

# On an inequality by N. Trudinger and J. Moser and related elliptic equations

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## Abstract

It has been shown by Trudinger and Moser that for normalized functions  $u$  of the Sobolev space  $W^{1,N}(\Omega)$ , where  $\Omega$  is a domain in  $\mathbb{R}^N$ , the integral  $\int_{\Omega} \exp(u^{\alpha N/(N-1)}) dx$  remains uniformly bounded. Carleson and Chang proved that there exists a corresponding extremal function in the case that  $\Omega$  is the unit ball in  $\mathbb{R}^N$ . In this paper we give a new proof, a generalization, and a new interpretation of this result. In particular, we give an explicit sequence which is maximizing for the above integral among all "normalized" concentrating sequences.

## 1 Introduction

### 1.1 Critical growth in $W^{1,q}$ : the case $q = N$ versus $1 < q < N$

We recall the following facts: let  $W_0^{1,q}(\Omega)$  denote the Sobolev space over a bounded domain  $\Omega \subset \mathbb{R}^N$ , with norm  $\|u\|_q^q = \int_{\Omega} |\nabla u|^q dx$ . Then, for  $1 < q < N$  critical growth can be expressed by the following relations: Let

$$\sup_{\|u\|_q=1} \int_{\Omega} |u|^p dx = s_{N,q}(p); \quad (1)$$

then

$$\begin{aligned} s_{N,q}(p) < \infty & \quad \text{for } 1 < p \leq q^* = \frac{qN}{N-q} \\ s_{N,q}(p) = +\infty & \quad \text{for } p > q^* \end{aligned}$$

The value of the *best Sobolev constant*  $s_{N,q}(q^*)$  is explicit and independent of the domain  $\Omega$ , and it is known that it is *never attained* in a smooth domain different from  $\mathbb{R}^N$ :  $s_{N,q}(q^*, \Omega) = s_{N,q}(q^*, \mathbb{R}^N)$ , and only  $s_{N,q}(q^*, \mathbb{R}^N)$  is attained.

If  $q = N$ , then critical growth is given by the Trudinger-Moser inequality, which can be expressed as

$$\sup_{\|u\|_N=1} \int_{\Omega} (e^{\alpha u^{N/(N-1)}} - 1) dx = c_N(\alpha) \cdot |\Omega| \quad (2)$$

(note the dependence on  $|\Omega|$ ); then, denoting by  $\omega_{N-1}$  the  $(N-1)$ -dimensional surface of the unit sphere in  $\mathbb{R}^N$ , one has

$$\begin{aligned} c_N(\alpha) < +\infty & \quad \text{for} \quad 0 < \alpha \leq \alpha_N = N\omega_{N-1}^{1/(N-1)} \\ c_N(\alpha) = +\infty & \quad \text{for} \quad \alpha > \alpha_N \end{aligned}$$

In this note we consider general nonlinearities with subcritical and critical growth in the case  $q = N$ . We will suppose throughout this paper that

- F1)  $F \in \mathcal{C}^1(\mathbb{R})$
- F2)  $F$  is increasing on  $\mathbb{R}^+$ , and  $F(t) = F(|t|)$
- F3)  $0 \leq F(t) \leq e^{\alpha_N |t|^{N/(N-1)}} - 1$ , for all  $t \in \mathbb{R}$

Then we say, for dimension  $N$

$$F \text{ has subcritical growth if } \lim_{t \rightarrow \infty} \frac{F(t)}{e^{\alpha_N |t|^{N/(N-1)}}} = 0 ;$$

otherwise we say that  $F$  has *critical growth*; in this case we normalize to  $\lim_{t \rightarrow \infty} F(t)e^{-\alpha_N |t|^{N/(N-1)}} = 1$ , i.e. we say

$$F \text{ has critical growth if } \lim_{t \rightarrow \infty} \frac{F(t)}{e^{\alpha_N |t|^{N/(N-1)}}} = 1 .$$

As is known from the cases  $1 < q < N$ , the notion of criticality is closely related to the behaviour of the functional on *concentrating sequences*, i.e. (in the case  $q = N$ ) sequences  $\{u_n\}$  converging weakly to 0 in  $W_0^{1,N}(\Omega)$  and such that  $|\nabla u_n|^N$  converges to a Dirac delta-function in measure. We make the

**Definition 1** A sequence  $\{u_n\} \subset W_0^{1,N}(\Omega)$  is a *normalized concentrating sequence*, if

- a)  $\|u_n\|_N = 1$
- b)  $u_n \rightharpoonup 0$ , weakly in  $W_0^{1,N}(\Omega)$
- c)  $\exists x_0 \in \Omega$  such that  $\forall \rho > 0 : \int_{\Omega \setminus B_\rho(x_0)} |\nabla u_n|^N dx \rightarrow 0$

We first consider the behaviour of the functional  $\int_{\Omega} F(u_n)dx$ , with *subcritical* nonlinearities  $F$ , for normalized concentrating sequences  $\{u_n\}$ . We will prove:

**Theorem 2** *If  $F$  has subcritical growth, then  $\lim_{n \rightarrow \infty} \int_{\Omega} F(u_n)dx = 0$ , for any normalized concentrating sequence  $\{u_n\}$ .*

By a concentration-compactness alternative of P.L. Lions it will be easy to conclude by Theorem 2 that

**Theorem 3** *If  $F$  has subcritical growth, then*

$$\sup_{\|u\|_N=1} \int_{\Omega} F(u)dx$$

*is attained.*

Next, we consider critical growth; here we restrict the considerations to the case  $\Omega = B_1(0)$ , the unit ball in  $\mathbb{R}^N$ . Studying again the behaviour of the functional on normalized concentrating sequences, we find

**Theorem 4** *Let  $\Omega = B_1(0)$ . If  $F$  has critical growth, then*

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(u_n)dx \in [0, e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} |\Omega|]$$

*for any normalized concentrating sequence  $\{u_n\}$ . In particular, there exists an explicit normalized concentrating sequence  $\{y_n\}$  with*

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(y_n)dx = e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} |\Omega|.$$

We now turn to the question whether the supremum is attained in the case of critical growth. In an interesting and intricate paper Carleson and Chang [3] have shown that

$$\sup_{\|u\|_N=1} \int_{B_1(0)} e^{\alpha_N u^{N/(N-1)}} dx$$

is attained. For  $N = 2$ , this result was extended to general bounded domains  $\Omega$  by Flucher [4], using symmetrization and conformal deformations.

Here we will prove more generally

**Theorem 5** *Suppose that  $F$  satisfies F1-F3 and*

$$F4) \quad F_\lambda(t) \geq e^{\alpha_N t^{N/(N-1)}} - 1 - \lambda t^{N/(N-1)}.$$

*Then, for  $\lambda < \alpha_N$ , one has*

$$C_{N,\lambda} = \sup_{\|u\|_N=1} \int_{B_1(0)} F_\lambda(u) dx > e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} |\Omega|,$$

*and  $C_{N,\lambda}$  is attained.*

## 1.2 Application to the existence problem for related PDE

The relations (1) and (2) are important with regard to the solvability of the related differential equations with critical growth:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

respectively

$$\begin{cases} -\Delta u = h(u)e^{4\pi u^2} = e^{4\pi u^2 + \log(h(u))} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

where  $g$  and  $h$  are functions with subcritical growth:

$$\frac{g(s)}{s^{2^*-1}} \rightarrow 0 \quad \text{as } s \rightarrow \infty$$

respectively

$$\frac{\log(h(s))}{s^2} \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

Solutions of (3) are given as critical points of the related functional

$$I(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} - G(u) dx$$

where  $G(s) = \int_0^s g(r) dr$ . Due to the fact that the supremum in (1) is not attained, one finds for the functional  $I(u)$  *certain levels* at which the compactness condition of Palais-Smale fails. In the famous paper [2] of Brezis-Nirenberg it was shown that one can overcome this obstacle and prove existence if one shows, *using properties of the lower order term  $g$* , that the

critical levels of the functional avoid these non-compactness levels. To obtain such statements, one uses special sequences of functions obtained from the *maximizing sequence* for (1), which are explicit concentrating functions converging weakly to zero.

It has been shown in [1] by Adimurthi and in [5] that similar methods can be employed to prove existence results also for equation (2). Indeed, considering the associated functional to (4)

$$J(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(u) \, dx \quad (5)$$

where  $F(s) = \int_0^s h(r) e^{4\pi r^2} dr$ , one finds again levels of non-compactness; however, due to the fact that the best constant  $c_2(4\pi)$  is attained, there is no natural concentrating sequence to be used to show that these levels are avoided. Thus, it is difficult to obtain *optimal existence results*. The sequence used in [1] and [5] is the so-called *Moser sequence*

$$z_n(t) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if } 0 \leq |x| \leq \frac{1}{n} \\ \frac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \text{if } \frac{1}{n} \leq |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad (6)$$

which was proposed by J. Moser in [8] to prove that the inequality (2) is sharp with respect to the constant  $4\pi$  in the exponent. This sequence satisfies

$$\lim_{n \rightarrow \infty} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = 2\pi . \quad (7)$$

while the *explicit concentrating sequence*  $\{y_n\}$  with  $\|y_n\| = 1$  mentioned in Theorem 4 satisfies

$$\lim_{n \rightarrow \infty} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = e\pi . \quad (8)$$

Since the proof of existence of solutions for equation (4) as done in [5] depends on the value of the limit (7) above, we can improve the result obtained using the limit (8). We will prove

**Theorem 6** *Assume that  $h \in C(\mathbb{R})$  and let  $f(s) = h(s)e^{4\pi s^2}$ . Assume that*

*H1)  $f(0) = 0$*

*H2)  $\exists s_0 > 0, \exists M > 0$  such that*

$$0 < F(s) = \int_0^s f(r)dr \leq M|f(s)|, \quad \forall |s| \geq s_0$$

$$H3) 0 < F(s) \leq \frac{1}{2}f(s)s, \quad \forall s \in \mathbb{R} \setminus \{0\}, \quad \forall x \in \Omega$$

Then equation (4) has a solution provided that

$$\lim_{s \rightarrow \infty} h(s)s = \beta > \frac{1}{e\pi} \quad (9)$$

We recall that in [5] it was proved that (4) has a solution provided that  $\lim_{s \rightarrow \infty} h(s)s = \beta > \frac{1}{2\pi}$ . We refer to [6] for non-existence results concerning equation (4).

## 2 Proofs

### 2.1 Proof of Theorem 2:

Step 1. Let  $\{u_n\}$  denote a normalized concentrating sequence as defined above. We may suppose that the concentration point is  $0 \in \Omega$ , and that  $u \geq 0$  (since both  $\|u\|_N$  and  $F(u)$  do not change replacing  $u$  by  $|u|$ ). We apply symmetrization (following J. Moser [8]), defining the radially symmetric function  $u^*$  as follows: let

$$m\{x \mid u^*(x) > \rho\} = m\{x \in \Omega \mid u(x) > \rho\} \quad \text{for every } \rho > 0.$$

Then  $u^*$  is a decreasing function in  $|x|$ , with  $u^*(|x|) = 0$  for  $|x| > R$ , where  $m(B_R(0)) = m(\Omega)$ . By construction

$$\int_{B_R} F(u_n^*)dx = \int_{\Omega} F(u_n)dx,$$

and it is known that

$$1 = \int_{\Omega} |\nabla u_n|^N dx \geq \int_{B_R} |\nabla u_n^*|^N dx.$$

Setting  $z_n = \frac{u_n^*}{\|u_n^*\|_N} \geq u_n^*$  we thus find, using the monotonicity of  $F(t)$

$$\int_{B_R} F(u_n^*)dx \leq \int_{B_R} F(z_n)dx$$

Hence it suffices to show that  $\int_{B_R} F(z_n)dx = \omega_{N-1} \int_0^R F(z_n(\rho))\rho^{N-1}d\rho \rightarrow 0$ .

Step 2. To prove that  $\int_0^R F(z_n)\rho^{N-1}d\rho \rightarrow 0$ , we perform a change of variable which transforms the radial integral on  $[0, R]$  into an integral on the half-line  $[0, +\infty)$ . Let

$$\rho = R e^{-t/N} \quad \text{and} \quad w_n(t) = N^{(N-1)/N} \omega_{N-1}^{1/N} z_n(\rho) = \alpha_N^{(N-1)/N} z_n(\rho) .$$

Then  $w_n(t)$  is an increasing function on  $[0, \infty)$ . One checks easily that

$$\int_0^\infty |w'_n(t)|^N dt = w_{N-1} \int_0^R \left| \frac{d}{d\rho} z_n(\rho) \right|^N \rho^{N-1} d\rho = \int_{B_R} |\nabla z_n(x)|^N dx$$

and

$$\begin{aligned} \int_0^\infty F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt &= \frac{N}{R^N} \int_0^R F(z_n(\rho)) \rho^{N-1} d\rho \quad (10) \\ &= \frac{1}{m(B_R)} \int_{B_R} F(z_n(x)) dx \end{aligned}$$

Clearly, since the sequence  $u_n$  is concentrating in  $x_0$ , the sequence  $z_n$  is concentrating in 0 and the sequence  $w_n$  in  $+\infty$ , *i.e.* for any fixed  $A > 0$  we have  $\int_0^A |w'_n|^N dt \rightarrow 0$ .

We now distinguish the cases:

a) there exists  $a_n \in (0, +\infty)$  with  $w_n^{N/(N-1)}(a_n) = a_n - 2 \log(a_n)$ . Since for a given interval  $[0, A]$  we have, for  $\eta > 0$  arbitrarily small, that

$$w_n^{N(N-1)}(t) \leq t \left( \int_0^A |w'_n|^N dt \right)^{1/(N-1)} \leq \eta t, \quad \text{for } n \text{ large,}$$

it follows that  $a_n \rightarrow \infty$ . Thus we have by assumption F3

$$\begin{aligned} \int_0^A F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt &\leq \int_0^A (e^{w_n^{N/(N-1)}(t)} - 1) e^{-t} dt \leq \int_0^A (e^{\eta t} - 1) e^{-t} dt \\ &= \frac{1}{1-\eta} (1 - e^{(\eta-1)A}) - 1 + e^{-A} \end{aligned}$$

Since  $A$  is arbitrarily large and  $\eta > 0$  arbitrarily small, we conclude that  $\int_0^A F\left(\frac{1}{\alpha_N} w_n\right) e^{-t} dt \rightarrow 0$  as  $n \rightarrow \infty$ .

Next, consider

$$\begin{aligned} \int_A^{a_n} F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt &\leq \int_A^{a_n} (e^{t-2 \log t} - 1) e^{-t} dt \\ &= \frac{1}{A} - \frac{1}{a_n} + e^{-a_n} - e^{-A} \rightarrow 0, \quad \text{as } n \rightarrow \infty \end{aligned}$$

Finally, on  $[a_n, +\infty)$  we have  $w_n^{N(N-1)}(t) \geq a_n - 2 \log(a_n)$ , and hence, by the assumption of subcriticality, we have for any  $\epsilon > 0$  fixed that  $F(s) \leq \epsilon(e^{\alpha_N s^{N/(N-1)}} - 1)$ , for all  $s \geq a_n$  with  $n$  sufficiently large; thus, for  $n$  sufficiently large

$$\int_{a_n}^{\infty} F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt \leq \epsilon \int_{a_n}^{\infty} e^{w_n^{N(N-1)}(t)-t} dt \leq \epsilon c$$

Hence, the theorem is proved in case a).

b) for all  $t \in \mathbb{R}$  holds:  $w_n^{N/(N-1)}(t) < t - 2 \log^+ t$ . Then, arguing as in a) we have

$$\begin{aligned} & \int_0^{\infty} F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt \\ & \leq \int_0^A F\left(\frac{1}{\alpha_N^{(N-1)/N}} w_n(t)\right) e^{-t} dt + \int_A^{\infty} (e^{t-2 \log t} - 1) e^{-t} dt \end{aligned}$$

which can be made arbitrarily small.

## 2.2 Concentration-compactness

For the subsequent proofs we rely on the following concentration-compactness result of P.L. Lions [7]:

**Proposition 7** (*P.L. Lions*). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ , and let  $\{u_n\}$  be a sequence in  $W_0^{1,N}(\Omega)$  such that  $\|u_n\|_N \leq 1$  for all  $n$ . We may suppose that  $u_n \rightharpoonup u$  weakly in  $W_0^{1,N}(\Omega)$ ,  $|\nabla u_n|^N \rightarrow \mu$  weakly in measure.*

*Then either*

*(i)  $\mu = \delta_{x_0}$ , the Dirac measure of mass 1 concentrated at some  $x_0 \in \overline{\Omega}$ , and  $u \equiv 0$ , or*

*(ii) there exists  $\beta > \alpha_N$  such that the family  $u_n = e^{u_n^{N/(N-1)}}$  is uniformly bounded in  $L^\beta(\Omega)$  and thus*

*$\int_{\Omega} e^{\alpha_N |u_n|^{N/(N-1)}} \rightarrow \int e^{\alpha_N |u|^{N/(N-1)}}$  as  $n \rightarrow \infty$ . In particular, this is the case if  $u$  is different from 0.*



### 2.3 Proof of Theorem 3:

Since  $\sup_{\|u\|_N=1} \int_{\Omega} F(u) dx > 0$ , we conclude by Theorem 2 that there cannot exist a normalized concentrating sequence which is maximizing. By the concentration-compactness alternative of P.L. Lions we infer that the supremum is attained.

### 2.4 Proof of Theorem 4:

We proceed by the following steps:

1) if  $\{u_n\}$  is any normalized concentrating sequence in  $W_0^{1,N}(B_1)$ , then

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(u_n) dx \leq e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} |\Omega| ;$$

2) give an *explicit* normalized concentrating sequence  $\{y_n\}$  with

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(y_n) dx = e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} |\Omega| ;$$

**1. Upper bound:** In this proof we follow Carleson-Chang [3]. First note that by F3) and the transformation in section 2.1 it suffices to show that for any normalized concentrating sequence  $\{u_n\} \in C^1[0, \infty)$ , i.e  $\int_0^{\infty} |u'_n|^N = 1$ ,  $\int_0^A |u'_n|^N \rightarrow 0$ , with  $u_n(0) = 0$ ,  $u'_n(t) \geq 0$  holds

$$\overline{\lim}_{n \rightarrow \infty} \int_0^{\infty} (e^{u_n^{N/(N-1)-t}} - 1) dt \leq e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} , \quad (11)$$

More precisely, we show:

If  $\overline{\lim}_{n \rightarrow \infty} \int_0^{\infty} e^{u_n^{N/(N-1)-t}} > 2$ , then  $\{u_n\}$  has the following properties (cf. [3], p.117)

(a) if  $a_n \in [1, \infty)$  denotes the first point with  $u_n^{N/(N-1)}(a_n) = a_n - 2 \log a_n$ , then  $a_n \rightarrow \infty$

$$(b) \quad \overline{\lim}_{n \rightarrow \infty} \int_{a_n}^{\infty} e^{u_n^{N/(N-1)-t}} dt \leq e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}$$

$$(c) \quad \lim_{n \rightarrow \infty} \int_0^{a_n} e^{u_n^{N/(N-1)-t}} dt = 1$$

**Proof:**

Estimate (11) follows clearly from (b) and (c).

Property a): following [3], we note that the point  $a_n$  exists, for  $n$  large enough; if not,  $u_n^{N/(N-1)}(t) < t - 2 \log^+(t)$  for all  $t$  and thus

$\lim_{n \rightarrow \infty} \int_0^\infty e^{u_n^{N/(N-1)} - t} \leq 2$ , contradicting the assumption.

Next, for each  $A > 0$ , we can choose numbers  $n_0$  and  $\eta > 0$  such that for all  $t \in [0, A]$  and all  $n \geq n_0$

$$u_n^{N(N-1)}(t) \leq t \left( \int_0^A |u_n'|^N dt \right)^{1/(N-1)} \leq \eta t < t - 2 \log^+ t. \quad (12)$$

This implies that  $a_n \geq A$  for all  $n \geq n_0$ , and since  $A$  is arbitrary, we have  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Property c): Note that (12) implies also that  $u_n \rightarrow 0$  uniformly on compact sets. Using that  $a_n$  is the first point where  $u_n^{N/(N-1)}(t) = t - 2 \log^+(t)$  we have the following estimate from above

$$\begin{aligned} \int_0^{a_n} e^{u_n^{N/(N-1)}(t) - t} dt &\leq e^\varepsilon \int_0^A e^{-t} dt + \int_A^{a_n} e^{-2 \log t} dt \\ &= e^\varepsilon (1 - e^{-A}) + \left( \frac{1}{A} - \frac{1}{a_n} \right) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0, A \rightarrow \infty \end{aligned}$$

On the other hand, we can estimate from below

$$\int_0^{a_n} e^{u_n^{N/(N-1)}(t) - t} dt \geq \int_0^{a_n} e^{-t} dt = 1 - e^{-a_n} \rightarrow 1 \quad \text{as } a_n \rightarrow \infty$$

Property b): We recall that in [3], Lemma 2, the following result was proved (note the misprint there: the factor  $e^{(c^n/n)((n-1)/n)^n \beta_n}$  should read  $e^{(c^n/n)((n-1)/n)^{n-1} \beta_n}$ ):

**Proposition 8** (*Carleson-Chang*). *For  $a > 0$  and  $\delta > 0$  given, suppose that  $\int_a^\infty |w'|^N \leq \delta$ ; then*

$$\begin{aligned} &\int_a^\infty e^{w^{N/(N-1)}(t) - t} dt \\ &\leq \frac{1}{1 - \delta^{\frac{1}{N-1}}} \exp \left( w^{\frac{N}{N-1}}(a) \left[ 1 + \frac{1}{N-1} \frac{\delta}{(1 - \delta^{\frac{1}{N-1}})^{(N-1)}} \right] - a \right) \cdot e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} \end{aligned}$$

**Proof.** Apply this to  $w(t) = u_n(t)$ ,  $a = a_n$ , and  $\delta = \delta_n = \int_{a_n}^\infty |u_n'|^N dt$ . Thus we have

$$\int_{a_n}^\infty e^{u_n^{N/(N-1)}(t) - t} dt \leq \frac{1}{1 - \delta_n^{\frac{1}{N-1}}} e^{K_n} \cdot e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} \quad (13)$$

where  $K_n = u_n^{N/(N-1)}(a_n)[1 + \frac{1}{N-1} \frac{\delta}{(1-\delta_n^{1/(N-1)})^{N-1}}] - a_n$ . By

$$u_n^{N/(N-1)}(a_n) \leq a_n \left( \int_0^{a_n} |u_n'|^N dt \right)^{1/(N-1)}$$

we have, using  $\int_0^\infty |u_n'|^{N/(N-1)} = 1$ , that

$$u_n^{N/(N-1)}(a_n) \leq a_n(1 - \delta_n)^{1/(N-1)}.$$

Thus

$$\begin{aligned} \delta_n &\leq 1 - \frac{u_n^N(a_n)}{a_n^{N-1}} = 1 - \left(1 - \frac{2 \log^+(a_n)}{a_n}\right)^{N-1} \\ &\leq (N-1) \frac{2 \log^+(a_n)}{a_n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} K_n &= u_n^{N/(N-1)}(a_n) \left[1 + \frac{1}{N-1} \frac{\delta_n}{(1 - \delta_n^{1/(N-1)})^{N-1}}\right] - a_n \\ &= (a_n - 2 \log^+(a_n)) \left(1 + \frac{1}{N-1} \delta_n + O(\delta_n^{N/(N-1)})\right) - a_n \\ &= -2 \log^+(a_n) + a_n \frac{1}{N-1} \delta_n + O\left(\left(\frac{\log^N(a_n)}{a_n}\right)^{1/(N-1)}\right) \\ &\leq O\left(\left(\frac{\log^N(a_n)}{a_n}\right)^{1/(N-1)}\right) \end{aligned}$$

Since  $a_n \rightarrow \infty$  we see that  $\overline{\lim}_{n \rightarrow \infty} K_n \leq 0$ . Thus, we get by (13)

$$\overline{\lim}_{n \rightarrow \infty} \int_{a_n}^\infty e^{u_n^{N/(N-1)}(t)-t} dt \leq e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}$$

■

**2. The maximizing concentrating sequence:** We first consider the function  $F(t) = e^{\alpha_N t^{N/(N-1)}}$  with  $\Omega = B_1(0)$ . We want to produce an explicit normalized concentrating sequence of functions  $\{y_n\} \in C[0, \infty)$ , piecewise differentiable, with  $y_n(0) = 0$  and  $y_n'(t) \geq 0$  and such that

$$\lim_{n \rightarrow \infty} \int_0^\infty e^{y_n^{N/(N-1)}(t)-t} dt \rightarrow 1 + e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}$$

For  $n \in \mathbb{N}$  set  $\delta_n = \frac{2 \log n}{n}$ , and let

$$y_n(t) = \begin{cases} \frac{t}{n^{1/N}} (1 - \delta_n)^{\frac{N-1}{N}} & , \quad 0 \leq t \leq n \\ \frac{N-1}{(n(1-\delta_n))^{1/N}} \log \frac{A_n+1}{A_n+e^{-(t-n)/(N-1)}} + (n(1-\delta_n))^{\frac{N-1}{N}} & , \quad n \leq t \end{cases} \quad (14)$$

First note that  $y_n(t)$  is continuous and piecewise differentiable; furthermore we have

$$\int_0^n |y'_n(t)|^N dt = (1 - \delta_n)^{N-1}$$

We now choose  $A_n$  in (14) such that  $\int_0^\infty |y'_n(t)|^N dt = 1$ , i.e.

$$\int_n^\infty |y'_n(t)|^N dt = 1 - (1 - \delta_n)^{N-1} = (N-1)\delta_n + \sigma_N(\delta_n^2) \quad , \quad (15)$$

where

$$\sigma_N(s) = \begin{cases} 0 & , \quad N = 2 \\ O(s) & , \quad N \geq 3 \end{cases}$$

We show that such a choice for  $A_n$  is possible, with

$$A_n = \frac{1}{n^2} \frac{1}{e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}} + \begin{cases} O(1/n^4) & , \quad N = 2 \\ O(\log^2(n)/n^3) & , \quad N \geq 3 \end{cases} \quad (16)$$

Indeed, using the change of variables  $s = t - n$ , we have

$$\begin{aligned} & \int_n^\infty |y'_n(t)|^N dt \\ &= \int_n^\infty \frac{(N-1)^N}{n(1-\delta_n)} \left| \frac{d}{dt} [\log(A_n+1) - \log(A_n+e^{-(t-n)/(N-1)})] \right|^N dt \\ &= \int_n^\infty \frac{(N-1)^N}{n(1-\delta_n)} \left| \frac{\frac{1}{N-1} e^{-(t-n)/(N-1)}}{A_n+e^{-(t-n)/(N-1)}} \right|^N dt \end{aligned}$$

Next, setting  $r = e^{s/(N-1)}$  yields

$$\begin{aligned} & \frac{(N-1)^N}{n(1-\delta_n)} \int_0^\infty \left| \frac{\frac{1}{N-1} e^{-s/(N-1)}}{A_n+e^{-s/(N-1)}} \right|^N ds \\ &= \frac{(N-1)^N}{n(1-\delta_n)} \int_1^\infty \left| \frac{\frac{1}{N-1}}{A_n r + 1} \right|^N \frac{N-1}{r} dr \\ &= \frac{N-1}{n(1-\delta_n)} \int_1^\infty \frac{1}{(A_n r + 1)^N r} dr \quad . \end{aligned}$$

Finally, set  $\rho = \frac{1}{r}$  to obtain

$$\begin{aligned} & \frac{N-1}{n(1-\delta_n)} \int_1^\infty \frac{1}{(A_n r + 1)^N r} dr \\ &= \frac{N-1}{n(1-\delta_n)} \int_0^1 \frac{\rho^{N-1}}{(A_n + \rho)^N} d\rho \\ &= \frac{N-1}{n(1-\delta_n)} \left( \log \frac{A_n + 1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n + 1)^{N-k}} \right) \end{aligned}$$

By (15) this gives the condition

$$\begin{aligned} & \frac{N-1}{n(1-\delta_n)} \left( \log \frac{A_n + 1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n + 1)^{N-k}} \right) \\ &= (N-1)\delta_n + \sigma_N(\delta_n^2) = (N-1)\frac{2 \log n}{n} + \sigma_N\left(\frac{\log^2 n}{n^2}\right), \end{aligned}$$

that is

$$\log \frac{A_n + 1}{A_n} - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n + 1)^{N-k}} = \left( 2 \log n + \sigma_N\left(\frac{\log^2 n}{n}\right) \right) (1 - \delta_n)$$

and hence

$$\frac{A_n + 1}{A_n} \exp \left( - \sum_{k=1}^{N-1} \frac{1}{(N-k)(A_n + 1)^{N-k}} \right) = n^2 \left( 1 + \sigma_N\left(\frac{\log^2 n}{n}\right) \right), \quad (17)$$

and finally that

$$\frac{A_n + 1}{A_n n^2} = e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} + \begin{cases} O(1/n^2), & N = 2 \\ O(\log^2(n)/n), & N \geq 3 \end{cases}, \quad \text{for } n \text{ large} \quad (18)$$

This yields (16).

**3. The limit:** We now calculate the limit of  $\int_0^\infty e^{y_n^{N/(N-1)} - t} dt$ .

**Proposition 9** *Let  $\{y_n\}$  denote the sequence (14). Then*

$$\int_0^\infty e^{y_n^{N/(N-1)}(t) - t} \rightarrow 1 + e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}, \quad \text{as } n \rightarrow \infty$$

**Proof.** a) By the upper bound proved above we have (since  $y_n$  is a normalized concentrating sequence) that

$$\overline{\lim}_{n \rightarrow \infty} \int_0^\infty e^{y_n^{N/(N-1)}(t)-t} \leq 1 + e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}$$

We now prove the other inequality:

b) We first prove that there exists some  $c > 0$  such that

$$\int_0^n e^{y_n^{N/(N-1)}(t)-t} \geq 1 + \frac{\Gamma(1 + \frac{N}{N-1})}{n^{1/(N-1)}} - c \frac{\log n}{n^{N/(N-1)}}, \quad \text{as } n \rightarrow \infty, \quad (19)$$

where  $\Gamma$  denotes the standard gamma-function. Indeed

$$\begin{aligned} \int_0^n e^{y_n^{N/(N-1)}(t)-t} dt &= \int_0^n e^{(1-\delta_n) (t^N/n)^{1/(N-1)}-t} dt \\ &\geq \int_0^n \left( 1 + \frac{1-\delta_n}{n^{1/(N-1)}} t^{N/(N-1)} \right) e^{-t} dt \\ &\geq 1 - e^{-n} + \frac{1-\delta_n}{n^{1/(N-1)}} \left( \int_0^\infty t^{N/(N-1)} e^{-t} dt - \int_n^\infty e^{-t/2} dt \right) \\ &\geq 1 + \frac{\Gamma(1 + \frac{N}{N-1})}{n^{1/(N-1)}} - c \frac{\log n}{n^{N/(N-1)}}, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

by the definition of the gamma-function.

c) Next we prove

$$\int_n^\infty e^{y_n^{N/(N-1)}(t)-t} dt \geq e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N = 2 \\ O(\frac{\log n}{n}) & , \quad N \geq 3 \end{cases} \quad (20)$$

We perform the change of variables  $s = t - n$ , and set

$$z_n(s) = \frac{N-1}{n^{1/N}(1-\delta_n)^{1/N}} \log \frac{A_n + 1}{A_n + e^{-s/(N-1)}},$$

and

$$d_n = n(1 - \delta_n).$$

Then  $\int_n^\infty e^{y_n^{N/(N-1)}(t)-t} dt$  becomes

$$\begin{aligned}
& \int_0^\infty \exp \left( [d_n^{\frac{N-1}{N}} + z_n(s)]^{\frac{N}{N-1}} - s - n \right) ds & (21) \\
& \geq \int_0^\infty \exp \left( d_n + \frac{N}{N-1} z_n(s) d_n^{\frac{1}{N-1}} - s - n \right) ds \\
& = \int_0^\infty \exp \left( \frac{N}{N-1} z_n(s) n^{\frac{1}{N}} (1 - \delta_n)^{\frac{1}{N}} - n\delta_n - s \right) ds \\
& = \int_0^\infty \exp \left( N \log \frac{A_n + 1}{A_n + e^{-s/(N-1)}} - n\delta_n - s \right) ds \\
& = \frac{1}{n^2} \int_0^\infty \left( \frac{1 + A_n}{A_n + e^{-s/(N-1)}} \right)^N e^{-s} ds \\
& = \frac{(1 + A_n)^N}{n^2} \int_0^\infty \frac{1}{(1 + A_n e^{s/(N-1)})^N e^{-sN/(N-1)}} e^{-s} ds \\
& = \frac{(1 + A_n)^N}{n^2} \int_0^\infty \frac{e^{s/(N-1)}}{(1 + A_n e^{s/(N-1)})^N} ds \\
& = \frac{(1 + A_n)^N}{n^2} (N-1) \int_1^\infty \frac{1}{(1 + A_n r)^N} dr \\
& = \frac{(1 + A_n)^N}{n^2} \frac{1}{A_n (1 + A_n)^{N-1}} \\
& = \frac{1 + A_n}{n^2 A_n} = e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , \quad N = 2 \\ O(\frac{\log^2 n}{n}) & , \quad N \geq 3 \end{cases}
\end{aligned}$$

by relation (18). Hence the claim. ■

**4. General nonlinearities  $F$  :** Suppose now that  $F(t)$  is a general nonlinearity with critical growth, satisfying hypotheses F1-F3. Then we may write

$$F(t) = e^{\alpha_N t^{N/(N-1)}} - 1 + G(t)$$

with

$$\frac{G(t)}{e^{\alpha_N t^{N/(N-1)}}} \rightarrow 0 \quad , \quad \text{as } t \rightarrow \infty$$

Then, by Theorems 2 and 4 we have for any normalized concentrating sequence  $\{u_n\}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} F(u_n) dx &= \lim_{n \rightarrow \infty} \int_{\Omega} (e^{\alpha_N u_n^{N/(N-1)}} - 1) dx + \lim_{n \rightarrow \infty} \int_{\Omega} G(u_n) dx \\ &\leq e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} |\Omega| \end{aligned}$$

while for the sequence  $\{y_n\}$  given in (14) holds

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(y_n) = \lim_{n \rightarrow \infty} \int_{\Omega} (e^{\alpha_N y_n^{N/(N-1)}} - 1 + G(y_n)) dx = e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} |\Omega|$$

## 2.5 Proof of Theorem 5:

We show that under condition F4

$$C_{N,\lambda} = \sup_{\int_0^\infty |u'|^N = 1} \int_0^\infty F_\lambda\left(\frac{1}{\alpha_N^{(N-1)/N}} u\right) e^{-t} dt > e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}}$$

Indeed, by the estimates (19) and (21) we have for  $n$  sufficiently large

$$\begin{aligned} &\int_0^\infty e^{y_n^{N/(N-1)}(t-t)} \\ &= \int_0^n e^{y_n^{N/(N-1)}(t-t)} + \int_n^\infty e^{y_n^{N/(N-1)}(t-t)} \geq \\ &\geq 1 + \frac{\Gamma(1 + \frac{N}{N-1})}{n^{1/(N-1)}} - c \frac{\log n}{n^{N/(N-1)}} + e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} + \begin{cases} O(\frac{1}{n^2}) & , N = 2 \\ O(\frac{\log^2 n}{n}) & , N \geq 3 \end{cases} . \end{aligned}$$

Furthermore, we can estimate the term

$$\begin{aligned} \lambda \int_0^\infty |y_n|^{\frac{N}{N-1}} e^{-t} dt &\leq \frac{\lambda}{n^{1/(N-1)}} \int_0^n |t|^{N/(N-1)} e^{-t} dt + c \int_n^\infty n e^{-t} dt \\ &\leq \lambda \frac{\Gamma(1 + \frac{N}{N-1})}{n^{1/(N-1)}} + c \frac{\log n}{n^{N/(N-1)}} . \end{aligned}$$

Hence we obtain for  $\lambda < 1$  and a suitably large  $n$

$$\begin{aligned} C_{N,\lambda} &= \sup_{\int |u'|^N = 1} \int_0^\infty F_\lambda\left(\frac{1}{\alpha_N^{(N-1)/N}} u\right) e^{-t} dt \geq \int_0^\infty F_\lambda\left(\frac{1}{\alpha_N^{(N-1)/N}} y_n\right) e^{-t} dt \\ &\geq e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} + (1 - \lambda) \frac{\Gamma(1 + \frac{N}{N-1})}{n^{1/(N-1)}} - c \frac{\log n}{n^{N/(N-1)}} + \begin{cases} O(\frac{1}{n^2}) & , N = 2 \\ O(\frac{\log^2 n}{n}) & , N \geq 3 \end{cases} \\ &> e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}} . \end{aligned}$$



Then, since by Theorem 4 there cannot exist a normalized concentrating sequence which is *maximizing* for  $C_{N,\lambda}$ , we conclude by the concentration-compactness theorem of P.L. Lions that  $C_{N,\lambda}$  is attained.

**Open problem:** Show that  $\sup_{\|u\|=1} \int_{B_1} F(u) dx$  is *not attained* for  $F(t)$  of the form

$$F(t) = e^{\alpha_N t^{N/(N-1)}} - g(t)$$

with  $g$  subcritical, and

$$g(t) \geq t^{N/(N-1)} \quad .$$

## 2.6 Proof of Theorem 6:

We restrict attention to the radial case, i.e  $\Omega = B_1(0) \subset \mathbb{R}^2$ . Consider the functional  $I(u) = \int_{B_1} [\frac{1}{2} |\nabla u|^2 - F(u)] dx$ , where  $F(s)$  is as in Theorem 6. Then we know by [5] that this functional satisfies the Palais-Smale condition  $(PS)_c$  for  $c < \frac{1}{2}$ . By the remarks in section 2.1, we may assume that  $u_n$  is radially symmetric, and we can rewrite the functional in radial coordinates:

$$\int_0^1 [\frac{1}{2} |u_r|^2 - F(u)] 2\pi r dr$$

Cancelling the factor  $2\pi$  we see that  $\int_0^1 [\frac{1}{2} |u_r|^2 - F(u)] r dr$  satisfies  $(PS)_c$  for  $c < \frac{1}{4\pi}$ . Next, we perform a change of variables to transform the interval  $(0, 1)$  to the interval  $(0, +\infty)$ : Let

$$r = e^{-t/2}, \quad dr = -\frac{1}{2} e^{-t/2} dt, \quad u_t = u_r \frac{dr}{dt} = -\frac{1}{2} u_r e^{-t/2}$$

and hence we obtain

$$\int_0^{+\infty} [\frac{1}{2} |2u_t e^{t/2}|^2 - F(u)] \frac{1}{2} e^{-t} dt$$

Multiplying by 2 we see that the functional

$$\int_0^{+\infty} [2|u_t|^2 - F(u)e^{-t}] dt$$

satisfies  $(PS)_c$  for  $c < \frac{1}{2\pi}$ . Finally, substitute  $y = 2\sqrt{\pi}u$  and multiply by  $\pi$  to obtain

$$J(y) = \int_0^{+\infty} [\frac{1}{2} |y_t|^2 - \pi F(\frac{1}{2\sqrt{\pi}}y) e^{-t}] dt \quad (22)$$

which satisfies again  $(PS)_c$  for  $c < \frac{1}{2}$ .

### 2.6.1 Estimates for (P-S)

We now show that the functional  $J(u)$  given by (22) (which satisfies  $(PS)_c$  for  $c < \frac{1}{2}$ ) has a critical level  $c$  with  $c < \frac{1}{2}$  provided  $h$  satisfies condition (9).

**Theorem 10** *Suppose that  $f(s) = h(s)e^{4\pi s^2}$  satisfies H1-H3, and assume that  $\lim_{s \rightarrow \infty} h(s)s > \frac{1}{e\pi}$ . Then  $J(u)$  has a critical level below  $\frac{1}{2}$ .*

**Proof:** As in [5], the critical level is given by the mountain pass theorem. To prove that the mountain pass level is below  $\frac{1}{2}$  it suffices to show that there is a  $w \in H_0^1$ ,  $\|w\| = 1$ , such that  $\max_{t \geq 0} J(tw) < \frac{1}{2}$ . In [5] we used the Moser sequence to show this. Here we use the sequence  $\{y_n\}$  given in (14). So we assume, by way of contradiction, that for all  $n \in \mathbb{N}$

$$\max_{s \geq 0} J(sy_n) = \int_0^\infty \left[ \frac{1}{2} s_n^2 |y_n'|^2 - \pi F\left(\frac{1}{2\sqrt{\pi}} s_n y_n\right) e^{-t} \right] dt \geq \frac{1}{2}.$$

This implies  $s_n^2 \geq 1$ . Furthermore, since  $\frac{d}{ds} J(sy_n)|_{s=s_n} = 0$  we have for  $n$  sufficiently large, using condition (9)

$$\begin{aligned} s_n^2 &= \pi \int_0^\infty f\left(\frac{s_n}{2\sqrt{\pi}} y_n\right) \frac{s_n}{2\sqrt{\pi}} y_n e^{-t} \\ &= \pi \int_0^\infty h\left(\frac{s_n}{2\sqrt{\pi}} y_n\right) \frac{s_n}{2\sqrt{\pi}} y_n \cdot e^{s_n^2 y_n^2 - t} \\ &\geq (\beta - \epsilon) \pi \int_n^\infty e^{s_n^2 (n-2 \log n) - t}. \end{aligned}$$

We show that  $s_n^2 \rightarrow 1$ ; assume that this is not so, i.e. suppose that there exists a subsequence of  $s_n$  with  $s_n^2 \geq 1 + \delta$ , for some  $\delta > 0$ . Then we have

$$s_n^2 \geq (\beta - \epsilon) \pi \int_n^\infty e^{(1+\delta)(n-2 \log n) - t} = (\beta - \epsilon) \pi e^{\delta n - (1+\delta)2 \log n}$$

This would imply that  $s_n^2 \rightarrow +\infty$ , which then yields a contradiction. Hence we must have

$$s_n^2 \rightarrow 1$$

We now estimate more precisely; fix  $A > 0$  and set  $[0, b_n) = \{t \in [0, \infty) : s_n y_n(t) < A\}$ . Since  $y_n(t) = \frac{t}{\sqrt{n}} \sqrt{1 - \delta_n} \rightarrow 0$ , for every fixed  $t \geq 0$ , we conclude that  $b_n \rightarrow \infty$ . Then we have

$$\begin{aligned} s_n^2 &\geq (\beta - \epsilon) \pi \int_0^\infty e^{s_n^2 y_n^2 - t} + \pi \int_0^{b_n} f\left(\frac{s_n}{2\sqrt{\pi}} y_n\right) \frac{s_n}{2\sqrt{\pi}} y_n \\ &\quad - (\beta - \epsilon) \pi \int_0^{b_n} e^{s_n^2 y_n^2 - t} \end{aligned} \quad (23)$$

The last integral in (23) goes to 1: indeed, we have

$$\int_0^{b_n} e^{-t} \leq \int_0^{b_n} e^{s_n^2 u_n^2 - t} = \int_0^{b_\epsilon} e^{s_n^2 u_n^2 - t} + \int_{b_\epsilon}^{b_n} e^{s_n^2 u_n^2 - t},$$

where we choose for given  $\epsilon > 0$  the number  $b_\epsilon > 0$  such that

$$\int_{b_\epsilon}^{b_n} e^{s_n^2 y_n^2 - t} \leq e^{s_n^2 A^2} \int_{b_\epsilon}^{b_n} e^{-t} \leq \epsilon/2, \quad \forall n$$

Next, using that  $y_n(t) \leq \tau_n \rightarrow 0$  uniformly on  $[0, b_\epsilon]$ , choose  $N_\epsilon$  sufficiently large such that

$$\int_0^{b_\epsilon} e^{s_n^2 y_n^2 - t} \leq e^{s_n^2 \tau_n^2} \int_0^{b_\epsilon} e^{-t} \leq 1 + \epsilon/2, \quad \text{for } n \geq N_\epsilon$$

The second integral in (23) is positive, and in fact goes to zero (as can be seen using a similar argument). Hence we have in the limit, using theorem 4 for  $N = 2$

$$\begin{aligned} 1 = \lim_{n \rightarrow \infty} s_n^2 &= (\beta - \epsilon)\pi \left[ \lim_{n \rightarrow \infty} \int_0^\infty e^{s_n^2 y_n^2 - t} dt - 1 \right] \\ &\geq (\beta - \epsilon)\pi \left[ \lim_{n \rightarrow \infty} \int_0^\infty e^{y_n^2 - t} dt - 1 \right] \\ &= (\beta - \epsilon) \pi e \end{aligned}$$

Thus, for  $\beta > \frac{1}{e\pi}$  given, we obtain a contradiction, choosing  $\epsilon > 0$  sufficiently small.

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