On the Behavior of a Rod's End

Aloisio F. Neves*
IMECC – UNICAMP - CP 6065
13081–970, Campinas, SP, Brazil

Abstract

In this paper we discuss some properties of a one dimensional wave equations. Our main purpose is to study how the excitations are transmitted through the system's rod. More precisely, we will obtain properties giving the precise behavior of one end of the rod from data given in the other end.

1. Introduction

In this paper we discuss some properties of the following nonhomogeneous mixed boundary value problem:

$$
\begin{gathered}
u_{tt} - u_{xx} = f(t,x); \quad 0 < x < \ell, \;\; t > 0, \\
\text{BC} \begin{cases} u(t,0) = \mu(t) \\ u_x(t,\ell) = \eta(t), \end{cases} \qquad \text{IC} \begin{cases} u(0,x) = \varphi(x) \\ u_t(0,x) = \psi(x). \end{cases}
\end{gathered} \tag{1.1}
$$

We study this problem looking for results that give us the behavior of the end $x = \ell$ of the rod, from data given in the other end $x = 0$. More precisely, we want information about the velocity $u_t(t,\ell)$ and the stress $u_x(t,\ell)$ in the end of the rod. We suppose that the function $\mu(t)$ is given, and in the other side, the function $\eta(t) \in L^2(0,T)$ is unknown a priori. We use the ideas of Lions and Magenes [5], of solutions in transposition sense, to prove existence and uniqueness of solution for our case that has mixed boundary condition. We obtain sharp properties about the behavior of the ends of the rod, in particular, an auxiliary function, that depends on $\mu(t)$, that gives the precise behavior of the other end. The techniques used are well known and the paper is finished with an application to a model of a mechanical system.

---

*E-mail: aloisio@ime.unicamp.br - Research partially supported by PRONEX and FAPESP, Brazil.

2. Existence and Uniqueness of Solutions

We will start by given the definition of solution in sense of transposition (cf. Lions and Magenes, [5]) for case where the boundary conditions are mixed.

In the homogeneous case:

$$\begin{aligned} &u_{tt} - u_{xx} = f(t,x); \quad 0 < x < \ell, \;\; t > 0, \\ &u(t,0) = 0, \\ &u_x(t,\ell) = 0, \end{aligned} \tag{2.1}$$

as usual in the context of wave equations, we introduce the variable $u_t = v$ and look at the problem as an evolution equation in Hilbert space

$$\mathcal{H} = H_{1,0} \times L^2(0,\ell),$$

where $H_{1,0}$ is the Hilbert Space

$$H_{1,0} = \{u \in H^1(0,\ell) : u(0) = 0\}$$

with the inner product

$$(u_1, u_2) = \int_0^\ell u_1' u_2' dx.$$

We have, by standard methods of semigroups, the following result:

Proposition 2.1 *The operator A is the infinitesimal generator of a C^0 Group, therefore for every initial condition $(\varphi,\psi) \in \mathcal{H}$ and $f \in L^1(0,T;L^2(0,\ell))$ there exists a unique solution of (2.1), $u \in C(0,T;H_{1,0}) \cap C^1(0,T;L^2(0,\ell))$. Moreover, there exists a constant $C = C(T)$ such that*

$$\|u_t\|_{L^2(0,\ell)} + \|u_x\|_{L^2(0,\ell)} \le C\left[\|(\varphi,\psi)\|_{\mathcal{H}} + \|f\|_{L^1(0,T;L^2(0,\ell))}\right]$$

and also

$$u(t,\ell)^2 + \int_0^T u_x(t,0)^2 dt \le C\left[\|(\varphi,\psi)\|^2_{\mathcal{H}} + \|f\|^2_{L^1(0,T;L^2(0,\ell))}\right]. \tag{2.2}$$

The estimate (2.2) can be obtained as follows: The first term is estimated using Schwarz and Poincaré inequalities

$$u(t,\ell)^2 = \int_0^\ell \frac{d}{dx}(u^2)dx = \int_0^\ell 2uu_x dx \le c\|u_x\|^2_{L^2(0,\ell)}$$

and the second term is estimated using *multiplier techniques*. We multiply the differential equation in (2.1) by $q(x)u_x$, with $q \in C^1([0,\ell])$ such that $q(0) = -1$ and $q(\ell) = 0$, and integrate by parts with respect to x and t, see J.L. Lions, [3], for details.

The problem with nonhomogeneous boundary conditions:

$$u_{tt} - u_{xx} = 0; \quad 0 < x < \ell, \ \ t > 0,$$
$$\text{BC} \begin{cases} u(t,0) = \mu(t) \\ u_x(t,\ell) = \eta(t), \end{cases} \qquad \text{IC} \begin{cases} u(0,x) = \varphi(x) \\ u_t(0,x) = \psi(x), \end{cases} \tag{2.3}$$

in the case where μ and η are smooth functions, can be treated as usual through an appropriate change of variables. In fact by taking

$$u = U + w,$$

where

$$U = \left(1 + \frac{x}{\ell} - \frac{x^2}{2\ell^2}\right)\mu(t) + \frac{x^2}{2\ell}\eta(t),$$

one can see that w will satisfy the problem (2.1).

The general case, when μ and η are L^2 functions, the solutions will be understood in the sense of transposition, see [3], [4], [5]. For every $f \in L^1(0,T;L^2(0,\ell))$, in view of the time-reversibility and as a consequence of the previous Proposition, the problem

$$\bar{u}_{tt} - \bar{u}_{xx} = f(t,x); \quad 0 < x < \ell, \ \ t > 0,$$
$$\text{BC} \begin{cases} \bar{u}(t,0) = 0 \\ \bar{u}_x(t,\ell) = 0, \end{cases} \qquad \text{IC} \begin{cases} \bar{u}(T,x) = 0 \\ \bar{u}_t(T,x) = 0, \end{cases}$$

has a unique solution $\bar{u} \in C(0,T;H_{1,0}) \cap C^1(0,T;L^2(0,\ell))$ satisfying

$$\|\bar{u}(t,\ell)\|_{L^2(0,T)} + \|\bar{u}_x(t,0)\|_{L^2(0,T)} \le C\|f\|_{L^1(0,T;L^2(0,\ell))}. \tag{2.4}$$

Multiplying by $\bar{u}$ the differential equation in (2.3) and integrating formally by parts with respect to x and t, we obtain the following identity:

$$\begin{aligned} \int_0^T \int_0^\ell uf\,dxdt &= -\int_0^\ell \varphi(x)\bar{u}_t(0,x)dx + \langle \psi, \ \bar{u}(0)\rangle \\ &\quad + \int_0^T \eta(t)\bar{u}(t,\ell)dt + \int_0^T \mu(t)\bar{u}_x(t,0)dr. \end{aligned} \tag{2.5}$$

The right hand side of (2.5) is well defined considering $\varphi \in L^2(0,\ell)$, $\mu, \eta \in L^2(0,T)$ and understanding $\langle \psi, \ \bar{u}(0)\rangle$ in the sense of the duality of $H_{1,0}$, applying $\psi \in (H_{1,0})'$ to the elements $\bar{u}(0) \in H_{1,0}$.

Definition 2.1 *A function $u : [0,T] \to L^2(0,\ell)$ is a (weak) solution of (2.3) in the sense of transposition if (2.5) holds for every $f \in L^1(0,T;L^2(0,\ell))$*

We have the following result.

Proposition 2.2 *For every $\mu, \eta \in L^2(0,T)$, $\varphi \in L^2(0,\ell)$ and $\psi \in (H_{1,0})'$ there exists a unique solution u (in the sense of transposition) of (2.3) in the class*

$$u \in C(0,T;L^2(0,\ell)) \tag{2.6}$$

$$u_t \in C(0,T;(H_{1,0})'). \tag{2.7}$$

Moreover there exists a positive constant C such that

$$\|u\|_{L^\infty(0,T;L^2(0,\ell))} \le C\left[\|\mu\|_{L^2} + \|\eta\|_{L^2} + \|\varphi\|_{L^2} + \|\psi\|_{(H_{1,0})'}\right] \tag{2.8}$$

Proof. Under the hypotheses and in view of the estimate (2.4) the right hand side of (2.5) defines a linear and continuous form L on $f \in L^1(0,T;L^2(0,\ell))$. Therefore, there exists a unique $u \in L^\infty(0,T;L^2(0,\ell))$ satisfying (2.5). Furthermore, since $\|u\|_{L^\infty(0,T;L^2(0,\ell))} = \|L\|$ we have the estimate (2.8).

The properti (2.6) can be proved by a density argument. When the data μ and η are smooth, the solution of (2.3) satisfies (2.6), therefore, one can use (2.8) to prove (2.6). The regularity property (2.7) needs a more refined argument. It can be proved proceeding as in [4, theorem 4.2, p. 46]. ∎

Hereafter, we will be assuming the function $\mu(t)$ regular, C^2 for instance, the functions $\eta \in L^2_{loc}(0,\infty)$, $\varphi \in H^2(0,\ell)$ and $\psi \in H^1(0,\ell)$. We will also be assuming the compatibility conditions:

$$\varphi(0) = \mu(0), \quad \psi(0) = \dot\mu(0) \quad \text{and} \quad \eta(0) = \varphi'(\ell).$$

In order to use the D'Alembert formula as a useful tool in our problems, we will extend the Initial Conditions $\varphi(x)$ and $\psi(x)$ in such way that the Boundary Conditions in (2.3) will be automatically satisfied:

$$\begin{array}{ll} \varphi 1) & \varphi(-t) = -\varphi(t) + \mu(t) + \mu(0) - \dot\mu(0)t \\ \varphi 2) & \varphi(\ell+t) = \varphi(\ell-t) + \int_0^t \eta(s)ds + \eta(0)t \\ \psi 1) & \psi(-t) = -\psi(t) + \dot\mu(t) + \dot\mu(0) \\ \psi 2) & \psi(\ell+t) = \psi(\ell-t) + \eta(t) - \eta(0). \end{array} \tag{2.9}$$

It is easy to see that (2.9) extends $\varphi(x)$ and $\psi(x)$ to the whole line in an unique way. Therefore, hereafter we will suppose that $\varphi(x)$ and $\psi(x)$ are defined in whole line and we can state the following result:

Theorem 2.1 *If $f(t,.)$ is extended, outside of $[0,\ell]$, odd with respect to $x=0$ and even with respect to $x=\ell$, then the function $u(t,x)$ defined by*

$$\begin{aligned} u(t,x) &= \frac{\varphi(x+t)+\varphi(x-t)}{2}+\frac{1}{2}\int_{x-t}^{x+t}\psi(s)ds \\ &+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(\tau,\xi)d\xi d\tau \end{aligned} \tag{2.10}$$

is the solution of the problem (1.1).

Proof: The D'Alembert formula implies that $u(t,x)$ given by (2.10) satisfies $u_{tt}-u_{xx}=f(t,x)$, and the initial conditions.

Integrating $\psi 1)$ of (2.9) from 0 to t we obtain

$$\int_{-t}^{t}\psi(s)ds=\mu(t)-\mu(0)+\dot{\mu}(0)t, \tag{2.11}$$

and, from $\varphi 1)$,

$$\varphi(t)+\varphi(-t)=\mu(t)+\mu(0)-\dot{\mu}(0)t \tag{2.12}$$

therefore, adding (2.11) and (2.12), and using that $f(\tau,.)$ is odd with respect to $x=0$, we can see that $u(t,x)$ satisfies

$$u(t,0)=\mu(t).$$

On the other hand, differentiating $\varphi 2)$ we obtain

$$\varphi'(\ell+t)+\varphi'(\ell-t)=\eta(t)+\eta(0),$$

from $\psi 2)$,

$$\psi(\ell+t)-\psi(\ell-t)=\eta(t)-\eta(0).$$

Therefore

$$\begin{aligned} u_x(t,\ell) &= \frac{\varphi'(\ell+t)+\varphi'(\ell-t)}{2}+\frac{\psi(\ell+t)-\psi(\ell-t)}{2} \\ &+\frac{1}{2}\int_0^t(f(\tau,\ \ell+(t-\tau))-f(\tau,\ \ell-(t-\tau))d\tau \\ &= \eta(t), \end{aligned}$$

since $f(\tau,.)$ is even with respect to $x=\ell$. The theorem is proved. ∎

3. Results on the Boundary Behavior

Theorem 3.1 *Setting $\xi(t) = u_x(t,0)$, the solutions of (1.1) satisfy for $t > \ell$:*

$$u_x(t,\ell) = \frac{\dot\mu(t+\ell)+\xi(t+\ell)}{2} + \frac{\xi(t-\ell)-\dot\mu(t-\ell)}{2} \tag{3.1}$$

$$\begin{aligned} u_t(t,\ell) &= \frac{\dot\mu(t+\ell)+\xi(t+\ell)}{2} - \frac{\xi(t-\ell)-\dot\mu(t-\ell)}{2} \\ &\quad + \int_0^t f(\tau,\ \ell+(t-\tau)d\tau \end{aligned} \tag{3.2}$$

where $f(t,.)$ is extended in order to be odd with respect to $x=0$ and even with respect to $x=\ell$

Proof: It is enough to prove the case where $f \equiv 0$. From (2.10) we have

$$\xi(t) = u_x(t,0) = \frac{\varphi'(t)+\varphi'(-t)}{2} + \frac{\psi(t)-\psi(-t)}{2}. \tag{3.3}$$

¿From $\varphi 1)$ and $\psi 1)$ of (2.9), we have respectively

$$\varphi'(-t) = \varphi'(t) - \dot\mu(t) + \dot\mu(0) \quad \text{and} \quad \psi(-t) = -\psi(t) + \dot\mu(t) + \mu(0).$$

Therefore (3.3) implies that,

$$\varphi'(t) + \psi(t) = \dot\mu(t) + \xi(t). \tag{3.4}$$

Plugging (3.4) in (3.3) we obtain

$$\varphi'(-t) - \psi(-t) = \xi(t) - \dot\mu(t). \tag{3.5}$$

Since

$$u_x(t,\ell) = \frac{\varphi'(\ell+t)+\psi(\ell+t)}{2} + \frac{\varphi'(\ell-t)-\psi(\ell-t)}{2} \tag{3.6}$$

and

$$u_t(t,\ell) = \frac{\varphi'(\ell+t)+\psi(\ell+t)}{2} - \frac{\varphi'(\ell-t)-\psi(\ell-t)}{2}, \tag{3.7}$$

the theorem follows from (3.4) and (3.5). ∎

Now we will study the influence of $\mu(t)$ on the behavior of the other side. Therefore, we will consider the problem (1.1) with null forcing term, $f=0$, and also with null initial condition, $\varphi=\psi=0$. Consider also the following auxiliary function $a(t)$, $t \geq 0$, defined recursively by:

$$a(t) = \dot\mu(t-\ell) - |a(t-2\ell)|, \quad t \geq 2\ell, \tag{3.8}$$

where $a(t)$ for $0 \le t \le 2\ell$ is given by:

$$a(t) = \begin{cases} 0, & 0 \le t \le \ell \\ \dot{\mu}(t-\ell), & \ell \le t \le 2\ell \end{cases} \tag{3.9}$$

We have the following result.

Theorem 3.2 *If $\eta(t) = 2a^-(t)$, then the solution u of (2.3) with $\varphi = \psi = 0$ satisfies*

$$u_t(t,\ell) = 2a^+(t),$$

where a^+ and a^- are respectively the positive and negative part of the function a.

Proof: From (3.6) and (3.7) we have that

$$u_t(t,\ell) - u_x(t,\ell) = -\varphi'(\ell - t) + \psi(\ell - t) \tag{3.10}$$

and also, since $\varphi'(\ell - t) = \psi(\ell - t) = 0$, for $t \in [0, \ell]$,

$$u_t(t,\ell) = u_x(t,\ell), \quad \forall t \in [0,\ell]$$

that is

$$u_t(t,\ell) = 2a(t) + \eta(t) = 2a^+(t), \quad \forall t \in [0,\ell].$$

Next one can see using the equalities (2.9), that

$$\varphi'(\ell + t) + \psi(\ell + t) = -\varphi'(\ell - t) + \psi(\ell - t) + 2\eta(t) \tag{3.11}$$

and, for $t \ge \ell$,

$$\begin{aligned} &\varphi'(\ell + t) + \psi(\ell + t) = -[\varphi'(t-\ell) + \psi(t-\ell)] + 2\dot{\mu}(t-\ell) + 2\eta(t), \\ &\varphi'(\ell - t) - \psi(\ell - t) = \varphi'(t-\ell) + \psi(t-\ell) - 2\dot{\mu}(t-\ell). \end{aligned} \tag{3.12}$$

Therefore, when $t \ge \ell$

$$u_t(t,\ell) = -[\varphi'(t-\ell) + \psi(t-\ell)] + 2\dot{\mu}(t-\ell) + \eta(t), \tag{3.13}$$

but $\varphi'(t-\ell) = \psi(t-\ell) = 0$, for $t \in [\ell,\ 2\ell]$, then

$$u_t(t,\ell) = 2\dot{\mu}(t-\ell) + \eta(t) = 2a(t) + \eta(t) = 2a^+(t), \quad \forall t \in [\ell, 2\ell].$$

Now, since $\eta(t) = 2a^-(t)$, we have

$$a(t) + \eta(t) = |a(t)|, \tag{3.14}$$

therefore, using the definition of the function $a(t)$,

$$\begin{aligned} 2a(t)+\eta(t) &= 2\dot{\mu}(t-\ell)-2|a(t-2\ell)|+\eta(t) \\ &= 2\dot{\mu}(t-\ell)-2a(t-2\ell)-2\eta(t-2\ell)+\eta(t). \end{aligned}$$

Then, for $t\in[2\ell,\ 3\ell]$, since

$$a(t-2\ell)=0 \ \text{ and } \ \varphi'(3\ell-t)=\psi(3\ell-t)=0,$$

we obtain from (3.13), using (3.11) with $t-2\ell$ instead of t, that

$$u_t(t,\ell)=2a(t)+\eta(t)=2a^+(t), \quad \forall t\in[2\ell,3\ell].$$

To complete the prove we will show that

$$u_t(t,\ell)=2a(t)+\eta(t) \tag{3.15}$$

remains true for every $t>3\ell$.

The proof will be proceeded by induction. Suppose (3.15) holds for $t\in[0,T_0]$, where $T_0\geq 3\ell$. We will prove (3.15) for $t\in[0,T_0+2\ell]$. In fact, using (3.13) and (3.14)

$$\begin{aligned} 2a(t)+\eta(t) &= 2\dot{\mu}(t-\ell)-2|a(t-2\ell)|+\eta(t) \\ &= u_t(t,\ell)+\varphi'(t-\ell)+\psi(t-\ell)-2|a(t-2\ell)| \\ &= u_t(t,\ell)+\varphi'(t-\ell)+\psi(t-\ell)-2a(t-2\ell)-2\eta(t-2\ell), \end{aligned}$$

changing t by $t-2\ell$ in the equalities (3.12),

$$2a(t)+\eta(t)=u_t(t,\ell)-\varphi'(3\ell-t)+\psi(3\ell-t)-2a(t-2\ell),$$

and using our induction hypothesis, we obtain

$$2a(t)+\eta(t)=u_t(t,\ell)+[-\varphi'(3\ell-t)+\psi(3\ell-t)]-[u_t(t-2\ell,\ell)-u_x(t-2\ell,\ell)].$$

The result now follows from (3.10). ∎

4. Application

Consider the problem that has different levels of stress, 0 and m depending on the direction of the movement, More precisely:

$$\begin{gathered} u_{tt}-u_{xx}=0 \quad 0<x<\ell,\ \ t>0, \\ \text{BC}\begin{cases} u(t,0)=\mu(t) \\ u_x(t,\ell)\in\rho(u_t(t,\ell)), \end{cases} \qquad \text{IC}\begin{cases} u(0,x)=\varphi(x) \\ u_t(0,x)=\psi(x), \end{cases} \end{gathered} \tag{4.1}$$

where ρ is a real multi valued function whose graph has the following shape:

The problem (4.1) has existence and uniqueness of solutions, the proof can be done using the Theory of Maximal Monotone Operators, see [2], and, it can be considered as a model of pump mechanisms, see [1] and the references therein.

We should observe that the function ρ, for $x < 0$, doesn't need to be necessarily a straight line as it was showed in the above diagram, but a decreasing graph. We should also observe that, $u_t(t, \ell) \geq 0$, $t > 0$, in (4.1), is equivalent to the estimate $0 \leq u_x(t, \ell) \leq m$ for the stress. What we want is to obtain a condition on $\mu(t)$ that implies this kind of behavior. We have the following result.

Theorem 4.1 *Let u be the solution of (4.1), with null initial condition, $\varphi = \psi = 0$. Then $\mu(t)$ satisfying $2a(t) \geq -m$ for $t \in [0, T]$ is a necessary and sufficient condition to $u_t(t, \ell) \geq 0$, for t in this interval. Moreover the solution u satisfies:*

$$\begin{aligned} u_t(t, \ell) &= 2a^+(t), \\ u_x(t, \ell) &= 2a^-(t), \quad \forall t \in [0, T], \end{aligned} \tag{4.2}$$

in particular $u(t, \ell) = \int_0^t 2a^+(s) ds$, where a^+ and a^- are respectively the positive and negative part of the function a.

Proof. Suppose $2a(t) \geq -m$ for $t \in [0, T]$ and let $\tilde{u}$ be the solution of (2.3) with null initial condition and $\eta(t) = 2a^-(t)$, then, using the result of theorem 3.2, we have that the couple $(\tilde{u}_t(t, \ell),\ \tilde{u}_x(t, \ell))$ satisfies $\tilde{u}_x(t, \ell) \in \rho(\tilde{u}_t(t, \ell))$ for $t \in [0, T]$. Therefore, from the uniqueness of solutions of the two problems, we have that the two solutions $\tilde{u}$ and u coincide in the interval $[0, T]$. Then, u satisfies (4.2), and, in particular, $u_t(t, \ell) \geq 0$ for t in this interval.

On the other hand, setting $\eta(t) = u_x(t, \ell)$, we have, according to Trace Theorem p. 15 of [6], that $\eta(t) \in L^2(0, T)$. So, in particular, the formula (2.10), with $f \equiv 0$, can be applied to our problem.

First of all observe that for $t \in [0, \ell]$, $u_x(t, \ell) = u_t(t, \ell) = 0$, this can be seen using (3.10) with $\varphi'(\ell - t) = \psi(\ell - t) = 0$, to show that $u_x(t, \ell) = u_t(t, \ell)$, and then the boundary condition

$$u_t(t,\ell) = \rho(u_t(t,\ell)),$$

implies the observation.

For $t \in [\ell, 2\ell]$, we have, doing the same computation we did in the theorem 3.2,

$$u_t(t,\ell) = 2a(t) + \eta(t) \tag{4.3}$$

but then, since $u_t(t,\ell) \geq 0$ implies $\eta(t) \leq m$, we have from (4.3)

$$2a(t) \geq -m, \quad \text{for} \quad t \in [\ell, 2\ell].$$

In particular, (4.2) holds, and,

$$a(t) + \eta(t) = |a(t)|, \quad \forall \ t \in [0, 2\ell] \cap [0, T]. \tag{4.4}$$

Of course, if $T \leq 2\ell$ the theorem is proved, otherwise, using (4.3), (4.4) and the argument used in the theorem 3.2, we can prove that (4.3) remains true for $t \in [0, 4\ell]$. Therefore repeating the argument we just have done we will go one step further, and thus so on. ∎

References

[1] Boldrini, J.L. and Neves, A.F., *Periodic motions of sucker rod pumping systems*, Nonlinear Analysis T.M.A., vol. 22, nº 7, (1994), 797-808.

[2] Brézis, H., *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[3] Lions, J.-L., *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev. vol. 30, nº 1, 1988.

[4] Lions, J.-L., *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*, Tome 1, Masson, Paris, 1988.

[5] Lions J.-L. and Magenes, E., *Problèmes aux limites non homogènes et applications*, vols. 1 and 2, Dunod, Paris, 1968.

[6] Nečas, J., *Les Méthodes Directes en Thérie des Équations Elliptiques*, Masson, 1967