The equations of a viscous incompressible chemically active fluid: existence and uniqueness of strong solutions in unbounded domain

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Abstract

In this paper we established the existence and uniqueness of strong solutions for the viscous incompresssible chemically active fluid in unbounded domain differing somewhat from those previously known.

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1 Introduction

In this work we study the initial boundary value problem for the equations that describe the motion of a viscous-chemically-active fluid in a bounded or unbounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary Γ in the time interval $[0,T]$, $0 < T < \infty$.

In the Oberbeck-Boussinesq approximation, the state of such a system is described by the equations (see [1]):

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$$
\left.
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \tfrac{1}{\rho}\nabla p = \mathbf{g}(\beta_\theta(\widetilde{\theta}-\theta_c) + \beta_\psi(\widetilde{\psi}-\psi_c)) + Q_1 \\
&\frac{\partial \widetilde{\theta}}{\partial t} + (\mathbf{u}\cdot\nabla)\widetilde{\theta} - \alpha\Delta\widetilde{\theta} = Q_2 \\
&\frac{\partial \widetilde{\psi}}{\partial t} + (\mathbf{u}\cdot\nabla)\widetilde{\psi} - D\Delta\widetilde{\psi} = Q_3 \\
&\text{div } \mathbf{u} = 0 \qquad\qquad\qquad \text{in} \quad U_T,
\end{aligned}
\right\} \tag{1}
$$

where $U_T = [0,T]\times\Omega$, also we denote $\Gamma_T = [0,T]\times\Gamma$..

Here $\mathbf{u} = (u_1,u_2,u_3) \in I\!R^3, \widetilde{\theta} \in I\!R, \widetilde{\psi} \in I\!R$ and $p \in I\!R$ denote the unknown velocity vector, temperature, concentration of material in the fluid and the pressure at a point $x \in \Omega$ at time $t \in [0,T]$; $Q_1(t,x), Q_2(t,x), Q_3(t,x)$ and $\mathbf{g}(t,x)$ are given source functions (usually, $\mathbf{g} = g\mathbf{e}$, where g is the free-fall acceleration); θ_c and ψ_c are the characteristic temperature and concentration; ρ is the mean density; ν is the kinematic viscosity; D is the diffusion coefficient; α is the thermal conductance. The quantities $\theta_c, \beta_\theta, \psi_c$ and β_ψ are assumed to be constant.

On the boundary , we assume that

$$
\mathbf{u}|_{\Gamma_T} = 0 \ ; \ \widetilde{\theta}|_{\Gamma_T} = \theta_1 \ , \ \widetilde{\psi}|_{\Gamma_T} = \psi_1 \tag{2}
$$

where θ_1 , ψ_1, are known functions , and the initial conditions are expressed by

$$
\mathbf{u}(0,x) = \mathbf{u}_0(x); \ \widetilde{\theta}(0,x) = \widetilde{\theta}_0(x); \ \widetilde{\psi}(0,x) = \widetilde{\psi}_0(x), \tag{3}
$$

where, $\mathbf{u}_0(x), \widetilde{\theta}_0(x)$ and $\widetilde{\psi}_0(x)$ are given functions on the variable $x \in \Omega$.

The expressions $\nabla, \triangle$ and div , as usual, denote the gradient, Laplacian and divergence operators, respectively; the ith componenet of $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is given by $[(\mathbf{u}\cdot\nabla)\mathbf{u}]_i = \sum_j u_j \frac{\partial u_i}{\partial x_j}$; $(\mathbf{u}\cdot\nabla)\phi = \sum_j u_j \frac{\partial \phi}{\partial x_j}$, for $\phi = \widetilde{\theta}$ or $\widetilde{\psi}$.

The main goal in this paper is to show the existence and uniqueness of strong solutions. Our arguments are true for bounded or unbounded domains. To prove our result, we use an iterative procedure together with results due to [2], [3] on nonstationary Stokes problem and parabolic problem. These procedure was used early by [5], and [6] in another class of problems.

When chemical reactions are absent ($\widetilde{\psi} \equiv 0$), the problem (1.1)-(1.3) is equivalent to the classical Boussinesq problem, which has been investigated by several authors, see for instance [7], [8], [9], [10] in the case of bounded domains, [11], [12], Hishida [13], [14] in the case of exterior domains. For $\Omega = I\!R^3$, see [15].

[16] studied the stationary model, [17], the stability of the solutions of the system (1.1)-(1.2) with different boundary conditions. [18], [19], [20], [21] studied in bounded domains. For case of infinite vertical strip or tube, see [22]

This paper is organized as follows: in Section 2 we state some preliminaries results that will be useful in the rest of the paper; we also state the results of existence and uniqueness of strong solutions from apriori estimates that form the theorical basis for the problem. In Section 3 we study the linear problems associated a (1.1) and (1.3). Finally, in Section 4 we prove our result.

To simplicity the notation in the expressions we will denote by c, C_0, M_0 generics finites positives constants depending only on Ω and the other fixed parameters of the problem (like the initial data) that may have different values in different expressions. To emphasize the fact that the constants are different we use $C_1, C_2, ..., M_1, M_2, \cdots$ and so on.

2 Preliminaries

We use the classical notations and results of the Sobolev spaces. For $k = 0, 1, 2, ...$ and $1 \leq p \leq \infty$,

$$W_p^k(\Omega) = \{\mathbf{u} \in L_p(\Omega) \ / \ \sum_{|\alpha| \leq k} \|D_x^\alpha \mathbf{u}\|_{L_p(\Omega)} < \infty\}$$

$$W_p^{2,1}(U_T) = \{\mathbf{u} \in L_p(U_T) \ / \ \|\mathbf{u}\|_{W_p^{2,1}(U_T)} = \|\mathbf{u}_t\|_{L_p(U_T)} + \sum_{|\alpha| \leq 2} \|D_x^\alpha \mathbf{u}\|_{L_p(U_T)} < \infty\},$$

where $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}$.

It is known that the values of the function from $W_p^{2,1}(U_T)$ on the hyperplane $t = k$, where k is a constant, belong for $\forall \ t \in [0, T]$ to the Slobodetskii-Besov space $W_p^{2-\frac{2}{p}}(\Omega)$ and depend continuously on t in the norm of $W_p^{2-\frac{2}{p}}(\Omega)$, defined by

$$\|\mathbf{u}\|_{W_p^{2-\frac{2}{p}}(\Omega)} = \left(\sum_{|\alpha|\le 1} \|D_x^\alpha \mathbf{u}\|^p_{L_p(\Omega)} + \sum_{|\alpha|=1} \int_\Omega \int_\Omega \frac{| D_x^\alpha \mathbf{u}(x) - D_y^\alpha \mathbf{u}(y) |^p}{| x-y |^{1+p}} dxdy \right)^{\frac{1}{p}}.$$

Moreover, we have the inequality

$$\|\mathbf{u}(\cdot,t)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \le \|\mathbf{u}(\cdot,0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\mathbf{u}\|_{W_p^{2,1}(U_t)},$$

where the constant $\widehat{c}$ does not depend on t.

We will transform problem (1.1) - (1.3) into another one with homogeneous boundary value. In order to do it, we introduce the new variables $\widetilde{\theta} = \theta + \theta_2$, $\widetilde{\psi} = \psi + \psi_2$, where θ_2, ψ_2 satisfies

$$\begin{aligned} \partial_t\theta_2 - \alpha\triangle\theta_2 &= 0 \text{ in } U_T, \\ \theta_2 &= \theta_1 \text{ on } \Gamma_T, \\ \theta_2(x,0) &= \theta_0(x) \end{aligned}$$

and

$$\begin{aligned} \partial_t\psi_2 - D\triangle\psi_2 &= 0 \text{ in } U_T, \\ \psi_2 &= \psi_1 \text{ on } \Gamma_T, \\ \psi_2(x,0) &= \psi_0(x) \end{aligned}$$

where θ_0 and ψ_0 are functions defined by

$$\begin{aligned} \triangle\theta_0 &= 0, \text{ in } \Omega, \\ \triangle\psi_0 &= 0, \text{ in } \Omega, \end{aligned}$$

with $\theta_2(0) = \theta_1(0)$ and $\psi_2(0) = \psi_1(0)$ on Γ(the existence of such θ_2 and ψ_2 are ensured by assuming some smoothness condition on θ_1 and ψ_1).

Consequently, we obtain

$$\left.\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \tfrac{1}{\rho}\nabla p = \mathbf{g}(\beta_\theta(\theta-\theta_c) + \beta_\psi(\psi-\psi_c)) + \widetilde{Q}_1 \\ &\frac{\partial\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - \alpha\Delta\theta = Q_2 - (\mathbf{u}\cdot\nabla)\theta_2 \\ &\frac{\partial\psi}{\partial t} + (\mathbf{u}\cdot\nabla)\psi - D\Delta\psi = Q_3 - (\mathbf{u}\cdot\nabla)\psi_2 \\ &\text{div } \mathbf{u} = 0 \qquad\qquad \text{in} \quad U_T \end{aligned}\right\} \qquad (4)$$

$$\mathbf{u}|_{\Gamma_T} = 0 \ ; \ \theta|_{\Gamma_T} = 0 \ , \ \psi|_{\Gamma_T} = 0 \tag{5}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x); \ \theta(0,x) = \theta_0(x); \ \psi(0,x) = \psi_0(x), \tag{6}$$

where $\widetilde{Q}_1 = Q_1 + \mathbf{g}(\beta_\theta\theta_2 + \beta_\psi\psi_2), \theta_0(x) = \widetilde{\theta}_0(x) - \theta_2(0)$ and $\psi_0(x) = \widetilde{\psi}_0(x) - \psi_0(0)$.

In what follows, we will concentrate our analysis on (2.1)-(2.3), instead (1.1)-(1.3). From now on we assume that $\theta_c = 0$ and $\psi_c = 0$, for simplicity.

Theorem 1 *Let $p > 3$. assume that*

$$\mathbf{u}_0(x) \in W^{2-\frac{2}{p}}(\Omega), \mathbf{u}_0|_\Gamma = 0, \ div \ \mathbf{u}_0 = 0,$$

$$\theta_0(x), \psi_0(x) \in W_p^{2-\frac{2}{p}}(\Omega), \theta_0|_\Gamma = 0, \psi_0|_\Gamma = 0,$$

$$\widetilde{Q}_1, Q_2, Q_3 \in L_p(U_T),$$

$$\nabla\theta_2, \nabla\psi_2, \mathbf{g} \in L_\infty(U_T).$$

Then there exists $T_1 \in (0,T]$ such that problem (2.1)-(2.3) has a unique solution ($\mathbf{u}$,θ, ψ,p) which satisfies

$$\mathbf{u} \in W_p^{2,1}(U_{T_1}),$$

$$\nabla p \in L_p(U_{T_1})$$

$$\theta, \psi \in W_p^{2,1}(U_{T_1}).$$

3 Linear problems

In this section, we give some results of the linear problems associated with (2.1)-(2.3). The first lemma is proved in [2]

Lemma 1 *Let $F(x,t) \in L_p(U_T)$ and $\mathbf{u}_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$ with $\mathbf{u}_0|_\Gamma = 0$ and $div \ \mathbf{u}_0 = 0$, then the following problem*

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\triangle\mathbf{u} + \nabla p &= F, \\ div \ \mathbf{u} &= 0, \\ \mathbf{u}|_{\Gamma_T} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0(x) \end{aligned}$$

has a unique solution $\mathbf{u} \in W_p^{2,1}(U_T)$*, and* $\nabla p \in L_p(U_T)$ *satisfying*

$$\|\mathbf{u}\|_{W_p^{2,1}(U_T)} + \|\nabla p\|_{L_p(U_T)} \leq K_1(T)(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|F\|_{L_p(U_T)}),$$

where $K_1(\cdot)$ *is an increasing function of* T.

The following result is a special case of the result for parabolic system given in [3], (sse also [4].

Lemma 2 *Let* $G(x,t) \in L_p(U_T)$ *and* $\phi_0(x) \in W_p^{2-\frac{2}{p}}(\Omega)$ *with* $\phi_0|_\Gamma = 0$, $\mu > 0$, *then the following problem*

$$\frac{\partial \phi}{\partial t} - \mu \triangle \phi = G,$$

$$\begin{aligned} \phi|_{\Gamma_T} &= 0, \\ \phi(0) &= \phi_0(x) \end{aligned}$$

has a unique solution $\phi \in W_p^{2,1}(U_T)$*, satisfying*

$$\|\phi\|_{W_p^{2,1}(U_T)} \leq K_2(T)(\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|G\|_{L_p(U_T)}),$$

where $K_2(\cdot)$ *is an increasing function of* T.

4 An auxiliar result

We construct an approximate solution inductively

$$\mathbf{u}^{(0)} = \mathbf{0},\ \theta^{(0)} = 0, \psi^{(0)} = 0$$

and for $k = 1, 2, 3, ...,$ $\{\mathbf{u}^{(k)}, p^{(k)}\}$ and $\{\theta^{(k)}\}, \{\psi^{(k)}\}$ are respectively, the solutions of problems

$$\begin{aligned} \frac{\partial \mathbf{u}^{(k)}}{\partial t} - \nu \triangle \mathbf{u}^{(k)} + \frac{1}{\rho}\nabla p^{(k)} &= \mathbf{g}(\beta_\theta \theta^{(k-1)} + \beta_\psi \psi^{(k-1)}) + \widetilde{Q_1} \\ &\quad -(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}, \\ \text{div}\ \mathbf{u}^{(k)} &= 0, \\ \mathbf{u}^{(k)}|_{\Gamma_T} &= 0, \\ \mathbf{u}^{(k)}(0) &= \mathbf{u}_0(x), \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \theta^{(k)}}{\partial t} - \alpha \triangle \theta^{(k)} &= Q_2 - (\mathbf{u}^{(k-1)} \cdot \nabla)\theta_2 - (\mathbf{u}^{(k-1)} \cdot \nabla)\theta^{(k-1)}, \\
\theta^{(k)}|_{\Gamma_T} &= 0, \\
\theta^{(k)}(0) &= \theta_0(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \psi^{(k)}}{\partial t} - D \triangle \psi^{(k)} &= Q_3 - (\mathbf{u}^{(k-1)} \cdot \nabla)\psi_2 - (\mathbf{u}^{(k-1)} \cdot \nabla)\psi^{(k-1)}, \\
\psi^{(k)}|_{\Gamma_T} &= 0, \\
\psi^{(k)}(0) &= \psi_0(x),
\end{aligned}
$$

where $\widetilde{Q}_1 = Q_1 + \mathbf{g}(\beta_\theta \theta_2 + \beta_\psi \psi_2)$.

Now, we prove the boundness of above sequence.

Lemma 3 *For sufficiently small $T_1 \in (0, T]$, the sequence $\{\mathbf{u}^{(k)}, p^{(k)}, \theta^{(k)}, \psi^{(k)}\}$ is bounded in $W_p^{2,1}(U_{T_1}) \times L_p(U_{T_1}) \times W_p^{2,1}(U_{T_1}) \times W_p^{2,1}(U_{T_1})$.*

Proof. Let

$$
\Phi^{(k)}(T) = \|\mathbf{u}^{(k)}\|_{W_p^{2,1}(U_T)} + \|\theta^{(k)}\|_{W_p^{2,1}(U_T)} + \|\psi^{(k)}\|_{W_p^{2,1}(U_T)} + \|\nabla p^{(k)}\|_{L_p(U_T)}.
$$

From Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
\Phi^{(k)}(T) \le\ & K_1(T)(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\mathbf{g}(\beta_\theta \theta^{(k-1)} + \beta_\psi \psi^{(k-1)})\|_{L_p(U_T)} \\
& + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}\|_{L_p(U_T)} + \|\widetilde{Q}_1\|_{L_p(U_T)}) \\
& + K_2(T)(\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_2\|_{L_p(U_T)} + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\theta^{(k-1)}\|_{L_p(U_T)} \\
& + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\theta_2\|_{L_p(U_T)}) \\
& + K_3(T)(\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_3\|_{L_p(U_T)} + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\psi^{(k-1)}\|_{L_p(U_T)} \\
& + \|(\mathbf{u}^{(k-1)} \cdot \nabla)\psi_2\|_{L_p(U_T)}).
\end{aligned}
$$

Now, we estimate the right-hand side of the above inequality.

The following estimate was obtained in [5] (see also [6])

$$
\|(\mathbf{u}^{(k-1)} \cdot \nabla)\mathbf{u}^{(k-1)}\|_{L_p(U_T)} \le C[\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^\delta \Phi^{(k-1)}(T)^2]
$$

with some positive constant δ and $C \geq 2$.

We will prove

$$\|(\mathbf{u}^{(k-1)} \cdot \nabla)\phi^{(k-1)}\|_{L_p(U_T)} \leq C_2[\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\phi_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + T^\alpha \Phi^{(k-1)}(T)^2],$$

where $\alpha > 0$.

In fact, we have

$$\|(\mathbf{u}^{(k-1)} \cdot \nabla)\phi^{(k-1)}\|^p_{L_p(U_T)} \leq \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(U_T)} \|\nabla\phi^{(k-1)}\|^p_{L_p(U_T)}.$$

We observe that

$$\begin{aligned}\|\nabla\phi^{(k-1)}(t)\|_{L_p(\Omega)} &\leq \|\phi^{(k-1)}(t)\|_{W_p^1(\Omega)} \\ &\leq \|\phi^{(k-1)}(t)\|^a_{W_p^2(\Omega)} \|\phi^{(k-1)}(t)\|^{(1-a)}_{L_\infty(\Omega)},\end{aligned}$$

where $a = \frac{p-3}{2p-3}$.

By other hand, based on [5], we have

$$\|\phi^{(k-1)}\|_{L_\infty(U_T)} \leq C_3(\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))$$

and

$$\|\mathbf{u}^{(k-1)}\|_{L_\infty(U_T)} \leq C_4(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T)).$$

Consequently,

$$\begin{aligned}\|(\mathbf{u}^{(k-1)} \cdot \nabla)\phi^{(k-1)}\|^p_{L_p(U_T)} &\leq \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(U_T)} \int_0^T \|\nabla\phi^{(k-1)}(t)\|^p_{L_p(\Omega)} dt \\ &\leq \|\mathbf{u}^{(k-1)}\|^p_{L_\infty(U_T)} \|\phi^{(k-1)}\|^{(1-a)p}_{L_\infty(U_T)} \int_0^T \|\phi^{(k-1)}(t)\|^{pa}_{W_p^2(\Omega)} dt.\end{aligned}$$

But,

$$\int_0^T \|\phi^{(k-1)}(t)\|^{ap}_{W_p^2(\Omega)} dt \leq \left(\int_0^T 1^s dt\right)^{\frac{1}{s}} \left(\int_0^T \|\phi^{(k-1)}(t)\|^{apr}_{W_p^2(\Omega)} dt\right)^{\frac{1}{r}}$$

since $a < 1$, we take $\frac{1}{s} = 1 - a$, $\frac{1}{r} = a$ then $\frac{1}{r} + \frac{1}{s} = 1$ and thus in the last inequality, we have

$$\begin{aligned}\int_0^T \|\phi^{(k-1)}(t)\|^{ap}_{W_p^2(\Omega)} dt &\leq T^{1-a} \left(\int_0^T \|\phi^{(k-1)}(t)\|^p_{W_p^2(\Omega)} dt\right)^a \\ &\leq T^{1-a} \|\phi^{(k-1)}\|^{ap}_{W_p^{2,1}(U_T)} \\ &\leq T^{1-a} \left(\Phi^{(k-1)}(T)\right)^{ap}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\phi^{(k-1)}\|_{L_p(U_T)} &\leq T^{\frac{1-a}{p}}C_4(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))\\ &\quad\times(\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))^{1-a}\times\left(\Phi^{(k-1)}(T)\right)^a.\end{aligned}$$

We observe that

$$\begin{aligned}&(\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))^{1-a}\times T^{\frac{1-a}{p}}\left(\Phi^{(k-1)}(T)\right)^a\\ \leq\ &2^{-a}(\|\phi_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{(1-\frac{1}{p})(1-\frac{3}{p})(1-a)}(\Phi^{(k-1)}(T))^{1-a})\times T^{\frac{1-a}{p}}\left(\Phi^{(k-1)}(T)\right)^a\\ =\ &2^{-a}(\|\phi_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a+T^{\delta_1}\Phi^{(k-1)}(T)),\end{aligned}$$

where $\delta_1=(1-\frac{1}{p})(1-\frac{3}{p})(1-a)+\frac{1-a}{p}$.

Also,

$$\|\phi_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a\leq\frac{\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}}{\frac{1}{1-a}}+\frac{T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T)}{\frac{1}{a}},$$

where we use the inequality $x^{\frac{1}{r}}y^{\frac{1}{s}}\leq\frac{x}{r}+\frac{y}{s}$, $\frac{1}{s}+\frac{1}{r}=1$, and therefore,

$$\begin{aligned}\|\phi_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a &\leq (1-a)\|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+aT^{\frac{1-a}{ap}}\Phi^{(k-1)}(T)\\ &\leq \|\phi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+aT^{\frac{1-a}{ap}}\Phi^{(k-1)}(T).\end{aligned}$$

Consequently,

$$\|(\mathbf{u}^{(k-1)}\cdot\nabla)\phi^{(k-1)}\|_{L_p(U_T)}\leq C_2(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)}+\|\phi_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{\alpha}\Phi^{(k-1)}(T)^2),$$

where $T^{\alpha}=T^{2(1-\frac{1}{p})(1-\frac{3}{p})}+(T^{\frac{1-a}{ap}}+T^{\delta_1})^2$. Similarly, we obtain

$$\|(\mathbf{u}^{(k-1)}\cdot\nabla)\theta_2\|_{L_p(U_T)}\leq C_5\|\nabla\theta_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{\delta_2}\Phi^{(k-1)}(T)),$$

where $T^{\delta_2}=T^{\frac{1-a}{ap}}+T^{\delta_1}$.

Indeed, we have

$$\|(\mathbf{u}^{(k-1)}\cdot\nabla)\theta_2\|^p_{L_p(U_T)} \le \|\nabla\theta_2\|^p_{L_\infty(U_T)}\int_0^T \|\mathbf{u}^{(k-1)}(t)\|^p_{L_p(\Omega)}dt,$$

but

$$\begin{aligned}\|\mathbf{u}^{(k-1)}(t)\|_{L_p(\Omega)} &\le \|\mathbf{u}^{(k-1)}(t)\|_{W^1_p(\Omega)}\\ &\le \|\mathbf{u}^{(k-1)}(t)\|^a_{W^2_p(\Omega)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)}_{L_\infty(\Omega)}\end{aligned}$$

with $a=\frac{p-3}{2p-3}$.

Therefore,

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\theta_2\|^p_{L_p(U_T)} &\le \|\nabla\theta_2\|^p_{L_\infty(U_T)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)p}_{L_\infty(U_T)}\int_0^T\|\mathbf{u}^{(k-1)}(t)\|^{ap}_{W^2_p(\Omega)}dt\\ &\le \|\nabla\theta_2\|^p_{L_\infty(U_T)}\|\mathbf{u}^{(k-1)}(t)\|^{(1-a)p}_{L_\infty(U_T)}T^{1-a}\|\mathbf{u}^{(k-1)}(t)\|^{ap}_{W^2_p(U_T)}.\end{aligned}$$

Then, we have

$$\begin{aligned}\|(\mathbf{u}^{(k-1)}\cdot\nabla)\theta_2\|_{L_p(U_T)} &\le C_4T^{\frac{1-a}{p}}\|\nabla\theta_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\ &\quad+T^{(1-\frac{1}{p})(1-\frac{3}{p})}\Phi^{(k-1)}(T))^{1-a}\Phi^{(k-1)}(T)^a\\ &\le C_4\|\nabla\theta_2\|_{L_\infty(U_T)}2^{-a}(\|\mathbf{u}_0\|^{1-a}_{W_p^{2-\frac{2}{p}}(\Omega)}(T^{\frac{1-a}{ap}}\Phi^{(k-1)}(T))^a+T^{\delta_1}\Phi^{(k-1)}(T))\\ &\le C_5\|\nabla\theta_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{\delta_2}\Phi^{(k-1)}(T)).\end{aligned}$$

Analogously, we obtain

$$\|(\mathbf{u}^{(k-1)}\cdot\nabla)\psi_2\|_{L_p(U_T)} \le C_6\|\nabla\psi_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{\delta_2}\Phi^{(k-1)}(T)).$$

By other hand,

$$\begin{aligned}\|\mathbf{g}(\beta_\theta\theta^{(k-1)}+\beta_\psi\psi^{(k-1)})\|_{L_p(U_T)} &\le \beta_\theta\|\mathbf{g}\|_{L_\infty(U_T)}\|\theta^{(k-1)}\|_{L_p(U_T)}\\ &\quad+\beta_\psi\|\mathbf{g}\|_{L_\infty(U_T)}\|\psi^{(k-1)}\|_{L_p(U_T)}\\ &\le C_7\|\mathbf{g}\|_{L_\infty(U_T)}(\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}+T^{\delta_2}\Phi^{(k-1)}(T)).\end{aligned}$$

The above estimates imply the following inequality

$$\begin{aligned}
\Phi^{(k)}(T) &\leq K_1(T)\{\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\widetilde{Q}_1\|_{L_p(U_T)} + C_1[\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{\delta}\Phi^{(k-1)}(T)^2] \\
&\quad + C_7\|\mathbf{g}\|_{L_\infty(U_T)}(\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{\delta_2}\Phi^{(k-1)}(T))\} \\
&\quad + K_2(T)\{\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_2\|_{L_p(U_T)} + C_2[\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\theta_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + T^{\alpha}\Phi^{(k-1)}(T)^2] + C_5\|\nabla\theta_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{\delta_2}\Phi^{(k-1)}(T))\} \\
&\quad + K_3(T)\{\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_3\|_{L_p(U_T)} + C_2[\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\psi_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + T^{\gamma}\Phi^{(k-1)}(T)^2] + C_6\|\nabla\psi_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + T^{\delta_2}\Phi^{(k-1)}(T))\}.
\end{aligned}$$

And, by choosing

$$\begin{aligned}
A_1 &\geq K_1(T_1)\{\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\widetilde{Q}_1\|_{L_p(U_T)} + C_1(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + 1) \\
&\quad + C_7\|\mathbf{g}\|_{L_\infty(U_T)}(\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + 1)\} \\
&\quad + K_2(T_1)\{\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_2\|_{L_p(U_T)} + C_2(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\theta_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + 1) \\
&\quad + C_5\|\nabla\theta_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + 1)\} \\
&\quad + K_3(T_1)\{\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_3\|_{L_p(U_T)} + C_2(\|\mathbf{u}_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\psi_0\|^2_{W_p^{2-\frac{2}{p}}(\Omega)} + 1) \\
&\quad + C_6\|\nabla\psi_2\|_{L_\infty(U_T)}(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + 1)\},
\end{aligned}$$

and define

$$T_1 = \min\{A_1^{-\frac{2}{\delta}}, A_1^{-\frac{1}{\delta_2}}, A_1^{-\frac{2}{\alpha}}, A_1^{-\frac{2}{\gamma}}\}.$$

Then, $\Phi^{(k)}(T_1) \leq A_1$ holds provided that $\Phi^{(k-1)}(T_1) \leq A_1$.

Since

$$\begin{aligned}
\Phi^{(1)}(T_1) &\leq K_1(T_1)\{\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\widetilde{Q}_1\|_{L_p(U_{T_1})}\} \\
&\quad + K_2(T_1)\{\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_2\|_{L_p(U_{T_1})}\} \\
&\quad + K_3(T_1)\{\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|Q_3\|_{L_p(U_{T_1})}\} \\
&\leq A_1,
\end{aligned}$$

the assertion of the lemma comes out.

5 Proof of the Theorem

Setting $\mathbf{u}^{(n,s)}(t) = \mathbf{u}^{(n+s)}(t) - \mathbf{u}^{(n)}(t), p^{(n,s)}(t) = p^{(n+s)}(t) - p^{(n)}(t)$, $\theta^{(n,s)}(t) = \theta^{(n+s)}(t) - \theta^{(n)}(t), \psi^{(n,s)}(t) = \psi^{(n+s)}(t) - \psi^{(n)}(t)$, we have

$$\begin{aligned} \frac{\partial \mathbf{u}^{(n,s)}}{\partial t} - \nu \triangle \mathbf{u}^{(n,s)} + \frac{1}{\rho}\nabla p^{(n,s)} &= F^{(n,s)}, \\ \text{div } \mathbf{u}^{(n,s)} &= 0, \\ \mathbf{u}^{(n,s)}|_{\Gamma_{T_1}} &= 0, \\ \mathbf{u}^{(n,s)}(0) &= 0, \end{aligned} \tag{7}$$

where $F^{(n,s)} = \mathbf{g}(\beta_\theta \theta^{(n-1,s)} + \beta_\psi \psi^{(n-1,s)}) - (\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{(n+s-1)} - (\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,s)}$,

$$\begin{aligned} \frac{\partial \theta^{(n,s)}}{\partial t} - \alpha \triangle \theta^{(n,s)} &= G^{(n,s)} \\ \theta^{(n,s)}|_{\Gamma_{T_1}} &= 0, \\ \theta^{(n,s)}(0) &= 0, \end{aligned} \tag{8}$$

where $G^{(n,s)} = -(\mathbf{u}^{(n-1,s)} \cdot \nabla)\theta_2 - (\mathbf{u}^{(n+s-1)} \cdot \nabla)\theta^{(n-1,s)} - (\mathbf{u}^{(n-1,s)} \cdot \nabla)\theta^{(n-1)}$ and

$$\begin{aligned} \frac{\partial \psi^{(n,s)}}{\partial t} - D \triangle \psi^{(n,s)} &= H^{(n,s)} \\ \psi^{(n,s)}|_{\Gamma_{T_1}} &= 0, \\ \psi^{(n,s)}(0) &= 0, \end{aligned} \tag{9}$$

where $H^{(n,s)} = -(\mathbf{u}^{(n-1,s)} \cdot \nabla)\psi_2 - (\mathbf{u}^{(n+s-1)} \cdot \nabla)\psi^{(n-1,s)} - (\mathbf{u}^{(n-1,s)} \cdot \nabla)\psi^{(n-1)}$

Let

$$\Lambda^{(n,s)}(t) = \|\mathbf{u}^{(n,s)}\|_{W_p^{2,1}(U_t)} + \|\theta^{(n,s)}\|_{W_p^{2,1}(U_t)} + \|\psi^{(n,s)}\|_{W_p^{2,1}(U_t)} + \|\nabla p^{(n,s)}\|_{L_p(U_t)}.$$

Then, it follows that for $t \in (0, T_1]$,

$$\begin{aligned} \|F^{(n,s)}\|^p_{L_p(U_t)} \leq\ & c(\|\mathbf{g}(\beta_\theta \theta^{(n-1,s)} + \beta_\psi \psi^{(n-1,s)})\|^p_{L_p(U_t)} + \|\mathbf{u}^{(n-1,s)} \cdot \nabla \mathbf{u}^{(n+s-1)}\|^p_{L_p(U_t)} \\ & + \|(\mathbf{u}^{(n-1)} \cdot \nabla)\mathbf{u}^{(n-1,s)}\|^p_{L_p(U_t)}. \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{(n-1+s)}\|^p_{L_p(U_t)} &\le \int_0^t \|\nabla\mathbf{u}^{(n-1+s)}(\tau)\|^p_{L_p(\Omega)}\|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)}d\tau\\
&\le \sup_{0\le s\le t}\|\nabla\mathbf{u}^{(n-1+s)}(\tau)\|^p_{L_p(\Omega)}\int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)}d\tau\\
&\le \sup_{0\le s\le t}\|\mathbf{u}^{(n-1+s)}(\tau)\|^p_{W^1_p(\Omega)}\int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W^1_p(\Omega)}d\tau\\
&\le \sup_{0\le s\le t}\|\mathbf{u}^{(n-1+s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}\int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}d\tau\\
&\le (\|\mathbf{u}^{(n-1+s)}(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\quad +\widehat{c}\|\mathbf{u}^{(n-1+s)}\|_{W_p^{2,1}(U_t)})^p\int_0^t \widehat{c}^p\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)}d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\|(\mathbf{u}^{(n-1)}\cdot\nabla)\mathbf{u}^{(n-1,s)}\|^p_{L_p(U_t)} &\le \int_0^t d\tau\int_\Omega |\mathbf{u}^{(n-1)}(\tau)|^p|\nabla\mathbf{u}^{(n-1,s)}(\tau)|^p dx\\
&\le \|\mathbf{u}^{(n-1)}\|^p_{L_\infty(U_t)}\int_0^t \|\nabla\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_p(\Omega)}d\tau\\
&\le \sup_{0\le\tau\le t}\|\mathbf{u}^{(n-1)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}\int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W^1_p(\Omega)}d\tau\\
&\le \sup_{0\le\tau\le t}\|\mathbf{u}^{(n-1)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}\int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)}d\tau\\
&\le (\|\mathbf{u}^{(n-1)}(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\quad +\widehat{c}\|\mathbf{u}^{(n-1)}\|_{W_p^{2,1}(U_t)})^p\widehat{c}^p\int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)}d\tau
\end{aligned}
$$

and

$$
\|\mathbf{g}(\beta_\theta\theta^{(n-1,s)}+\beta_\psi\psi^{(n-1,s)})\|^p_{L_p(U_t)}\le C\|\mathbf{g}\|^p_{L_\infty(U_t)}\int_0^t\|(\theta^{(n-1,s)}+\psi^{(n-1,s)})\|^p_{L_p(U_\tau)}d\tau,
$$

consequently

$$
\begin{aligned}
\|F^{(n,s)}\|^p_{L_p(U_t)} &\le C\|\mathbf{g}\|^p_{L_\infty(U_t)}\int_0^t\|(\theta^{(n-1,s)}+\psi^{(n-1,s)})\|^p_{L_p(U_\tau)}d\tau+(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}\\
&\quad +\widehat{c}\|\mathbf{u}^{(n-1+s)}\|_{W_p^{2,1}(U_t)})^p\int_0^t\widehat{c}^p\|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)}d\tau
\end{aligned}
$$

$$+(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)}$$
$$+\widehat{c}\|\mathbf{u}^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau.$$

Also, we have

$$\begin{aligned}
\|(\mathbf{u}^{(n+s-1)}\cdot\nabla)\theta^{(n-1,s)}\|^p_{L_p(U_t)} &\le \int_0^t d\tau \int_\Omega |\mathbf{u}^{(n+s-1)}(\tau)|^p |\nabla\theta^{(n-1,s)}(\tau)|^p dx \\
&\le \|\mathbf{u}^{(n+s-1)}\|^p_{L_\infty(U_t)} \int_0^t \|\nabla\theta^{(n-1,s)}(\tau)\|^p_{L_p(\Omega)} d\tau \\
&\le \sup_{0\le\tau\le t} \|\mathbf{u}^{(n+s-1)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\theta^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\le (\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad +\widehat{c}\|\mathbf{u}^{(n+s-1)}\|_{W_p^{2,1}(U_t)})^p \widehat{c}^p \int_0^t \|\theta^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau,
\end{aligned}$$

$$\begin{aligned}
\|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\theta^{(n-1)}\|^p_{L_p(U_t)} &\le \int_0^t \|\nabla\theta^{(n-1)}(\tau)\|^p_{L_p(\Omega)} \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\le \sup_{0\le s\le t} \|\nabla\theta^{(n-1)}(\tau)\|^p_{L_p(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\le \sup_{0\le s\le t} \|\theta^{(n-1)}(\tau)\|^p_{W_p^1(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^1(\Omega)} d\tau \\
&\le \sup_{0\le s\le t} \|\theta^{(n-1)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^{2-\frac{2}{p}}(\Omega)} d\tau \\
&\le (\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad +\widehat{c}\|\theta^{(n-1)}(\tau)\|_{W_p^{2,1}(U_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau,
\end{aligned}$$

$$\begin{aligned}
\|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\theta_2\|^p_{L_p(U_t)} &\le \int_0^t \|\nabla\theta_2(\tau)\|^p_{L_p(\Omega)} \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{L_\infty(\Omega)} d\tau \\
&\le \sup_{0\le s\le t} \|\nabla\theta_2(\tau)\|^p_{L_p(\Omega)} \int_0^t \|\mathbf{u}^{(n-1,s)}(\tau)\|^p_{W_p^1(\Omega)} d\tau \\
&\le (\|\theta_2(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad +\widehat{c}\|\theta^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau
\end{aligned}$$

It follows that for $t \in (0, T_1]$,

$$
\begin{aligned}
\|G^{(n,s)}\|^p_{L_p(U_t)} &\leq c(\|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\theta_2\|^p_{L_p(U_t)} + \|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\theta^{(n-1)}\|^p_{L_p(U_t)} \\
&\quad + \|(\mathbf{u}^{(n+s-1)} \cdot \nabla)\theta^{(n-1,s)}\|^p_{L_p(U_t)} \\
&\leq (\|\theta_2(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\theta^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau) \\
&\quad + c(\|\theta_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\theta^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau \\
&\quad + c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + \widehat{c}\|\mathbf{u}^{(n+s-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \|\theta^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau.
\end{aligned}
$$

Analogously, we can prove that

$$
\begin{aligned}
\|H^{(n,s)}\|^p_{L_p(U_t)} &\leq c(\|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\psi_2\|^p_{L_p(U_t)} + \|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\psi^{(n-1)}\|^p_{L_p(U_t)} \\
&\quad + \|(\mathbf{u}^{(n+s-1)} \cdot \nabla)\psi^{(n-1,s)}\|^p_{L_p(U_t)} \\
&\leq (\|\psi_2(0)\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\psi^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \widehat{c}^p \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau) \\
&\quad + c(\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \widehat{c}\|\psi^{(n-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \|\mathbf{u}^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau \\
&\quad + c(\|\mathbf{u}_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} \\
&\quad + \widehat{c}\|\mathbf{u}^{(n+s-1)}\|_{W_p^{2,1}(U_t)})^p \int_0^t \|\psi^{(n-1,s)}\|^p_{W_p^{2,1}(U_\tau)} d\tau.
\end{aligned}
$$

By using the above estimates and together with Lemma 4.1, we have for $t \in [0, T_1]$ and $p > 3$

$$
\Lambda^{(n,s)}(t) \leq c \left(\int_0^t \Lambda^{(n-1,s)}(\tau)^p \right)^{\frac{1}{p}}
$$

or

$$
\left[\Lambda^{(n,s)}(t)\right]^p \leq c^p \int_0^t \left[\Lambda^{(n-1,s)}(\tau)\right]^p d\tau, \tag{10}
$$

consequently $\Lambda^{(n,s)}(t) \to 0$ as $n \to \infty$, $\forall\ t \in [0, T_1]$. Firstly, we observe that $W_p^{2,1}(U_t)$ is a Banach space and consequently, we have there exist $\mathbf{u} \in$

$W_p^{2,1}(U_{T_1})$, $\theta, \psi \in W_p^{2,1}(U_{T_1})$ such that

$$\begin{aligned} \mathbf{u}^n &\to \mathbf{u} \text{ strongly in } W_p^{2,1}(U_{T_1}), \\ \theta^n &\to \theta \text{ strongly in } W_p^{2,1}(U_{T_1}) \\ \psi^n &\to \psi \text{ strongly in } W_p^{2,1}(U_{T_1}). \end{aligned}$$

Also, from of the completeness of $L_p(U_{t_1})$, there exist $p \in L_p(U_{T_1})$ such that

$$p^n \to p \text{ strongly in } L_p(U_{T_1}).$$

Now, the next step is to take the limit. But, once the above convergences have been established, this is a standard procedure to obtain $\mathbf{u}$,θ, ψ,p as a strong solution of the problem (2.1)-(2.3).

Now, we need only to argument the uniqueness of the solution in order to complete the proof of Theorem . Suppose that there exist another solution $\overline{\mathbf{u}}, \overline{\theta}, \overline{\psi}, \overline{p}$ of (2.1) and (2.3) with the same regularity as stated in the theorem. Define

$$U = \overline{\mathbf{u}} - \mathbf{u},\ \Theta = \overline{\theta} - \theta,\ \Psi = \overline{\psi} - \psi,\ \ P = \overline{p} - p.$$

These auxiliar functions verify a set of equations similar to (5.1)-(5.3). Repeat the argument used to obtain (5.4), we get for $\eta(t) = \|U\|^p_{W_p^{2,1}(U_t)} + \|\Theta\|^p_{W_p^{2,1}(U_t)} + \|\Psi\|^p_{W_p^{2,1}(U_t)} + \|P\|^p_{L_p(U_t)}$ an inequality of the following type

$$\eta(t) \leq c \int_0^t \eta(\tau) d\tau$$

which, by Gronwall's inequality, is equivalent to assert $U = 0, \Theta = 0, \Psi = 0, P = 0$ i.e.,

$$\overline{\mathbf{u}} = \mathbf{u},\ \overline{\theta} = \theta,\ \ \overline{\psi} = \psi.$$

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