On the Strongly Damped Wave Equation and the Heat Equation with mixed boundary conditions

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Abstract

In this paper we will study two one dimensional equations: the Strongly Damped Wave Equation and the Heat Equation, both with mixed boundary conditions. We will prove existence of global strong solutions and the existence of compact global attractors for these equations in two different spaces.

1. Introduction

In this paper we study existence of strong solutions and existence of global compact attractors for the following one dimensional problems:
The Strongly Damped Wave Equation,

$$\begin{aligned} & u_{tt} - u_{xx} - u_{txx} = g(t), \quad 0 < x < \ell, \quad 0 < t < T; \\ & \begin{cases} u(t,0) = 0 \\ u_x(t,\ell) + u_{tx}(t,\ell) = \rho(u_t(t,\ell)), \end{cases} \end{aligned} \tag{1.1}$$

and the Heat Equation

$$\begin{aligned} & z_t - z_{xx} + G(z) = h \\ & \begin{cases} z(0) = 0 \\ z_x(\ell) = \rho(z(\ell)). \end{cases} \end{aligned} \tag{1.2}$$

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Here ℓ and T are positive constants, $\rho : I\!R \to I\!R$ is a non-increasing and bounded function, $g, h \in L^1(0,T;L^2(0,\ell))$, and G is an operator from a sub space of H^1 into L^2. In the case where ρ is not continuous, we will understand $\rho(x_0)$, at a point of discontinuity x_0, as being the whole interval $[\rho(x_0+0),\ \rho(x_0-0)]$. In this case ρ will be a multi-valued function, and the "equal signs" in the last equations of (1.1) and 1.2 will be changed to "belong signs". So, the boundary conditions at $x = \ell$ will be written respectively as

$$u_x(t,\ell) + u_{tx}(t,\ell) \in \rho(u_t(t,\ell)) \text{ and } z_x(\ell) \in \rho(z(\ell))$$

or equivalently,

$$(u_t(t,\ell), \quad u_x(t,\ell) + u_{tx}(t,\ell)) \in \Gamma \text{ and } (z(\ell),\ z_x(\ell)) \in \Gamma$$

where Γ is the graph of the multi-valued function ρ

The existence of global solutions for these two problems can be obtained using the Theory of Monotone Operators. The problem 1.2 gives rise to a maximal monotone operator A that is of sub differential type, $A = \partial\varphi$, where φ is a lower semi continuous and convex functional. This problem was studied in [2] under some conditions on G, in particular the existence of strong solutions was proved.

Our goal is to obtain existence of global compact attractor. To reach this goal, first of all, we will obtain a relation between the solutions of the two problems. With this relation we can use one problem to get properties of the other, in particular this relation we will be used to prove the existence of strong solutions for the problem 1.1. Once we have existence of solutions, we will start working in order to get the existence of the attractors. For our purpose, we will study the problem 1.2 in two different spaces L^2 and H^1 and using the relation between the solutions we will prove the existence of attractors for the problems. More specifically, setting $u_t = v$, where $u(t)$ is solution operator given by 1.1, we will study the evolution of three operators, $z(t)$ given by 1.2, in the spaces L^2 and H^1, $u(t) + v(t)$ in the space H^1 and $v(t)$ in the space L^2.

To obtain the results we will use the following procedures: To prove the bounded dissipativeness of the problem 1.1 we will construct an appropriate equivalent norm in the space. The bounded dissipativeness of 1.2 in H^1 will be obtained using the Uniform Gronwall Lemma with some appropriate estimates. The prove of the compactness of the operators will be done using arguments of Aubin-Lion's type.

Asymptotic behavior of parabolic equations with monotone principal part was recently studied by Carvalho and Gentile in [4], the difference with our case, problem 1.2, is that our functional φ is not equivalent to the norm of the space.

2. Abstract Formulation and Existence of Solutions

As usual in wave equations context, setting $v = u_t$, the equation (1.1) can be seen as a system:

$$\begin{array}{l} u_t = v \\ v_t = (u+v)_{xx} + g, \qquad 0 < x < \ell, \quad 0 < t < T; \\ \left\{ \begin{array}{l} u(t,0) = 0 \\ u_x(t,\ell) + v_x(t,\ell) = \rho(v(t,\ell)), \end{array} \right. \end{array} \tag{2.1}$$

Therefore, our problem (1.1) can be viewed as an evolution equation

$$\dot{w} + Aw = f(t) \tag{2.2}$$

in the Hilbert space

$$\mathcal{H} = H_{1,0} \times L^2(0,\ell),$$
$$H_{1,0} = \{u \in H^1(0,\ell) : u(0) = 0\},$$

with the inner product:

$$\langle (u_1,v_1),(u_2,v_2)\rangle_{\mathcal{H}} = \int_0^\ell (u_1' u_2' + v_1 v_2)dx,$$

and

$$A : \mathcal{D}(A) \subset \mathcal{H} :\to \mathcal{H}$$

given by

$$A(u,v) = (-v,\ -(u+v)''),$$

on the domain

$$\begin{array}{rl} \mathcal{D}(A) = \{(u,v) \in H_{1,0} \times H_{1,0} : & \\ & (u+v) \in H^2(0,\ell) \text{ and } (u+v)'(\ell) \in \rho(v(\ell))\} \end{array} \tag{2.3}$$

Throughout the paper we will denote respectively by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the usual inner product e norm of L^2. We will use the terminology of [3, Brézis] and [5, Hale].

Lemma 2.1 *The operator A is maximal monotone.*

Proof: If $w_1 = (u_1,v_1)$ and $w_2 = (u_2,v_2)$ are in $\mathcal{D}(A)$, we have integrating by parts that

$$\begin{array}{l} \langle w_1 - w_2, Aw_1 - Aw_2 \rangle \\ = -(v_1(\ell) - v_2(\ell))[(u_1+v_1)'(\ell) - (u_2+v_2)'(\ell)] + \int_0^\ell (v_1' - v_2')^2 dx. \end{array}$$

Since ρ is non-increasing and $(u_i + v_i)'(\ell) \in \rho(v_i(\ell))$, $i = 1, 2$, we have

$$\langle w_1 - w_2, Aw_1 - Aw_2 \rangle \geq 0,$$

therefore, A is a monotone operator.

We will prove that A is maximal by showing that $R(I + A) = \mathcal{H}$. In fact, if $(f, g) \in \mathcal{H}$ we consider z as being the unique solution of the ODE problem:

$$\begin{cases} z - 2z'' = f + 2g := h \in L^2(0, \ell) \\ z(0) = 0, \quad z'(0) = a \in I\!R \end{cases}$$

where a will be chosen conveniently. Since $z \in H^2(0, \ell) \cap H_{1,0}$ and $f \in H_{1,0}$, setting

$$u = \frac{1}{2}(z + f) \quad \text{and} \quad v = \frac{1}{2}(z - f)$$

we have that $u, v \in H_{1,0}$, $u + v = z \in H^2(0, \ell)$ and

$$\begin{cases} u - v = f \\ v - (u + v)'' = g. \end{cases}$$

Therefore, it remains to be proved that $(u + v)'(\ell) \in \rho(v(\ell))$ or equivalently $z'(\ell) \in \tilde{\rho}(z(\ell))$, where

$$\tilde{\rho}(x) = \rho\left(\frac{1}{2}(x - f(\ell))\right).$$

We will obtain that condition choosing the constant a appropriately. Setting

$$M = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$$

we have from the variation constant formula

$$\begin{pmatrix} z(\ell) \\ z'(\ell) \end{pmatrix} = a e^{\ell M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2} \int_0^\ell e^{(\ell - s)M} \begin{pmatrix} 0 \\ h(s) \end{pmatrix} ds. \tag{2.4}$$

Since

$$e^{\ell M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \sinh(\ell/\sqrt{2}) \\ \cosh(\ell/\sqrt{2}) \end{pmatrix}$$

we have that the right hand side of (2.4) is a straight line in plane, parametrized by a, with positive slope. Therefore, there will be a unique a that gives the intersection with the non-increasing graph of $\tilde{\rho}$. The lemma is proved. ∎

Considering solutions in the sense of Brézis [3], that is

Definition 2.1 *Let f be in $L^1(0,T;\mathcal{H})$. A continuous function $w:[0,T]\to\mathcal{H}$ is a solution (or strong solution) of*

$$\dot{w}(t)+Aw(t)=f(t) \tag{2.5}$$

if w satisfies

(i) $w(t)\in\mathcal{D}(A),\ \forall t\in(0,T)$,

(ii) *$w(t)$ is absolutely continuous (AC) on every compact set $K\subset(0,T)$ (therefore $\dot{w}(t)$ exists a.e. in $(0,T)$),*

(iii) $\dot{w}(t)+A(w(t))=f(t)$, *a.e. in* $(0,T)$.

Moreover, $w\in C([0,T];\mathcal{H})$ is a weak solution of (2.5) if there exist sequences $(f_n)\in L^1(0,T;\mathcal{H})$ and $(w_n)\in C([0,T];\mathcal{H})$ such that w_n are strong solutions of

$$\dot{w}_n(t)+A(w_n(t))=f_n(t),$$

$f_n\to f$ in $L^1(0,T;\mathcal{H})$ and $w_n\to w$ uniformly in $[0,T]$.

We have from theorem 3.4 of [3] the existence of weak solution for the problem 2.1.

In order to prove that this weak solution is in fact strong we will look for a relation between the solutions of 2.1 and the solutions of 1.2.

The problem 1.2 was studied in [2], where G is an operator

$$G:H_{1,0}\to L^2(0,\ell)$$

not necessarily local and $h\in L^2(0,T;L^2(0,\ell))$. The problem can be written as the abstract evolution problem in $L^2(0,\ell)$

$$\dot{z}+\mathcal{A}z=F(t,z) \tag{2.6}$$

where $\mathcal{A}:\mathcal{D}(A)\subset H_{1,0}\to L^2(0,\ell)$ is the operator given by

$$\mathcal{A}z=-z''$$

on the domain

$$\mathcal{D}(\mathcal{A})=\{z\in H_{1,0}\cap H^2(0,\ell):z'(\ell)\in\rho(z(\ell))\}. \tag{2.7}$$

¿From lemmas 2.1 and 2.2 of [2] we have that the operator $\mathcal{A}$ is strictly monotone

$$\langle\mathcal{A}z_1-\mathcal{A}z_2,\ z_1-z_2\rangle\geq|z_1'-z_2'|^2 \tag{2.8}$$

and of sub-differential type, $\mathcal{A} = \partial\varphi$, where $\varphi : L^2(0,\ell) \to I\!R \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function defined by

$$\varphi(z) = \begin{cases} p(z(\ell)) + \frac{1}{2}\int_0^\ell z'(x)^2 dx, & \text{if } z \in H_{1,0} \\ +\infty \quad \text{otherwise,} \end{cases} \tag{2.9}$$

where p is given by

$$p(x) = \int_0^x -\rho(s)ds. \tag{2.10}$$

We should observe that φ may assume negative values, but the following estimate is true

$$|z'|^2 \leq k_1\varphi(z) + k_2, \quad \forall z \in H_{1,0}. \tag{2.11}$$

where $k_1,\ k_2$ are constants, in particular φ is bounded below.

Indeed, since $|\rho(s)|$ is bounded (by a constant k), we have for $z \in H_{1,0}$

$$p(z(\ell)) \geq -k|z(\ell)| = -k \mid \int_0^\ell z'(x)dx \mid \geq -k \int_0^\ell |z'(x)|dx$$

and then

$$\int_0^\ell \left(\frac{1}{4}z'(x)^2 - k^2\right) dx \leq \int_0^\ell \left(\frac{1}{2}z'(x)^2 - k|z'(x)|\right) dx \leq \varphi(z)$$

implies the estimate (2.11).

When G is Lipschtz continuous and $h \in L^2(0,T;L^2(0,\ell))$, it was proved, theorems 3.2 and 4.1 of [2], that the solutions of (1.2) are strong, in particular $z(t) \in \mathcal{D}(\mathcal{A})$, for every $t \in (0,T)$. Moreover, from theorem 3.6 of [3], the solution z satisfies:

$$\sqrt{t}\frac{dz}{dt}(t) \in L^2(0,T;L^2(0,\ell)) \tag{2.12}$$

and, when $z(0) \in \mathcal{D}(\varphi) = H_{1,0}$,

$$\frac{dz}{dt}(t) \in L^2(0,T;L^2(0,\ell)). \tag{2.13}$$

Consider the following relations between the problems (2.1) and (1.2):

$$z(t,x) = u(t,x) + v(t,x) - u(t,\ell)\xi(x), \tag{2.14}$$

$$G(z) = z(\ell)\xi) \tag{2.15}$$

$$h(t,x) = g(t,x) + v(t,x) + u(t,\ell)\xi''(x) \tag{2.16}$$

where $\xi : [0,\ell] \to I\!R$ is a smooth function satisfying $\xi(0) = 0$, $\xi(\ell) = 1$ and $\xi'(\ell) = 0$.

The operator G given in (2.15) can be considered with values in $L^2(0,\ell)$ and also with values in $H_{1,0}$, in both of these cases G is Lipschtz continuous and G also satisfies

$$|G(z)| \leq c\,|z'|, \tag{2.17}$$

since

$$|z(\ell)| = |\int_0^\ell z'(x)dx| \leq \|z'\|_{L^1}, \tag{2.18}$$

It is easy to see that if (u,v) is a solution of (2.1) than z, given by (2.14), is the a solution of (1.2) with h given by (2.16) and with initial condition $z(0) = u(0) + v(0)$.

Reciprocally, if z is a solution of (1.2), we consider the problem in $H_{1,0}$ given by

$$\begin{aligned} &\frac{du}{dt}(t) + u(t) - J(t)u(t) = 0 \\ &u(0) = 0 \end{aligned}$$

where $J(t)u(t) = G(u(t)) + z(t)$.

Since $J(t) : H_{1,0} \to H_{1,0}$, for $t > 0$, is globally Lipschtz, this problem has existence and uniqueness of solutions, see theorem 1.4 of [3]. If $u(t)$ is this unique solution, then considering $v(t)$ given by the relation (2.14) and g by the relation (2.16) we have that (u,v) satisfies the problem (2.1) with $u(0) = 0$ and $v(0) = z(0)$.

Under these condition we can prove the following result:

Theorem 2.1 *If $g \in L^2(0,T;L^2(0,\ell))$, then for every $w_0 = (u_0,v_0) \in \mathcal{H}$ there exists a unique strong solution $w = (u,v) \in C([0,T];\mathcal{H})$ of (2.1) such that $w(0) = w_0$. Moreover, the solution $w = (u,v)$ satisfy:*

$$\sqrt{t}\frac{d}{dt}(u+v)(t) \in L^2(0,T;L^2(0,\ell)) \tag{2.19}$$

and, for $v(0) \in H_{1,0}$,

$$\frac{d}{dt}(u+v)(t) \in L^2(0,T;L^2(0,\ell)). \tag{2.20}$$

Proof: Since z given by (2.14) is a strong solution (1.2), in particular $z(t) \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A})$ given in (2.7), for every $t \in (0,T)$. It is easy to see that (u,v) is a strong solution of (2.1).

¿From (2.14)

$$\frac{d}{dt}(u+v) = \frac{dz}{dt} + v(t,\ell)\xi \quad \text{and} \quad v(t,\ell) = z(t,\ell). \tag{2.21}$$

and $z(t,\ell) \in L^2(0,T)$, according to the Trace Theorem for Lipschitz domain, p.15 of [7], therefore (2.19) and (2.20) follow respectively from (2.12) and (2.13). The proof is complete. ∎

Although we are interested in the study the influence of the nonlinear boundary condition in the problems, we should observe that we have existence of strong solution in more general situation. In fact, we can consider

$$\begin{aligned} &u_{tt} - u_{xx} - u_{txx} + q(t,x,u,u_t) = 0, \quad 0 < x < \ell, \quad 0 < t < T; \\ &\begin{cases} u(t,0) = 0 \\ u_x(t,\ell) + u_{tx}(t,\ell) = \rho(u_t(t,\ell)), \end{cases} \end{aligned} \tag{2.22}$$

where

q_1) the application $(t,x) \to q(t,x,w)$ belongs to $L^2(0,T;L^2(0,\ell))$, for every fixed $w \in \mathcal{H}$;

q_2) there exists $k > 0$, such that

$$|q(t,x,w_1) - q(t,x,w_2)|_{L^2(0,\ell)} \leq k\|w_1 - w_2\|_{\mathcal{H}}, \quad \forall t \in [0,T], \quad \forall w_1, w_2 \in \mathcal{H}.$$

This problem can be viewed as an abstract evolution equation in the Hilbert space $\mathcal{H}$

$$\dot{w} + Aw + B(t,w) = 0 \tag{2.23}$$

where $B : [0,T] \times \mathcal{H} \to \mathcal{H}$ is given by

$$B = (0,q). \tag{2.24}$$

¿From the assumptions q_1 and q_2, we have that B satisfies:

B_1) for every $w \in \mathcal{H}$, the application $t \to B(t,w)$ belongs to $L^2(0,T;\mathcal{H})$;

B_2) there exists $k > 0$, such that

$$\|B(t,w_1) - B(t,w_2)\| \leq k\|w_1 - w_2\|, \quad \forall t \in [0,T], \quad \forall w_1, w_2 \in \mathcal{H}.$$

Under the above assumptions we have

Theorem 2.2 *For every $w_0 \in \mathcal{H}$, there exists a unique strong solution $w \in C([0,T];\mathcal{H})$ of (2.23)) satisfying $w(0) = w_0$.*

Proof: We will use the method of Brézis [3]. Since, for every $w \in C([0,T];\mathcal{H})$, $B(t,w(t)) \in L^2(0,T;\mathcal{H})$, we can consider the sequence w_n in $C([0,T];\mathcal{H})$, defined by $w_0(t) = w_0$ and w_{n+1} is the weak solution of

$$\begin{aligned}&\dot{w}_{n+1}(t) + A(w_{n+1}(t)) = -B(t, w_n(t))\\ &w_{n+1}(0) = w_0\end{aligned}$$

which exists by the theorem 2.1. Using the first inequality of lemma 3.1 of [3], we obtain

$$\begin{aligned}\|w_{n+1}(t) - w_n(t)\| &\leq \int_0^t \|B(\sigma, w_n(\sigma)) - B(\sigma, w_{n-1}(\sigma))d\sigma\\ &\leq k\int_0^t \|w_n(\sigma) - w_{n-1}(\sigma)\|d\sigma,\end{aligned}$$

therefore

$$\|w_{n+1}(t) - w_n(t)\| \leq \frac{(kt)^n}{n!}\|w_1 - w_0\|_{L^\infty}.$$

Thus, the sequence w_n converges uniformly to w in $[0,T]$, so w is a weak solution of

$$\begin{aligned}&\dot{w}(t) + A(w(t)) = -B(t, w(t))\\ &w(0) = w_0.\end{aligned}$$

Now, since $B(t,w(t)) = (0, q(t,\cdot,w(t)))$ and it is easy to see that $q(t,\cdot,w(t)) \in L^2(0,T;L^2(0,\ell))$, we have from theorem 2.1 that w is a strong solution of (2.23). The proof is completed. ∎

It is not difficult to see that the strong solutions, given by this theorem, depend on continuously of the initial data. More specifically, we have that there exists a positive constant c such that

$$\|w(t) - \tilde{w}(t)\|_{L^\infty([0,T];\mathcal{H})} \leq c\|w_0 - \tilde{w}_0\|_{\mathcal{H}},$$

where $w(t)$ and $\tilde{w}(t)$ are solutions of (2.23) with initial conditons w_0 and $\tilde{w}_0$, respectively.

3. Existence of Attractors in L^2

We will start by constructing an equivalent norm in the space $\mathcal{H}$

Lemma 3.1 *If $W(w)$ is given by*

$$W(w) = W(u,v) = \int_0^\ell \left[\frac{1}{2}(u')^2 + \frac{1}{2}v^2 + 2\beta uv\right] dx \tag{3.1}$$

where

$$0 < \beta < \frac{2}{2\ell^2 + 1} \tag{3.2}$$

then, $W^{1/2}$ is an equivalent norm in $\mathcal{H}$.

Moreover, there exists a positive constant λ such that

$$\int_0^\ell \left[(\beta\ell^2 - 1)f^2 - 2\beta g^2 - 2\beta f g\right] dx \leq -\lambda \left(|f|^2 + |g|^2\right)$$

for every $f, g \in L^2(0, \ell)$.

Proof: Using the Poincaré ($|u| \leq (\ell/\sqrt{2})|u'|$) and the Schwarz inequalities, we have

$$-\frac{\beta\ell}{\sqrt{2}}(|u'|^2 + |v|^2) \leq \int_0^\ell 2\beta uv dx \leq \frac{\beta\ell}{\sqrt{2}}(|u'|^2 + |v|^2).$$

Using (3.2) we can see that

$$\frac{\beta\ell}{\sqrt{2}} < \frac{1}{2}.$$

Therefore, if $\eta = 1/2 - \beta\ell/\sqrt{2}$, we have

$$\eta \left(|u'|^2 + |v|^2\right) \leq W(u, v) \leq \left(|u'|^2 + |v|^2\right),$$

then $W^{1/2}$ is an equivalent norm in $\mathcal{H}$.

The second part of the lemma follows noting that

$$\int_0^\ell \left[(\beta\ell^2 - 1)f^2 - 2\beta g^2 - 2\beta f g\right] dx \leq (\beta\ell^2 - 1)|f|^2 - 2\beta|g|^2 + 2\beta|f||g|$$

and, for β satisfying (3.2), the right hand side this inequality is a negative definite form. ∎

Theorem 3.1 *If $g, h \in L^\infty(\mathbb{R}^+; L^2(0, \ell))$, then the problems (2.1) and (1.2) are bounded dissipative. More precisely, if (u, v) and z are the solutions of (2.1) and (1.2), with initial conditions (u_0, v_0) and z_0, respectively, then there exist positive constants c_1, c_2 and μ such that*

$$\|(u(t),\ v(t))\|_{\mathcal{H}} \leq c_1 \|(u_0,\ v_0)\|_{\mathcal{H}}\, e^{-\mu t} + c_2 \tag{3.3}$$

$$|z(t)| \leq c_1 |z_0|\, e^{-\mu t} + c_2. \tag{3.4}$$

Moreover, for $z_0 \in H_{1,0}$ and r positive, there exist positive constants a, b, with $b = b(r)$ depending on r, such that

$$\int_t^{t+r} \varphi(z(s)) ds \leq a|z_0|^2\, e^{-\mu t} + b, \quad t \geq 0. \tag{3.5}$$

Proof: From the relation between the two problems the estimate (3.4) follows from (3.3). To prove (3.3) it is enough to consider initial data in the domain $\mathcal{D}(A)$. Using the equation (2.1) and the Poincaré inequality ($|v|^2 \leq (\ell^2/2)|v'|^2$), we obtain after an integration by parts that for almost every t

$$\begin{aligned} \dot{W}(t) &= \frac{d}{dt}W(u(t),v(t)) \leq \\ & \int_0^\ell \left[(\beta\ell^2-1)(v')^2 - 2\beta(u')^2 - 2\beta u'v'\right] dx + \qquad (3.6) \\ & [2\beta u(\ell)+v(\ell)](u+v)'(\ell) + \int_0^\ell [2\beta u + v] g(t) dx. \qquad (3.7) \end{aligned}$$

The first integral, line (3.6), can be estimated using lemma 3.1

$$\int_0^\ell \left[(\beta\ell^2-1)(v')^2 - 2\beta(u')^2 - 2\beta u'v'\right] dx \leq -\lambda\left(|u'|^2 + |v'|^2\right)$$

To estimate the terms in line 3.7, we observe that $(u+v)'(\ell)$ satisfies the boundary condition, so it is bounded by some constant M, then using (2.18) we can show that there exists a positive constant c, such that, for every $\delta > 0$

$$[2\beta u(\ell)+v(\ell)](u+v)'(\ell) \leq c\left(\delta\left(|u'|^2+|v'|^2\right) + \frac{1}{\delta}M^2\right).$$

Using the Poincaré inequality we obtain also

$$\int_0^\ell (2\beta u + v) g(t) dx \leq c\left(\delta\left(|u'|^2+|v'|^2\right) + \frac{1}{\delta}\|g\|^2\right).$$

Choosing δ sufficiently small, we obtain positive constants $\mu_i, = 1, 2$, and K, such that

$$\dot{W}(t) \leq -\mu_1(|u'|^2 + |v'|^2) + K \leq -\mu_2 W(t) + K.$$

Solving this differential inequality we obtain

$$W(t) \leq e^{-\mu_2 t} W(0) + \frac{K}{\mu_2}.$$

that implies (3.3)

To prove the last inequality. We have that $\mathcal{A}$ is the sub differential of the functional φ and $\varphi(0) = 0$, therefore $\varphi(z) \leq \langle \mathcal{A}z,\ z\rangle$. So, multiplying (1.2) by z we obtain

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \varphi(z) \leq -\langle G(z),\ z\rangle + \langle h,\ z\rangle.$$

The operator G satisfies (2.17), then, using (2.11), we obtain for every $\delta > 0$ a constant M depending on δ such that

$$|\langle G(z(t)),\ z(t)\rangle| \leq \delta\varphi(z(t)) + M(|z(t)|^2 + 1)$$

and, since

$$|\langle h(t),\ z(t)\rangle| \leq c\left(|z(t)|^2 + 1\right),$$

we have grouping the equivalent terms and choosing a convenient small value for δ, we obtain

$$\frac{d}{dt}|z(t)|^2 + \varphi(z(t)) \leq a_1 + a_2|z(t)|^2,$$

for some positive constants $a_1,\ a_2$. Integrating this inequality from t to $t + r$ we obtain

$$\int_t^{t+r} \varphi(z(s))ds \leq |z(t)|^2 + a_1 r + a_2 \int_t^{t+r} |z(s)|^2 ds.$$

This inequality and (3.4) imply (3.5). ∎

Theorem 3.2 *If $h \in L^\infty(I\!R^+; L^2(0,\ell))$, then the solution operator $T_h(t) : L^2(0,\ell) \to L^2(0,\ell)$, associated to the solution of (1.2), is a compact operator for $t > 0$.*

Proof: Multiplying the equation (1.2) by $\phi \in H_{1,0}$, we obtain

$$\langle z_t,\ \phi\rangle = z_x(t,\ell)\ \phi(\ell) - \langle z_x,\ \phi_x\rangle - \langle G(z),\ \phi\rangle + \langle h,\ \phi\rangle, \tag{3.8}$$

therefore (3.5) and (3.8) imply that $z_t \in L^2(0,T; H'_{1,0})$ and

$$\int_0^T \|z_t\|^2_{H'_{1,0}} dt \leq C(|z(0)|, T). \tag{3.9}$$

To prove the compactness it is enough to consider initial data in a dense sub set of $L^2(0,\ell)$. Let B be the bounded set $B = B(r) \cap H_{1,0}$, where $B(r)$ the ball of $L^2(0,\ell)$ with center in zero and radius r, and $T_h(t)(z_0)$ the solution of (1.2) with initial condition z_0.

¿From (3.5) and (3.9)

$$\bar{B} = \{T_h(.)(z_0);\ z_0 \in B\}$$

is a bounded set in the Banach Space

$$W = \{v \in L^2(0,T; H_{1,0});\ \ v_t = \frac{dv}{dt} \in L^2(0,T; H'_{1,0}\}.$$

Therefore, from [6, thm 5.1], $\bar{B}$ is precompact set in $L^2(0,T;L^2(0,\ell))$. Then, if (z_n) is a sequence in B, taking subsequences if necessary, we can suppose that $(T_h(\cdot)(z_n))$ converges to some function $z(\cdot) \in L^2(0,T;L^2(0,\ell))$, and also, for almost every $\tau \in (0,T)$,

$$T_h(\tau)(z_n) \to z(\tau), \quad \text{as } n \to \infty. \tag{3.10}$$

Consider now the evolution operator $S(\cdot)(z,h)$ given by

$$S(t)(z,h) = (T_h(t)z, h_t),$$

where h_t is the translation of h, $h_t(\tau) = h(t+\tau)$. ¿From [8] $S(t): \ t \geq 0$ is a dynamical system. Therefore, for $t > 0$, there exists $\tau \in (0,t)$ such that (3.10) is true, then

$$\begin{aligned}(T_h(t)z_n, h_t) &= S(t)(z_n,h) = S(t-\tau)S(\tau)(z_n,h) = S(t-\tau)(T_h(\tau)z_n, h_\tau) \\ &\to S(t-\tau)(z(\tau),h_\tau) = (T_{h_\tau}(t-\tau)z(\tau), \ h_t)\end{aligned}$$

implies the compactness of $T_h(t)$.

Denoting by $v_{u_0}(t)$ the dynamical system given by the problem 2.1, when the initial condition $u(0) = u_0 \in H_{1,0}$ is fixed, we have from the two previous theorems and the relation 2.14, the following result

Theorem 3.3 *Under the above conditions the two dynamical systems $z(t)$ and $v_{u_0}(t)$, have compact global attractors in $L^2(0,\ell)$.*

4. Existence of attractors in $H_{1,0}$

We will start doing some estimates of the solution $z(t)$ of (1.2) when the initial condition $z(0) \in H_{1,0}$. Using theorem 3.6 of [3], we have that $t \to \varphi(z(t))$ is absolutely continuous and

$$\frac{d}{dt}\varphi(z(t)) = \langle \mathcal{A}z(t), z_t(t)\rangle, \ \ a.e.$$

then

$$\frac{d}{dt}\varphi(z(t)) = -|\mathcal{A}z(t)|^2 + \langle \mathcal{A}z(t), -G(z) + h\rangle, \tag{4.1}$$

and integrating in t we obtain

$$\begin{aligned}&\int_0^t |\mathcal{A}z(s)|^2 ds + \varphi(z(t)) \\ &\leq \ \varphi(z(0)) + \int_0^t |\mathcal{A}z(s)| \, |h(s) - G(z(s))| ds \\ &\leq \ \varphi(z(0)) + \int_0^t \frac{1}{2}|\mathcal{A}z(s)|^2 ds + \int_0^t |G(z(s))|^2 ds + \int_0^t |h(s)|^2 ds.\end{aligned}$$

Using (2.11) and (2.17), we obtain, for $t \in [0,T]$, that

$$\frac{1}{2}\int_0^t |\mathcal{A}z(s)|^2 ds + \varphi(z(t)) \le \varphi(z(0)) + c_1 + c_2 \int_0^t \varphi(z(s))ds.$$

for some constants c_1, c_2.

Thus, from Gronwall inequality, there exists a constant $C(\varphi(z(0)),T)$ depending on $\varphi(z(0))$ and T such that

$$\varphi(z(t)) \le C(\varphi(z(0)),T) \tag{4.2}$$

$$\int_0^t |\mathcal{A}z(s)|^2 ds \le C(\varphi(z(0)),T), \tag{4.3}$$

in particular, we have $z \in L^\infty(0,T;H_{1,0}) \cap L^2(0,T;H^2(0,\ell))$.

Moreover, if $z_1(t)$ and $z_2(t)$ are solutions with initial condition on $H_{1,0}$, we have using (2.8)

$$\begin{aligned}|(z_1(t))' - (z_2(t))'|^2 &\le \langle \mathcal{A}z_1(t) - \mathcal{A}z_2(t),\ z_1(t) - z_2(t)\rangle = \\ &-\frac{1}{2}\frac{d}{dt}|z_1(t) - z_2(t)|^2 - \langle G(z_1(t)) - G(z_2(t)),\ z_1(t) - z_2(t)\rangle .\end{aligned}$$

Since G is Lipschtz, we obtain after an integration in t

$$\begin{aligned}&\frac{1}{2}\int_0^t |(z_1(s))' - (z_2(s))'|^2 ds + \frac{1}{2}|z_1(t) - z_2(t)|^2 \\ &\qquad \le \frac{1}{2}|z_1(0) - z_2(0)|^2 + c\int_0^t |z_1(s) - z_2(s)|^2 ds,\end{aligned}$$

therefore, from Gronwall inequality, there exists a constant C depending on T, such that

$$|z_1(t) - z_2(t)| \le C\ |z_1(0) - z_2(0)| \tag{4.4}$$

and

$$\int_0^t |(z_1(s))' - (z_2(s))'|^2 ds \le C\ |z_1(0) - z_2(0)|^2 \tag{4.5}$$

for $t \in [0,T]$.

Lemma 4.1 *If* $z(t) = T(t)z_0,\ t \ge 0$, *denotes the solution of the problem (1.2) with initial condition* $z(0) = z_0$, *then the following operators are continuous:*

$$\begin{array}{rcl} i)\ \mathbb{R}^+ & \to & H_{1,0} \\ t & \to & T(t)z_0, \end{array}$$

for fixed $z_0 \in H_{1,0}$*, and*

$$ii)\ \begin{array}{rcl} H_{1,0} & \to & H_{1,0} \\ z_0 & \to & T(t)z_0. \end{array}$$

for fixed $t > 0$.

Proof: Let (t_n) be a sequence in $I\!R^+$ converging to t, we know that $(z(t_n))$ converges to $z(t)$ in $L^2(0,\ell)$ and, using lemma 3.6 of [3], $(\varphi(z(t_n)))$ converges to $\varphi(z(t))$. Then, from (2.11), $|(z(t_n))'|$ is bounded, therefore, there exists a sub sequence of $(z(t_n))$, that we will keep denoting by $(z(t_n))$, that converges weakly to $z(t)$ in $H_{1,0}$.

First of all we claim that the weak convergence implies the convergence of $(z(t_n,\ell))$. In fact considering a smooth function ϕ such that $\phi(0) = 0$ and $\phi(\ell) \neq 0$, we obtain integrating by parts

$$\int_0^\ell z'(t_n,x)\phi(x)dx = z(t_n,\ell)\phi(\ell) - \int_0^\ell z(t_n,x)\phi'(x)dx$$

and

$$\int_0^\ell z'(t,x)\phi(x)dx = z(t,\ell)\phi(\ell) - \int_0^\ell z(t,x)\phi'(x)dx$$

Thus, passing to the limit, $z(t_n,\ell) \to z(t,\ell)$, what proves our claim. Next, since p is continuous and

$$\|z(t_n)\|^2_{H_{1,0}} = 2[\varphi(z(t_n)) - p(z(t_n,\ell))],$$

we have $\|z(t_n)\|_{H_{1,0}} \to \|z(t)\|_{H_{1,0}}$ that implies the strong convergence of $(z(t_n))$ to $z(t)$ and the continuity of the first operator is proved.

Now we will prove the continuity of the second operator, In fact, what we have is a stronger result:

Theorem 4.1 *If* (z_{0_n}) *is a bounded sequence in* $H_{1,0}$ *and converges to* z_0 *in the* $L^2(0,\ell)$*-norm, then the corresponding solutions of (1.2)* $z_n(t) = T(t)z_{0_n}$ *converges to* $z(t) = T(t)z_0$ *in* $H_{1,0}$*, for fixed* $t > 0$*, as* $n \to \infty$*. In particular the operator* $ii)$ *given in lemma 4.1 is a compact operator.*

Proof: We have $(\varphi(z_{0_n}))$ bounded, then from (4.2) and (2.11), both sequences $(\varphi(z_n(t)))$ and $(|(z_n(t))'|)$ are uniformly bounded for $t \in [0,T]$. The convergence $z_{0_n} \to z_0$ in $L^2(0,\ell)$ and (4.5) imply the convergence

$$z_n \to z \quad \text{in } L^2(0,T;H_{1,0}),$$

therefore $z_n(\tau) \to z(\tau)$ in $H_{1,0}$ for almost every $\tau \in [0,T]$.

For $t \in [0,T]$,

$$\begin{aligned} |\varphi(z_n(t)) - \varphi(z(t))| &\leq |\varphi(z_n(t)) - \varphi(z_n(\tau))| \\ &+ |\varphi(z_n(\tau)) - \varphi(z(\tau))| + |\varphi(z(\tau)) - \varphi(z(t))|.) \end{aligned} \quad (4.6)$$

The first term in the right hand side satisfies

$$\varphi(z_n(t)) - \varphi(z_n(\tau)) = \int_\tau^t \frac{d}{ds}\varphi(z_n(s))ds$$

and from (4.1)

$$\frac{d}{dt}\varphi(z(t)) \leq |G(z(t))|^2 + |h(t)|^2, \quad (4.7)$$

therefore the sequences $(d/dt(\varphi(z_n(t))))$ are a uniform bounded in $L^2(0,\ell)$ for every $t \in [0,T]$. Then (4.6) implies $\varphi(z_n(t)) \to \varphi(z(t))$ for every $t \in [0,T]$, as $n \to \infty$. Therefore, the same argument we have just used in the first part of the theorem implies that $z_n(t) \to z(t)$ in $H_{1,0}$-norm, as $n \to \infty$. ∎

Theorem 4.2 *If $h \in L^\infty(0,\infty; L^2(0,\ell))$, then there exists a bounded set in $H_{1,0}$ that attracts all the solutions of the problem (1.2) with initial condition in a sub set of $H_{1,0}$ that it is bounded in $L^2(0,\ell)$. In particular, the problem (1.2) is bounded bounded dissipative in $H_{1,0}$.*

Proof: If $z(t)$ is a solution of the problem (1.2) with initial condition in $H_{1,0}$ we have, using (4.7), that $\varphi(z(t))$ satisfies the differential inequality

$$\frac{d}{dt}\varphi(z(t)) \leq a_1\varphi(z(t)) + a_2 + |h(t)|^2, \quad t > 0 \quad (4.8)$$

where a_1, a_2 are constants.

For solution with initial conditions in $H_{1,0}$ and bounded in $L^2(0,\ell)$, (3.5) implies that $\int_t^{t+r} \varphi(z(s))ds$ is less than a fixed constant for t sufficiently large, then we can use the Uniform Gronwall Lema, see [9, pg. 89], to obtain the result of the theorem. ∎

As consequence of the two previous theorems and the relation 2.14 we have

Theorem 4.3 *Under the above conditions, the dynamical system $z(t)$ given by 1.2 has a compact global attractor in $H_{1,0}$. Moreover, for $v(0) \in H_{1,0}$, $u(t) + v(t)$, given by 2.1 has also a compact global attractor in $H_{1,0}$.*

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