ellipsoid of least volume containing C. Let $t > 0$ be the smallest number such that $\frac{1}{t}E \subset C$ and let $r = \frac{\frac{n}{t}-1}{n+1}$. By Fritz John's Theorem, $t \leq n$ so $r \geq 0$. Then all points in the ellipsoid

$$\hat{E} \equiv \{x|(x-\xi)^t R(x-\xi) \leq r^2\}$$

are Helly centers for C.

2.2.4 H Centers for LP

We close this section by showing that H centers can be used to solve linear programming problems in polynomial time in the same way that J centers can be used for this purpose. In order to show this we need a result for H centers analogous to Theorem 2.3 for J centers.

Let E denote an $n-$dimensional ellipsoid in $\Re^n$. For simplicity, we take E to be the unit ball. Consider the portion of E lying between two parallel planes situated at distances α and β, respectively, from the origin. Denote this set by E'. Assume $0 \in E'$. We are interested in the ellipsoid of least volume containing E'. Todd [19] has shown that this ellipsoid is E if $\alpha\beta \geq \frac{1}{n}$. We will use this result to prove a result for H centers analogous to John's Theorem for J centers.

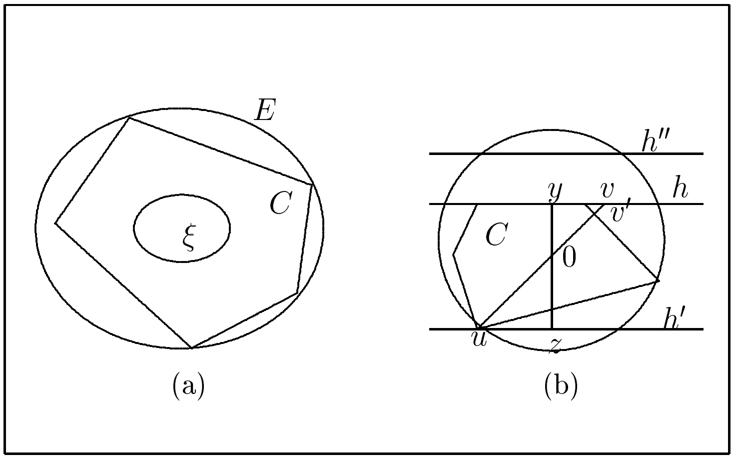

Figure 9: (a) and (b) : Illustrations for the proof of Theorem 1.7

Theorem 2.7 *Let C be a convex body in $\Re^n$ and let ξ be an H center for C. Let*

$$(x-\xi)^t R(x-\xi) = 1 \tag{15}$$

denote the ellipsoid of least volume centered at ξ and containing C. Then the ellipsoid

$$(x-\xi)^t R(x-\xi) = \frac{1}{n^3}$$

is contained entirely within C.

Proof :

To prove this result denote the ellipsoid (15) by E and use the transformation $y = R^{\frac{1}{2}}(x-\xi)$ to map E to the unit ball. Denote the transformed image of C by C again. In the unit ball let α denote the distance from 0 to the boundary of C and choose y in the boundary of C so that $\| y \| = \alpha$.

Since y is the point in the boundary of C nearest to 0, there is a hyperplane h supporting C at y. Let h' denote the hyperplane which is parallel to h and supports C is such a way that C is situated between h and h'. See Figure (9).

Let u be a point where h' intersects C. Choose $v \in h$ such that the line segment $[u, v]$ contains 0. Let v' denote the point where the line segment $[0, v]$

intersects the boundary of C. Note that v' is not necessarily distinct from v. Clearly, H points are preserved under affine transformations. Therefore 0 is an H point for C. And since $\| v \| \geq \| v' \|$, we have

$$\frac{\| v \|}{\| u \| + \| v \|} \geq \frac{\| v' \|}{\| u \| + \| v' \|} \geq \frac{1}{n+1}. \tag{16}$$

Let z be the point where the line through y and 0 meets h'. By construction we have

$$z = -ty \quad \text{and} \quad u = -\tau v \tag{17}$$

for some positive constants t and τ. Moreover, y is normal to the hyperplanes h and h'. Thus,

$$(z-u)^t y = 0 \quad \text{and} \quad y^t(y-v) = 0.$$

Combining this with (17) gives

$$(\tau - t) \| y \|^2 = 0$$

which implies $t = \tau$. It follows from (17) that

$$\frac{1}{t} = \frac{\| y \|}{\| z \|} = \frac{\| v \|}{\| u \|}.$$

Now observe that

$$\begin{aligned} \tfrac{\alpha}{\alpha + \| z \|} = \tfrac{\| y \|}{\| y \| + \| z \|} &= \frac{\| y \| / \| z \|}{\| y \| / \| z \| + 1} \\ &= \frac{\| v \| / \| u \|}{\| v \| / \| u \| + 1} = \frac{\| v \|}{\| u \| + \| v \|}. \end{aligned} \tag{18}$$

Combining this with (16) gives

$$\frac{\alpha}{\alpha + \| z \|} \geq \frac{1}{n+1}$$

which implies

$$\| z \| \leq n\alpha. \tag{19}$$

Let h'' be a translate of h in the direction opposite to h' so that h'' is a distance $\| z \|$ from 0. Consider the portion of the unit ball lying between h' and h''. Denote this region by F. Let E' denote the ellipsoid of least volume containing F. By symmetry arguments, E' has its center at 0. We claim that E' is just the unit ball. For if E' had a volume smaller than that of the unit ball the mapping

$$x = \xi + R^{\frac{-1}{2}} y$$

would transform E' into an ellipsoid centered at ξ and containing C and having a volume less than the volume of E. But this is impossible by the way E was chosen. Therefore, E' is the unit ball.

Since E' is identical with the unit ball it follows from a result due to Todd [19], which is illustrated in Figure (8), that

$$\| z \|^2 \ \geq \ \frac{1}{n}.$$

If we combine this with (19) we obtain

$$\alpha \geq \frac{1}{n^{\frac{3}{2}}}.$$

But α is the distance from 0 to the boundary of C in Figure 9(b). This means that the sphere $\| y \|^2 = (\frac{1}{n^{3/2}})^2 = \frac{1}{n^3}$ is contained entirely within C. Under the transformation $x = \xi + R^{\frac{-1}{2}} y$ this sphere maps into the ellipsoid

$$(x - \xi)^t R (x - \xi) = \frac{1}{n^3}$$

which must be contained entirely within C as shown in Figure 9(a). This is what we set out to prove.

□

It is now a simple matter to construct an algorithm for linear programming which uses H center in the way we used J centers for the construction of the linear programming algorithm in Section 2.2.1.

We have seen that H centers can be used to construct an efficient algorithm for linear programming. We have also seen that in general a convex body has a large set of H centers. This suggests that it is probably easy to construct an algorithm for computing H centers. But we have not been able to find such an algorithm which is efficient. Khachyan's ellipsoid algorithm can be used to compute the J center of polytopes in polynomial time [16] and we have seen that J centers are H centers. So, the ellipsoid algorithm is one technique for computing H centers. However, this algorithm is not very efficient for this purpose.

Because of this difficulty in computing H centers, we are going to concentrate on computing another type of center – the analytic center . Analytic centers are not necessarily H centers, but they behave like H centers in some respects. For example, it is possible to construct an ellipsoid about the analytic center of a polytope which when expanded by a small factor encloses the polytope. This is the property of all centers we have mentioned that make them useful for linear programming.

2.3 Analytic Centers

Let S be a polytope described by a system of linear inequalities $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for $i = 1, 2, \ldots, m$. The analytic center of S is the point $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \Re^n$ which solves the maximization problem

$$\text{Maximize} \quad \prod_{i=1}^{m}(b_i - \sum_{j=1}^{n} a_{ij}x_j)$$

$$\text{subject to}$$

$$x \in S. \tag{20}$$

Since S is bounded this maximization problem has a solution. It can also be shown that the solution is unique. Thus the analytic center of a polytope is well-defined.

The analytic center has found widespread use in interior point methods for linear programming. See for instance , [1], [3], [4], [5], [6], [8], [13], [14], [15]. It turns out that the analytic center solves a certain weighted linear least squares problem. We will now explain what this problem is. Later on it will be useful in helping us compute analytic centers.

Let ξ denote the analytic center of the polytope S defined above and define

$$f(x) = \log \prod_{i=1}^{m}(b_i - a_i^t x) = \sum_{i=1}^{m} \log(b_i - a_i^t x).$$

Since f attains its maximum value at $x = \xi$ we have

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} (\frac{1}{b_i - a_i^t x})(-a_{ij}) = 0. \tag{21}$$

Multiplying (21) by x_j and sum over j gives

$$\sum_{i=1}^{m} (\frac{-\sum_{j=1}^{m} a_{ij} x_j}{b_i - a_i^t \xi}) = 0. \tag{22}$$

Adding and subtracting b_i to the numerator in (22) gives

$$\sum_{i=1}^{m} (\frac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{b_i - a_i^t \xi}) = \sum_{i=1}^{m} (\frac{b_i}{b_i - a_i^t \xi}). \tag{23}$$

Observe that the right hand side of (23) is independent of x. If we take $x = \xi$ on the left hand side we see that the left hand side, and hence the right hand side equals m.

For a fixed $x \in S$, let d_i denote the distance from x to the hyperplane $a_i^t z = b_i$, and, let d_i^* denote the distance from ξ to this hyperplane. Then

$$\begin{aligned} m &= \sum_{i=1}^{m} (\frac{b_i - a_i^t x}{b_i - a_i^t \xi}) = \sum_{i=1}^{m} \frac{\frac{b_i - a_i^t x}{\|a_i\|}}{\frac{b_i - a_i^t \xi}{\|a_i\|}} \\ &= \sum_{i=1}^{m} \frac{d_i}{d_i^*}. \end{aligned}$$

It now follows from Schwartz's inequality that

$$m = \sum_{i=1}^{m} \frac{d_i}{d_i^*} \leq \sqrt{m} \sqrt{\sum_{i=1}^{m} \frac{d_i^2}{(d_i^*)^2}}.$$

This implies that

$$m \leq \sum_{i=1}^{m} \frac{d_i^2}{(d_i^*)^2},$$

where equality holds if $x = \xi$. This proves the following theorem.

Theorem 2.8 *The analytic center minimizes a weighted sum of the squares of the distances from a point in S to the hyperplanes defining S. The weights are given by $w_i = \frac{1}{d_i^2(\xi)}$, $i = 1, \ldots, m$, where ξ is the analytic center of S.*

2.3.1 Analytic Centers for Linear Programming

We will now show that analytic centers can be used to solve linear programming problems in polynomial time in the same way that John centers can be used for this purpose.

To show this, consider the ellipsoid

$$\sum_{i=1}^{m}(\frac{d_i - d_i^*}{d_i^*})^2 \quad \leq \quad 1. \tag{24}$$

Clearly this inequality implies that $d_i \geq 0, \quad i = 1, \ldots, m$. We can rewrite (24) as

$$\sum_{i=1}^{m}\{\frac{(b_i - a_i^t x) - (b_i - a_i^t \xi)}{(b_i - a_i^t \xi)}\}^2 \quad \leq \quad 1,$$

or

$$(x - \xi)^t \{\sum_{i=1}^{m} \frac{a_i a_i^t}{(b_i - a_i^t \xi)^2}\}(x - \xi) \quad \leq \quad 1.$$

Finally, we write this inequality as

$$(x - \xi)^t Q (x - \xi) \quad \leq \quad 1, \tag{25}$$

where $Q = \sum_{i=1}^{m} \frac{a_i a_i^t}{(b_i - a_i^t \xi)^2}$ is a symmetric positive definite matrix. In Section 2.2.1 we referred to this as the affine scaling ellipsoid centered at ξ.

When x satisfies (25) we have seen that $d_i = \frac{b_i - a_i^t x}{\|a_i\|} \quad \geq \quad 0, \quad i = 1, \ldots, m$. Therefore, (25) describes an ellipsoid contained in S. Now, we will show that if we expand this ellipsoid by a factor of m, this new ellipsoid will contain the set S. Thus (25) resembles the ellipsoids described in Theorems 2.5 and 2.7.

Lemma 2.1 *The ellipsoid*

$$(x - \xi)^t Q (x - \xi) \quad \leq \quad m^2$$

contains S.

Proof :
Let $x \in S$ and $d_i = \frac{b_i - a_i^t x}{\|a_i\|}$. Then $d_i \geq 0, i = 1, 2, \ldots, m$. Observe that

$$\begin{aligned}
(x-\xi)^t Q (x-\xi) &= \sum_{i=1}^{m} (\frac{d_i - d_i^*}{d_i^*})^2 = \sum_{i=1}^{m} (\frac{d_i}{d_i^*} - 1)^2 \\
&= \sum_{i=1}^{m} \{(\frac{d_i}{d_i^*})^2 - 2\frac{d_i}{d_i^*} + 1\} \\
&= \sum_{i=1}^{m} (\frac{d_i}{d_i^*})^2 - 2m + m \\
&\leq \quad (\sum_{i=1}^{m} \frac{d_i}{d_i^*})^2 - m \\
&= m^2 - m \quad < \quad m^2
\end{aligned}$$

as claimed.

□

This result means that we can solve linear programming by using analytic centers in the same way that we use John centers in Section 2.2.1.

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