$(1,2)$-Symplectic metrics on flag manifolds and tournaments

N. Cohen*
Department of Applied Mathematics

Caio J. C. Negreiros
Department of Mathematics

Luiz A. B. San Martin†
Department of Mathematics
Imecc
Universidade Estadual de Campinas
Cx. Postal 6065
13.083-970, Campinas – SP, Brasil

December 14, 2000

Abstract

It has been recently shown by Mo and Negreiros [11] that a necessary condition for an invariant almost-complex structure on the complex full flag manifold $\mathbb{F}(n)$ to admit a $(1,2)$-symplectic invariant metric is that its associated tournament is cone-free.

In this paper we find a canonical stair-shaped form for such tournaments and apply it to show that the condition is also sufficient. In doing this we describe all the associated $(1,2)$-symplectic metrics, and get, in particular, a different and self-contained proof of a theorem of Gray and Wolf [17] asserting that the Cartan-Killing metric on $\mathbb{F}(n)$ is not $(1,2)$-symplectic for $n>3$.

*Supported by CNPq grant no. 300019/96-3
†Supported by CNPq grant n$^{\underline{o}}$ 301060/94-0

AMS 2000 subject classification: 53C55, 05C20, 58E20
Key words: Flag manifold, tournament, Harmonic maps and Hermitian geometry.

1 Introduction

Denote by $\mathbb{F}(n)$ the usual manifold of full flags of subspaces of $\mathbb{C}^n$ and endow it with an almost-complex structure J and a Riemannian metric ds^2. Let M^2 be an arbitrary closed Riemann surface. A theorem due to Lichnerowicz [9], proved independently by Gray [6], states: "If $\phi : M^2 \to (\mathbb{F}(n), J, ds^2)$ is J-holomorphic and ds^2 is $(1,2)$-symplectic with respect to J, then ϕ is harmonic". The importance of this theorem rests in the fact that it furnishes solutions of the second order Euler-Lagrange partial differential equations, satisfied by the harmonic maps from the solutions of first order partial differential equations, the Cauchy-Riemann equations. This theorem drives the attention to the problem of understanding the $(1,2)$-symplectic metrics on the flag manifolds. Apart from the relation to harmonic maps the $(1,2)$-symplectic metrics appear in twistor theory, as shown, for example, in Eells and Salamon [5].

Our approach to this problem is based on a method derived by Burstall and Salamon [4], relating harmonic maps on flag manifolds to tournament theory. It is well known that an invariant almost-complex structure J on the flag manifold $\mathbb{F}(n)$ is defined by means of a skew-hermitian sign matrix. This matrix can be considered as the incidence matrix of an n-player tournament T_J, providing a one-to-one correspondence between the set of invariant almost-complex structures on $\mathbb{F}(n)$ and the tournaments with n players. Relying on this natural association the method consists in studying properties of J through the combinatorics of T_J.

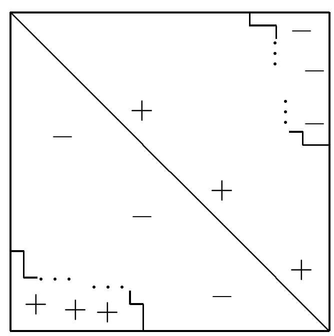
Recently Mo and Negreiros [10], [11] singled out the class of the so-called cone-free tournaments in relation to the $(1,2)$-symplectic metric problem. Namely, it was proved in [11] that a necessary condition for J to admit an invariant $(1,2)$-symplectic metric ds^2 is that the tournament T_J is cone-free, a property which involves the 4-subtournaments of T_J (see Definition 3.1 below for a precise statement). In this paper we prove that this condition is also sufficient, thus arriving at the following characterization.

Theorem 1.1 $(\mathbb{F}(n), J)$ *admits an invariant* $(1,2)$*-symplectic metric* Λ *if and only if the associated tournament* T_J *is cone-free.*

The sufficiency of the condition was studied by Paredes [14], [15], where an affirmative answer was obtained for certain classes of tournaments, including all tournaments with $5 \leq n \leq 7$ players. For $n = 3$ and 4 this result was proved in [11]. In Section 4 we offer a proof for arbitrary n (see Theorem 4.4).

In addition, given J consistent with Theorem 1.1), we exhibit an explicit n-dimensional parametrization for all the possible invariant $(1,2)$-symplectic metrics for $(\mathbb{F}(n), J)$.

Our approach is based on a concrete characterization of the structure of cone-free tournaments. Specifically, in Theorem 3.5 it is shown that a tournament T is cone-free if and only if its incidence matrix is permutation-similar to a stair-shaped incidence matrix of the following type



(1)

The problem discussed here has a natural extension to the framework of generalized flag manifolds associated to arbitrary complex semi-simple Lie groups. Similar results can be proved in this more general set up, and will appear elsewhere.

2 Preliminaries

2.1 Tournaments

We define an n-player tournament as a complete directed graph $T = (N, E)$ (no loops or multiple edges) where N is an ordered set and $|N| = n$. For concreteness we shall always assume that $N = \{1, \ldots, n\}$. Two tournaments are isomorphic if one is obtained from the other via a rearrangement of N. See Moon [12] for further details.

With each tournament T we assign its incidence matrix $\varepsilon = \varepsilon_T$, which is a real skew-symmetric matrix with all off-diagonal entries ± 1 (see [12]).

Tournament isomorphism amounts to permutation similarity between the associated incidence matrices.

If $(a, b) \in E$ we say that a wins against b and set $\varepsilon_{ab} = 1$, $\varepsilon_{ba} = -1$. By our conventions, T would then contain an arrow pointing from vertex a into vertex b.

We define the valency of a vertex a of T as the number of its wins. We also define, in a strict sense, a winner in T as a (unique) vertex with valency $n-1$, and a loser as a (unique) vertex with valency 0. Not every tournament has a winner or a loser; however, when it exists, it is not part of any cycle.

The descending sequence of valencies of T is called the *score vector* associated with T. Every non-increasing sequence of n non-negative integers with total sum $\binom{n}{2}$ is a score vector for some n-player tournament T. However, T need not be uniquely determined by its score vector.

Since T is complete, $|E| = n(n-1)/2$. Moreover, E splits as a disjoint union

$$E = E_1 \cup E_2,\ E_1 = \{(a,b) \in E :\ a < b\},\ E_2 = \{(a,b) \in E :\ a > b\}. \qquad (2)$$

2.2 Flag manifolds

In this section we discuss the correspondence between invariant almost complex structures on the full flag manifold and tournaments or, equivalently, their incidence matrices which are real skew-symmetric matrices with off-diagonal entries in $\{\pm 1\}$.

Consider the complex full flag manifold

$$\mathbb{F}(n) = \{(V_1, \ldots, V_n) : V_j \subset V_{j+1}, \dim V_j = j\}.$$

The natural action of the unitary group $\mathrm{U}(n)$ on $\mathbb{F}(n)$ is transitive, turning the flag manifold into the homogeneous space $\mathrm{U}(n)/M$ where M is any maximal torus of $\mathrm{U}(n)$, i.e., $M \cong \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$.

Let $\mathfrak{u}(n)$ be the Lie algebra of skew-hermitian matrices. It decomposes as

$$\mathfrak{u}(n) = \mathfrak{p} \oplus \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)$$

where $\mathfrak{p} \subset \mathfrak{u}(n)$ is the subspace of zero-diagonal matrices. Let E_{jk} be the canonical basis matrix $E_{jk} = (a_{rs})$ with $a_{rs} = 1$ if $(r,s) = (j,k)$ and zero otherwise. Put

$$\mathfrak{p}_{jk} = (\mathbb{C}E_{jk} + \mathbb{C}\mathbb{E}_{kj}) \cap \mathfrak{u}(n).$$

Of course, $\mathfrak{p} = \bigoplus_{j \neq k} \mathfrak{p}_{jk}$.

In order to give an invariant almost complex structure on $\mathbb{F}(n)$ it is enough to present $J : \mathfrak{p} \to \mathfrak{p}$, $J^2 = -1$, which commutes with the adjoint representation of the torus M on $\mathfrak{p}$. This condition implies that $J(\mathfrak{p}_{jk}) = \mathfrak{p}_{jk}$ for all $j \neq k$, which in turn guarantees that $J(A) = A'$ with $A'_{jk} = \varepsilon_{jk}\sqrt{-1}A_{jk}$ such that $\varepsilon_{jk} = \pm 1$ and satisfies $\varepsilon_{kj} = -\varepsilon_{jk}$. Thus an invariant almost complex structure is completely determined by a skew-symmetric matrix (ε_{jk}), with off-diagonal entries in $\{\pm 1\}$.

For the computations we work in the complexification V of $\mathfrak{p}$. It is easy to check that V is the subspace of complex matrices with zero-diagonal entries. It decomposes as $V = \bigoplus_{j \neq k} V_{jk}$, where $V_{jk} = \mathrm{span}_{\mathbb{C}}\{E_{jk}\}$.

An almost complex structure J on $\mathfrak{p}$ extends to a $\mathbb{C}$-linear operator on V, also denoted by J. Its eigenvalues are $\pm\sqrt{-1}$. If J is given by the incidence matrix ε_{jk} then the corresponding eigenspaces are

$$V^{10} = \bigoplus \{V_{jk} : \varepsilon_{jk} = 1\} \qquad \text{for} \qquad \sqrt{-1}$$

and

$$V^{01} = \bigoplus \{V_{jk} : \varepsilon_{jk} = -1\} \qquad \text{for} \qquad -\sqrt{-1}.$$

Let ds^2 be an $\mathrm{U}(n)$-invariant Riemannian metric on $\mathbb{F}(n)$. Like the invariant almost complex structures, ds^2 is completely determined by its value at the origin, that is, by an inner product $(\cdot,\cdot)$ in $\mathfrak{p}$, which is invariant under the adjoint action of M. To describe these inner products, start with the Cartan-Killing form on $\mathfrak{p}$:

$$\langle X, Y \rangle = -\mathrm{tr}\,(XY)\,,$$

which is an M-invariant inner product. Any other inner product on $\mathfrak{p}$ is of the form $(X,Y)_\Lambda = \langle \Lambda X, Y \rangle$ with $\Lambda : \mathfrak{p} \to \mathfrak{p}$ positive-definite with respect to $\langle \cdot,\cdot \rangle$. Furthermore, $(\cdot,\cdot)_\Lambda$ is M-invariant if and only if the elements of the standard basis $\sqrt{-1}\,(E_{jk} + E_{kj})$, $E_{jk} - E_{kj}$ are eigenvectors of Λ. Thus $\Lambda(E_{jk}) = \lambda_{jk}E_{jk}$ with $\lambda_{jk} > 0$ and $\lambda_{kj} = \lambda_{jk}$, and

$$(X,Y)_\Lambda = -\mathrm{tr}\,(\Lambda(X)\,Y)\,.$$

We denote by ds^2_Λ the invariant metric given by Λ.

Remark: Consider the $n \times n$ symmetric matrix, say A, whose entries are λ_{jk}, $j \neq k$, the eigenvalues of Λ, and $\lambda_{jj} = 0$. Of course, the inner product depends only on the entries A. Furthermore, $\Lambda(X) = A \circ X$ where $\circ$ stands for the Hadamard product of two $n \times n$ matrices. Therefore, $(X, Y)_\Lambda = -\mathrm{tr}\,((A \circ X)\,Y)$. In the sequel we also denote by Λ the matrix A.

The inner product $(\cdot,\cdot)_\Lambda$ admits a natural extension to a symmetric bilinear form on the complexification V of $\mathfrak{p}$. We use the same notation $(\cdot,\cdot)_\Lambda$ for this bilinear form as well as for the corresponding complexified map Λ. Here the two-dimensional real eigenspace $\mathfrak{p}_{jk}$ of Λ, whose basis is $\sqrt{-1}\,(E_{jk} + E_{kj})$, $E_{jk} - E_{kj}$, extends to complex spaces having basis E_{jk} and E_{kj}, respectively.

A special class of invariant inner products is given by those Λ satisfying

$$\lambda_{ij} + \lambda_{jk} = \lambda_{ik} \tag{3}$$

for all j between i and k, or equivalently satisfying the relation

$$\lambda_{ij} = \sum_{k=i}^{j-1} \lambda_{k,k+1}.$$

In this case Λ is defined by an adjoint operation as follows: Consider the real diagonal matrix

$$H_\Lambda = \mathrm{diag}\{\mu_1, \ldots, \mu_n\} \tag{4}$$

whose eigenvalues are defined (up to an additive constant) by the conditions $\mu_i - \mu_{i+1} = \lambda_{i,i+1}$, $i = 1, \ldots, n-1$. Then the action of Λ on the upper triangular matrices is given by

$$\Lambda(E_{jk}) = \mathrm{ad}\,(H_\Lambda)(E_{jk}) \qquad j < k$$

if Λ satisfy the conditions (3). Analogously, $\Lambda = -\mathrm{ad}\,(H_\Lambda)$ on the lower triangular matrices. In view of these relations a metric satisfying the equalities (3) is called of *adjoint type*.

Now, let J and ds^2_Λ be an invariant almost complex structure and a metric, respectively. Then it is easily checked that ds^2_Λ is Hermitian, that is,

$$ds^2_\Lambda(JX, JY) = ds^2_\Lambda(X, Y).$$

Let $\Omega = \Omega_{J,\Lambda}$ stand for the corresponding Kähler form

$$\Omega(X, Y) = ds^2_\Lambda(X, JY) = -\mathrm{tr}\,(\Lambda(X)\,J(Y)).$$

This form is also invariant under $\mathrm{U}(n)$. Relying on this invariance its exterior differential is easily calculated from the standard formula: If $X, Y, Z \in \mathfrak{p}$ are regarded as vector fields in $\mathbb{F}(n)$ then $d\Omega$ at the origin is given by

$$-\frac{1}{3} d\Omega(X, Y, Z) = \Omega([X, Y], Z) + \Omega([X, Z], Y) - \Omega([Y, Z], X)$$

(see [8]).

For all $i = 0, 1, 2, 3$ define the operator $d\Omega^{i,3-i}$ as follows:

1. $d\Omega^{i,3-i}(X, Y, Z) = d\Omega(X, Y, Z)$ as long as i of the three matrices X, Y, Z are in V^{10} and the remaining are in V^{01};

2. $d\Omega^{i,3-i}(X, Y, Z) = 0$ as long as j of the three matrices X, Y, Z are in V^{10} and the remaining are in V^{01}, with $j \neq i$.

Since V^{10} and V^{01} are orthogonal complements in V, it is readily seen that the four operators $d\Omega^{i,3-i}$ are well defined and moreover

$$d\Omega = d\Omega^{30} + d\Omega^{21} + d\Omega^{12} + d\Omega^{03}.$$

Moreover, by replacing X, Y, Z by X^T, Y^T, Z^T it can be seen that $d\Omega^{30} = -[d\Omega^{03}]^*$ and $d\Omega^{12} = -[d\Omega^{21}]^*$.

According to the annihilation of these forms the triple $(\mathbb{F}(n), J, ds^2)$ is distinguished as follows:

1. $(\mathbb{F}(n), J, ds^2)$ is *almost Kähler* if $d\Omega = 0$. It is *Kähler* if furthermore J is integrable.

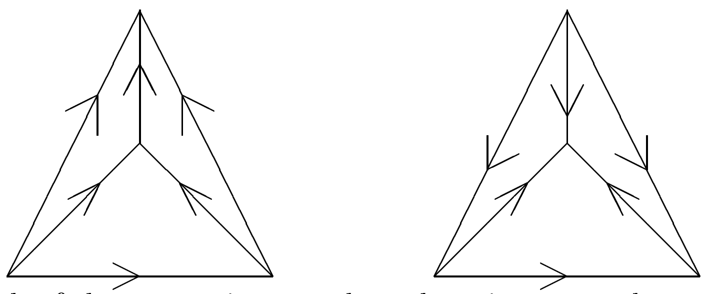
2. $(\mathbb{F}(n), J, ds^2)$ is (r, s)-*symplectic* if $d\Omega^{rs} = 0$.

The following theorem by Burstall and Salamon [2] ensures that a sufficient condition for J to be integrable is that $d\Omega = 0$.

Theorem 2.1 *$(\mathbb{F}(n), J, ds^2)$ is an almost Kähler manifold if and only if it is a Kähler manifold.*

3 Cone-free tournaments

Up to isomorphism, there are four distinct 4-player tournaments. The two of them which contain a single (directed) 3-cycle are called *coned* 3-*cycles.*



Each of them contains a cycle and a winner or a loser.

Definition 3.1 *An n-player tournament T is called cone-free if its restriction to any four vertices is never a coned 3-cycle.*

In this section we show that the incidence matrix of a cone-free tournament is permutation similar to a stair-shaped matrix of the type shown in (1). We start by providing a rigorous definition of stair-shaped tournaments.

We call a tournament T *transitive* if it contains no 3-cycle. In this case, T is obviously cone-free, and is isomorphic to the *canonical* transitive tournament T' for which the subset E_1 in (2) is E. The incidence matrix of T' has a triangular form.

In what follows we shall denote by T/U the restriction of T to U, which is an m-player tournament if $|U| = m$.

Definition 3.2 *A tournament T' is called* stair-shaped *if there are integers s, t (with $1 \leq s \leq t \leq n$) such that the axioms below are satisfied. For all $U \subset N$, with $|U| = m < n$.*

A) $T'/\{1, \ldots, t\}$ *is a maximal canonical-transitive subtournament of T';*

B) $T'/\{s, \ldots, n\}$ *is a maximal canonical-transitive subtournament of T';*

C) *If $(z, x) \in E_2$ then $x < s$ and $t < z$;*

D) *if $x' \leq x$ and $z \leq z'$ then $(z, x) \in E_2$ implies $(z', x') \in E_2$.*

A special case of a stair-shaped tournament is when $(s, t) = (1, n)$. In such a case T' is a canonical transitive tournament.

Lemma 3.3 *Stair-shaped tournaments are cone-free.*

Proof: Let T' be stair-shaped, and let s, t be the associated numbers. If $s = 1$ and $t = n$ then T' is transitive, hence contains no coned 3-cycles, and there is nothing to prove. So, we shall assume that $1 < s \leq t < n$ and subsequently, due to the maximality clause in axioms A and B, E_2 in (2) is not empty. The 3-cycles in T' are exactly the triples $x, y, z \in N$ with

$$x < a \leq b < z, \qquad x < y < z, \qquad (z, x) \in E_2. \tag{5}$$

Let $U = \{x, y, z, w\} \subset N$ with $|U| = 4$. Assume that x, y, z form a 3-cycle in T'. Due to axiom C, since $x < y < z$ then the cycle arrows are $x \to y \to z \to x$, and moreover $x < s \leq t < z$.

We want to show that T'/U is not a coned 3-cycle. To this end, we first show that w is not a winner in U. Indeed, if $z < w$ then by axiom B w loses to z; if $y < w < z$ then by axiom D w loses to y; if $x < w < y$ then by axiom D w loses to x; and if $w < x$ then by axiom D w loses to z. By a similar argument, w is not a loser in U either. Thus, T'/U is not coned, implying that T' is cone-free. □

Next we show our central result on the representation of cone-free tournaments. It requires the following.

Definition 3.4 *A subtournament T/U of a tournament $T = (N, E)$ is said to be 1-transitive if $T/(U \cup \{p\})$ is transitive for all $p \in T$.*

In particular, T/U itself must be transitive.

Theorem 3.5 *A tournament T is cone-free if and only if it is isomorphic to a stair-shaped tournament T'.*

Proof: The "if" part is covered by Lemma 3.3. We now prove the "only if" part. Let T be a cone-free n-tournament . If T is transitive, it is isomorphic to the canonical-transitive tournament, which is stair-shaped. Thus, we shall assume that T is not transitive, i.e. T contains 3-cycles.

Let T/U be a maximal 1-transitive subtournament of T. Since every 2-player tournament is transitive, $m := |U| > 0$. Since T is not transitive, $m < n$. By reordering N we obtain a new tournament T'', isomorphic to T, such that $T''/(U \cup \{p\})$ is the canonical-transitive $m+1$-player tournament for all $p \in N$.

In particular, T''/U is canonical-transitive. Let $s, t \in N$, $s \leq t$, be its winner and loser. Define the subsets $U_1 = \{1, 2, \ldots, t\}$ and $U_2 = \{s, s+1, \ldots, n\}$. It is easy to see that s is a winner in U_2 and t is a loser in U_1.

We claim that the subtournament T''/U_1 is transitive. Indeed, assume that there exists a cycle x, y, z with $1 \leq x < y < z \leq t$. Necessarily we have $z < t$ since t is a loser in U_1 and cannot form cycles there. But then $T''/\{x, y, z, t\}$ forms a coned 3-cycle, in contradiction to the assumption that T is cone-free.

So, T''/U_1 is transitive, and by an analogous argument, T''/U_2 is transitive. By a suitable reordering of N, T'' can be made isomorphic to a new tournament T', for which T'/U_1 and T'/U_2 are canonical-transitive. Note that this reordering leaves unchanged the set U.

We shall show that T', t, s as constructed above satisfy axioms A-D, namely T' is stair-shaped.

Let $U' := U_1 \cap U_2$. The subtournament T'/U' is transitive. In fact, it is 1-transitive, since for all $p \in N$ $U' \cup \{p\}$ is a subset of U_1 or U_2. Moreover, since by construction $U \subset U_1 \cap U_2$, and T/U is a maximal 1-transitive set, we conclude that

$$U = U' = \{s, s+1, \ldots, t\}.$$

In fact, T'/U_1 is a *maximal* transitive subtournament of T'. Indeed, assume that $U_1 \subset U_1'$ and T'/U_1' is transitive. It is easy to see that $T'/(U_1' \cap U_2)$ is 1-transitive. Since $U \subset U_1' \cap U_2$, and T'/U is a maximal 1-transitive subset, we conclude that $U_1' = U_1$, proving the maximality of T'/U_1.

By a similar argument, T'/U_2 is a maximal transitive subtournament of T'. Thus, T' satisfies axioms A and B.

In order to verify axiom C, we observe that without loss of generality every edge of E_2 belongs to a 3-cycle. Indeed, all the edges of E_2 without cycle can be inductively inverted without changing the sets U, U_1, U_2 and reducing E_2.

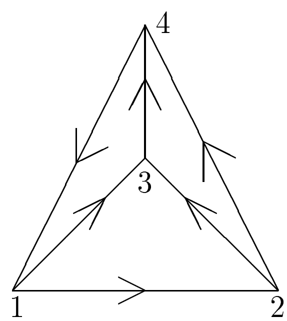
So, let $\{x, y, z\}$ be a 3-cycle in T, where $x < y < z$. Since U_2 is transitive, we have $x < s$. Since U_1 is transitive, we have $t < z$. Independently of the location of y, one of the two statements $x, y \in U_1$ or $y, z \in U_2$ must hold. Since T'/U_1 and T'/U_2 are canonical-transitive, we conclude that the arrows in the 3-cycle are directed as $x \to y \to z \to x$. Note that only the rightmost arrow represents an edge in E_2. Thus, T' satisfies axiom C.

Finally, we verify axiom D. If $(z,x) \in E_2$ and $z' < z$, consider any $y \in U$. Since x, y, z, z' cannot be a coned 3-cycle, z' cannot lose to all the others. Since z' loses to y and z, we must have $z' \to x$ as desired. Similarly, if $x' > x$ an analogous argument shows that $z \to x'$.

Thus, T' is a stair-shaped tournament isomorphic to T. $\square$

A natural question related to the above theorem is whether there exists uniqueness of the stair-shaped structure for a cone-free tournament. An affirmative answer of this question would lead to canonical forms for such tournaments.

In general, a cone-free tournament may have several maximal 1-transitive subtournaments of different sizes, leading to different, possibly pairwise non-similar, stair-shaped incidence matrices. For instance, the 4-tournament



has the following incidence matrices

$$\begin{pmatrix} 0 & + & - & - \\ - & 0 & + & + \\ + & - & 0 & + \\ + & - & - & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & + & + & - \\ - & 0 & + & + \\ - & - & 0 & + \\ + & - & - & 0 \end{pmatrix}$$

with respect to the orders $(1,2,3,4)$ and $(2,3,4,1)$, respectively. The maximal 1-transitive subtournaments are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ as well as $\{3 \to 4\}$. Both incidence matrices may show up in the construction made in the proof above. The first one appears if one takes $\{2\}$ as maximal 1-transitive tournament, whereas the second matrix comes out from $\{3 \to 4\}$.

We do not know whether two maximal 1-transitive tournaments *of the same size* necessarily lead to equal stair-shaped incidence matrices for a given tournament. The lack of uniqueness in the stair-shaped representation of a cone-free tournament T may be related to the co-existence of several non-isomorphic 1-transitive subtournaments for T.

4 Proof of Theorem 1.1

Recall that a metric is called $(i, 3-i)$-symplectic if $d\Omega^{i,3-i} = 0$. In this case, we also have $d\Omega^{3-i,i} = 0$. Theorem 1.1 in the introduction is formulated in terms of $(1,2)$-symplectic metrics. A special case of this theorem concerns the standard complex structure in $\mathbb{F}(n)$ and is already known (see Borel [2] and [11]). Denote by J_c the corresponding almost complex structure. Then in this particular case Theorem 1.1 reads:

Proposition 4.1 *Let J_c standard complex structure on $\mathbb{F}(n)$. Then J_c corresponds to the canonical transitive tournament and Λ is $(1,2)$-symplectic if and only if it is of adjoint type (see (3)).*

We now attack Theorem 1.1 in its full generality. The proof that the condition is sufficient is one of the main results of this paper. Before embarking on it we outline the necessity proof of [11]: Let $\omega = \left(\omega_{i\bar{j}}\right)$ be the matrix formed by the Maurer-Cartan form of $\mathrm{U}(n)$. The space of $(1,0)$-cotangent vectors at the origin identifies to $\mathfrak{p}_{(1,0)} = \mathrm{span}\{\omega_{i\bar{j}} : i \to j\}$. The key point in the proof is to compute $d\Omega$ using the moving frame method of Cartan. We find a permutation τ such that

$$d\Omega = \sum_{i<j<k} C_{\tau(i)\tau(j)\tau(k)} \psi_{\tau(i)\tau(j)\tau(k)}$$

where $C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}$, $\psi_{ijk} = \mathrm{Im}\left(\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}\right)$ and finally $\mu_{ij} = \varepsilon_{ij}\lambda_{ij}$. From this expression for $d\Omega$ it is not hard to prove that the tournament must be cone-free (see [11], Section 4).

Now, we proceed to the proof of the sufficiency of Theorem 1.1, namely that there are invariant $(1,2)$-symplectic metrics if the tournament is cone-free. First we calculate the conditions on J and Λ to be $(1,2)$-symplectic. Using the formulas

$$[E_{ij}, E_{rs}] = \delta_{jr}E_{is} - \delta_{si}E_{rj} \qquad \mathrm{tr}\left(E_{ij}E_{rs}\right) = \delta_{is}\delta_{jr}$$

a straighforward computation shows that $d\Omega\left(E_{ii'}, E_{jj'}, E_{kk'}\right) = -3\sqrt{-1}\alpha\beta$ where

$$\alpha = \delta_{ij'}\delta_{jk'}\delta_{ki'} - \delta_{ik'}\delta_{kj'}\delta_{ji'}$$

and

$$\beta = \varepsilon_{ii'}\lambda_{ii'} + \varepsilon_{jj'}\lambda_{jj'} + \varepsilon_{kk'}\lambda_{kk'}.$$

The expression for α shows that it is nonzero only when $i' = k$, $j' = i$, $k' = j$ or $i' = j$, $j' = k$, $k' = i$. Hence $d\Omega$ is completely determined by its values in the triples

$$(E_{ij}, E_{jk}, E_{ki}) \quad \text{and} \quad (E_{ik}, E_{kj}, E_{ji})\,, \tag{6}$$

given by the triples $\{i, j, k\}$. Note that these sets are transposed to each other so that if one of them contains a vector in V^{10} then the other contains a vector in V^{01}. Let us say that the triple $\{i, j, k\}$ is of type $\{3, 0\}$ if one of the corresponding sets (6) is contained in V^{10}. The triple is of type $\{1, 2\}$ otherwise. The proof of the following lemma is immediate from the definitions.

Lemma 4.2 *The triple $\{i, j, k\}$ is of type $\{3, 0\}$ if and only if the restriction to $\{i, j, k\}$ of the tournament corresponding to J is a cycle.*

Equivalently, $\{i, j, k\}$ is of type $\{1, 2\}$ if and only if the tournament is transitive in $\{i, j, k\}$.

Therefore, in order to check that an invariant metric Λ is $(1, 2)$-symplectic, it is enough to compute $d\Omega$ in one set of vectors (6), only for the transitive subtournaments $\{i, j, k\}$.

Proposition 4.3 *The invariant metric Λ is $(1, 2)$-symplectic if and only if for all transitive subtournaments $T_3 = \{i, j, k\}$ the following holds*

$$\lambda_{ik} = \lambda_{ij} + \lambda_{jk} \tag{7}$$

where i is the winner of T_3 while k is the loser.

Proof: For fixing ideas let us write $T_3 = \{a, b, c\}$ with $a < b < c$. From the computations performed we have $d\Omega\,(E_{ab}, E_{bc}, E_{ca}) = \sqrt{-1}\alpha\beta$ with

$$\beta = \varepsilon_{ab}\lambda_{ab} + \varepsilon_{bc}\lambda_{bc} + \varepsilon_{ca}\lambda_{ca}. \tag{8}$$

In order to have the values of ε_{rs}, we must run through the six transitive tournaments having vertices (a, b, c). Below we list the signs of $(\varepsilon_{ab}, \varepsilon_{bc}, \varepsilon_{ca})$ with the winners (w) and losers (l) for each transitive tournament:

1. $(+, +, -)$, $w = a$, $l = c$.
2. $(+, -, -)$, $w = a$, $l = b$.

3. $(+,-,+)$, $w=c$, $l=b$.
4. $(-,+,-)$, $w=b$, $l=c$.
5. $(-,+,+)$, $w=b$, $l=a$.
6. $(-,-,+)$, $w=c$, $l=a$.

By a direct inspection at these tournaments we see that in the right hand side of (8), λ_{wl} appears with a sign different from the other terms. Therefore, $\beta = 0$ if and only if $\lambda_{ik} = \lambda_{ij} + \lambda_{jk}$ with $w = i$ and $l = k$. Now the statement follows by Lemma 4.2. $\square$

Our central result gives an n-parameter description for all the $(1,2)$-symplectic metrics on $\mathbb{F}(n)$. We only cover the non-transitive case, given that the transitive case is covered by Proposition 4.1.

Theorem 4.4 *Let ε be the incidence matrix of a given complex structure J on $\mathbb{F}(n)$. Assume that ε is stair-shaped and not canonical-transitive. Then Λ corresponds to a $(1,2)$-symplectic metric on $(\mathbb{F}(n), J)$ if and only if there are n positive values λ_1 through λ_n such that for all $i < j$:*

(i) $\lambda_{ij} = \sum_{k=i}^{j-1} \lambda_k$ *if* $\varepsilon_{ij} = 1$, *and*

(ii) $\lambda_{ij} = \lambda_n + \sum_{k=1}^{i} \lambda_k + \sum_{k=j}^{n-1} \lambda_k$, *otherwise. Equivalently,*

$$\lambda_{ij} = q - \sum_{k=i+1}^{j-1} \lambda_k$$

where $q = \sum_{k=1}^{n} \lambda_n$.

Proof: Our starting point is Proposition 4.3. Dividing N into three subsets: $\{1, \ldots, s-1\}$, $\{s, \ldots, t\}$, and $\{t+1, \ldots, n\}$, we see that each triple $X = E_{ij}$, $Y = E_{jk}$, $Z = E_{ki}$ of basic matrices may be located in one of nine rectangular regions. The requirement $X, Y \in V^{10}$ and $Z \in V^{01}$ reduces the number of cases to the following (see (1)):

1. $i < j < k < s$;
2. $i < j < s \leq k \leq t$;

3. $i < s \leq j < k \leq t$;

4. $i < j \leq s \leq t < k$;

5. $i < s \leq j \leq t < k$;

6. $i < s \leq t < j < k$;

7. $s \leq i < j < k \leq t$;

8. $s \leq i < j \leq t < k$;

9. $s \leq i \leq t < j < k$;

10. $t < i < j < k$;

11. $k < s \leq t < i < j$;

12. $j < k < s \leq t < i$.

It can be seen that (i) is equivalent to the totality of conditions on Λ which emanate from cases 1-10, while (ii) emanates equally from cases 11 and 12. $\square$

The sufficiency of the condition stated in Theorem 1.1 now follows as an immediate corollary of Theorem 4.4.

Remark: Any metric Λ arising in the above theorem decomposes as $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_1 corresponds to the entries $\{(i,j) : i < j, \varepsilon_{ij} = +1\}$ and Λ_2 to region where $\varepsilon_{ij} = -1$, $i < j$. The entries λ_{ij} of both Λ_1 and Λ_2 are eigenvalues of $\mathrm{ad}\,(H)$ for some diagonal H. For Λ_1 take $H = H_\Lambda$ as in (4) while for Λ_2 take $H = H_{\Lambda_2}$ with

$$H_{\Lambda_2} = \mathrm{diag}\{\nu_1, \ldots, \nu_n\}$$

with $\nu_n - \nu_1 = \lambda_n$ and $\nu_{i+1} - \nu_i = -\lambda_{i,i+1}$ if $i = 1, \ldots, s$ or $i = t, \ldots, n-1$. Therefore the $(1,2)$-symplectic metrics are combinations of metrics of adjoint type, restricted to the proper regions.

Is is clear that the possible $(1,2)$-symplectic metrics Λ built in Theorem 4.4 cannot have all entries $\lambda_{ij} = 1$ if $n \geq 4$. As a consequence our theorem provides an independent proof of the following result by Wolf and Gray.

Corollary 4.5 *The Cartan-Killing metric on $\mathbb{F}(n)$ is not $(1,2)$-symplectic for any J if $n \geq 4$.*

Proof: In fact, the Cartan-Killing metric is given by $\Lambda = (\lambda_{ij})$ with $\lambda_{ij} = 1$ if $i \neq j$. $\square$

Remark: Notice that for invariant forms the annihilation of both $d\Omega^{30}$ and $d\Omega$ are equivalent to the nonexistence of cycles in the corresponding tournament. Therefore Ω is $(3,0)$ (or $(0,3)$) symplectic if and only if it is Kähler.

5 Partial flag manifolds

Let $\mathbb{F} = \mathbb{F}^N(n_1, \ldots, n_k)$ denote the manifold of generalized flags of subspaces of $\mathbb{C}^n$ viewed as a homogeneous space as $\mathrm{U}(N)/\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)$, where $n_1 + \cdots + n_k = N$. Its tangent space at the origin identifies with a subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{u}(N)$, which complements the Lie algebra $\mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_k)$ of the isotropy group. We can easily work with this apparently more general situation like in the full flag manifold case. We write the subspace of zero-diagonal complex matrices in blocks according to the partition determined by $n_1, \ldots, n_k$. Let V_{ij} be the subspace having nonzero block only in the position ij and put $V = \bigoplus_{ij} V_{ij}$, which is the complexification of $\mathfrak{p}$.

An invariant almost complex structure $J : \mathfrak{p} \to \mathfrak{p}$ is given by multiplication by $\pm\sqrt{-1}$ on each block. To such a structure we can associate, in the same way, a k-player tournament T_J as follows. Put $J(A_{ij}) = \left(A'_{ij}\right)$ and for $i < j$ choose the orientation $i \to j$ if $A'_{ij} = \sqrt{-1}A_{ij}$ and $i \leftarrow j$ if $A'_{ij} = -\sqrt{-1}A_{ij}$.

Analogous to the full flag manifold, an invariant metric is determined by its value at the origin, which is of the form $(X, Y)_\Lambda = \langle \Lambda X, Y \rangle$ with $\Lambda : \mathfrak{p} \to \mathfrak{p}$ positive-definite with respect to the Cartan-Killing metric $\langle \cdot, \cdot \rangle$. Again Λ is multiplication by λ_{ij} on each V_{ij} and hence is given by a $k \times k$ matrix $(\lambda_{\iota j})$ which is also denoted by Λ.

By working with the block decomposition of matrices the study of the generalized flag manifold $\mathbb{F}^N(n_1, \ldots, n_k)$ reduces to the understanding of $\mathbb{F}(k)$. In particular, $\mathbb{F}(2)$ codifies any Grassmannian of n_1 subspaces $\mathbb{F}^N(n_1, N - n_1)$. We note in passing that the Grassmannians are symmetric spaces as happens to $\mathbb{F}(2) = \mathbb{C}P^1$ (with the Fubini-Study metric), the only of the full flag manifolds that is a symmetric space.

6 Parabolic invariant almost complex structures

The *canonical parabolic* tournament T_P is defined by the incidence matrix $\gamma = (\gamma_{ij})$ given by

$$\gamma_{ij} = \begin{cases} -1 & \text{if } |i-j| \text{ is odd} \\ +1 & \text{if } |i-j| \text{ is even.} \end{cases} \tag{9}$$

A tournament is called parabolic if it is equivalent to T_P. Similarly an invariant almost complex structure is parabolic if this happens to its tournament.

It is easy to check that the canonical parabolic tournament is cone-free. In fact, a 4-player subtournament of T_P has nodes $a < b < c < d$ which are joined by the following rules: Take $i < j$. Then $i \to j$ if $i - j$ is even and $i \leftarrow j$ if $i - j$ is even.

According to the parity value of a, b, c, d one gets up to 16 distinct 4-player subtournaments, some of which may be pairwise similar. We leave to the reader the amusement of checking that none of these tournaments is a cone. Therefore T_P is cone-free. (Alternatively, in [11] $(1,2)$-symplectic metrics for parabolic J was exhibited. Hence the cone-free property of T_P follows from Theorem 1.1.)

We propose here to find a stair-shaped parabolic tournaments. Such a tournament has an incidence matrix of the form

$$\varepsilon = \begin{pmatrix} \mathrm{ct}_a & + & *_u \\ - & \mathrm{ct}_b & + \\ *_l & - & \mathrm{ct}_c \end{pmatrix} \tag{10}$$

where $a + b + c = n$ and ct_k stands for the canonical transitive k-player tournament. By the construction in Theorem 3.5, b is the size of a maximal 1-transitive tournament. Our method is based in the following remarks:

1. Let δ and $w\delta w^{-1}$ be the incidence matrix of two equivalent tournaments, where w is a permutation matrix. Let $v^\delta = \left(v_1^\delta, \ldots, v_n^\delta\right)$ be the row sum vector of δ:

   $$v_j^\delta = \sum_{k=1}^{n} \delta_{jk}.$$

   Clearly, $v^\delta = \delta e$, where $e = (1, \ldots, 1)$. Then the entries of $v^{w\delta w^{-1}}$ are obtained form v^δ by the permutation w. In fact, it is clear that $v^\delta = \delta e$, where $e = (1, \ldots, 1)$. Hence,

$$v^{w\delta w^{-1}} = w\delta w^{-1} e = w(\delta e).$$

2. The sum of the entries of the column $a + j$, $1 \le j \le b$, of the matrix ε in (10) is
$$a - b - c + 2(j-1) + 1. \qquad (11)$$

Now, the entries of the columns of the canonical parabolic tournament add up to 0 if n is odd, and to ± 1 if n is even. Hence, the expression in (11) ensures that, for a stair-shaped parabolic ε, we must have $b = 1$ if n is odd and $b = 1$ or 2 if n is even.

Lemma 6.1 *If n is even then the stair-shaped ε has $b = 2$.*

Proof: As mentioned above, b is the number of elements in a 1-maximal subtournament of T_P. Hence, it is enough to exhibit such a tournament having two elements. This is given by $\{1, n\}$. In fact in T_P the following 3-tournaments hold, for any even $e < n$ and odd $o > 1$:

$$\begin{array}{ccc} 1 & \leftarrow & e \\ & \nwarrow & \downarrow \\ & & n \end{array} \qquad\qquad \begin{array}{ccc} 1 & \rightarrow & o \\ & \nwarrow & \uparrow \\ & & n \end{array}$$

These tournaments are transitive, showing that $\{1, n\}$ is 1-transitive. $\square$

Now, we can describe stair-shaped parabolic tournaments.

Proposition 6.2 *Given the parabolic tournament T_P put $n = 2k + 1$ if n is odd and $n = 2(k+1)$ if n is even. Order $\{1, \ldots, n\}$ as follows:*

1. $\{2, 4, \ldots, n-1, 1, 3, \ldots, n-2, n\}$ *if n is odd.*
2. $\{2, 4, \ldots, n-2, n, 1, 3, \ldots, n-1\}$ *if n is even.*

In these orderings T_P has the following stair-shaped incidence matrices

$$\varepsilon = \begin{pmatrix} \mathrm{ct}_k & + & U \\ - & \delta & + \\ L & - & \mathrm{ct}_k \end{pmatrix}.$$

Here $\delta = (0)_{1\times 1}$ if $n = 2k+1$ and $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if $n = 2(k+1)$. Also, $U = (U_{ij})_{k\times k}$ is $U_{ij} = +1$ if $i > j$ and -1 otherwise, and $L = -U^T$.

Proof: The choice of the orderings was made after the construction of T' in the proof of Theorem 3.5. Here we took as maximal 1-transitive tournaments $\{1\}$ for n odd and $\{n \to 1\}$ for n even. Then the even numbers are the winners of 1 in the first case and of n in the second. In both cases the odd numbers are the losers of 1. It is straighforward to check that the incidence matrices are as stated. $\square$

Once we have the permutation realizing the stair-shaped equivalence the n-th parameter family of invariant $(1,2)$-symplectic is readily obtained from Theorem 4.4. Therefore we can recover Theorem 3.2 of [11] and get the invariant $(1,2)$-symplectic metrics for the parabolic almost complex structures. In order to write down these metrics we consider the permutations leading to the orderings of Proposition 6.2:

$$\sigma_o(i) = \begin{cases} 2i & \text{if } 1 \le i \le k \\ 2(i-k)-1 & \text{if } k+1 \le i \le n \end{cases}$$

for $n = 2k+1$ and

$$\sigma_e(i) = \begin{cases} 2i & \text{if } 1 \le i \le k \\ 2(i-k)-1 & \text{if } k+1 \le i \le n \end{cases}$$

for $n = 2k$.

Theorem 6.3 *Let J be the canonical parabolic invariant almost complex structure in $\mathbb{F}(n)$. Then the invariant $(1,2)$-symplectic metrics $\Lambda = (\lambda_{ij})$ are given by n parameters $a_1, \ldots, a_n$ as follows: Given $i < j$,*

$$\lambda_{\sigma(i)\sigma(j)} = \begin{cases} a_i + \cdots + a_{j-1} & \text{if } |i-j| \le k+1 \\ a_j + \cdots + a_n + \cdots + a_i & \text{if } |i-j| > k+1 \end{cases}$$

where $\sigma = \sigma_o$ if $n = 2k+1$ and $\sigma = \sigma_e$ if $n = 2k$.

References

[1] M. Black, *Harmonic maps into homogeneous spaces.* Pitman Research Notes, Math series 255. Longman, Harlow, 1991.

[2] A. Borel, Kählerian coset spaces of semi-simple Lie groups. *Proc. Nat. Acad. of Sci.* **40**:1147-1151, 1954.

[3] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces I. *Amer. J. Math.* **80**:458-538, 1958.

[4] F.E. Burstall and S. Salamon, Tournaments, flags and harmonic maps. *Math Ann.* **277**:249-265, 1987.

[5] J. Eells and S. Salamon, Twistorial constructions of harmonic maps of surfaces into four-manifolds. *Ann. Scuola Norm. Sup. Pisa (4)* **12**:589-640, 1985.

[6] A. Gray, Minimal varieties and almost Hermitian submanifolds, *Michigan Math. J.* **2**:273-287, 1965.

[7] A. Gray and L. M. Hervella, The sixteen classes of almost hermitian manifolds and their linear invariants. *Ann. Mat. Pura Appl* **123**:35-58, 1980.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience Publishers (1969).

[9] A. Lichnerowicz, Applications harmoniques et variétés Kähleriennes. *Symp. Math. 3* (Bologna), 341-402, 1970.

[10] X. Mo and C.J.C. Negreiros, Tournaments and geometry of full flag manifolds. *Proc. XI Brazilian Topology Meeting*, World Scientific 2000.

[11] X. Mo and C.J.C. Negreiros, $(1,2)$-Symplectic structure on flag manifolds. *Tohoku Math. J.*, **52**:271-282, 2000.

[12] J.W. Moon, *Topics on tournaments*. Holt, Reinhart and Winston, 1968.

[13] C.J.C. Negreiros, Some remarks about harmonic maps into flag manifolds. *Indiana Univ. Math. J.* **37**:617-636, 1988.

[14] M. Paredes, *Aspectos da geometria complexa das variedades bandeira.* Ph.D. thesis, State Univ. of Campinas, 2000.

[15] M. Paredes, On the existence of families of $(1,2)$-symplectic metrics on flag manifolds. Preprint.

[16] S. Salamon, *Harmonic and holomorphic maps.* Lec. Notes in Math. **1164**, Springer 1986.

[17] J. A. Wolf and A. Gray, Homogeneous spaces defined by Lie group automorphisms II. *J. Diff. Geom.* **2**:115-159, 1968.