# On the finiteness homological properties of some modules over metabelian Lie algebras

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**Abstract** *We characterise the modules $B$ of homological type $FP_m$ over a finitely generated Lie algebra $L$ such that $L$ is a split extension of an abelian ideal $A$ and an abelian subalgebra $Q$ and $A$ acts trivially on $B$. The characterisation is in terms of the invariant $\Delta$ introduced by R. Bryant and J. Groves and is a Lie algebra version of the still open generalised $FP_m$-Conjecture for metabelian groups. The case $m = 1$ is treated separately as there the characterisation is proved without restrictions on the type of the extension.*

## Introduction

The purpose of this paper is to formulate and establish in the split extension case the counterpart of the generalised $FP_m$–Conjecture suggested in [K 2, Conjecture 6] for finitely generated metabelian Lie algebras. The original $FP_m$–Conjecture [B-G 1] describes when a finitely generated metabelian group $G$ is of homological type $FP_m$ in terms of the invariant

$$\Sigma^1(G) \subseteq S(G) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in Hom(G, \mathbb{R}) \setminus \{0\}\}.$$

Though the $FP_m$–Conjecture for metabelian groups is still open it is known to hold in the following cases : $m = 2$ [B-S], $m = 3$ and $G$ a split extension of abelian groups [B-H], $G$ of finite Prufer rank [Å], $G$ a torsion analogue of a group of finite Prufer rank [K 1]. A proof of the pro-p version of the $FP_m$-Conjecture for finitely generated metabelian pro-p groups suggested in [King] could be found in [K 3].

The question of finite presentability of metabelian Lie algebras is addressed in [B-G 1] and [B-G 2] where R. Bryant and J. Groves give a characterization of finite presentability in terms of the invariant $\Delta$. The question what restrictions the finite presentability imposes on the structure of a Lie algebra is treated in [W] where links between finite presentability and an HNN-construction for Lie algebras are investigated. This approach gives the surprising result that for a finitely presented Lie algebra without free subalgebras of rank two the ideals of codimension one are finitely generated as subalgebras.

In this paper we examine some finiteness homological properties of modules over metabelian Lie algebras. A module over a Lie algebra $L$ (over a field $K$) is a module over its universal enveloping algebra $U(L)$. We are primary interested in modules $B$ over $L$ such that some abelian ideal $A$ of $L$ with $L/A$ abelian has the property that $A$ acts trivially on $B$. This includes the case of the trivial module $K$.

**Theorem A.** *Suppose $L$ is a finitely generated Lie algebra over a field $K$, $A$ is an abelian ideal in $L$ with $Q = L/A$ abelian and $B$ is a finitely generated (right) module over the universal algebra $U(Q)$ of $Q$. Then the following are equivalent:*

*1. $B$ is finitely presented as a module over $L$ (i.e. as an module over the universal algebra $U(L)$ of $L$) where the action of $L$ is via the canonical projection $\pi : L \to Q$.*

*2. $A \otimes_K B$ is finitely generated over the universal algebra $U(Q)$, where $U(Q)$ acts via the diagonal homomorphism $\partial : U(Q) \to U(Q) \otimes U(Q)$ sending $q \in Q$ to $q \otimes 1 + 1 \otimes q$.*

*3. $\Delta(Q, A) \cap -\Delta(Q, B) = 0$.*

**Corollary B.** *Suppose $L$ is a finitely generated Lie algebra over a field with an abelian ideal $A$ such that $Q = L/A$ is abelian. Then $L$ is finitely presented as a Lie algebra if and only if $A$ is finitely presented as a module over $U(L)$.*

The group counterpart of the equivalence of conditions 1 and 3 from Theorem A is considered in [K 2, Prop 4]. There only the case of split extension groups is solved leaving the question for non-split groups open.

The main result of this paper is the proof of the Lie algebra version of the generalised $FP_m$–Conjecture suggested in [K 2, Conjecture 6]. In the Lie algebra case the Bryant-Groves invariant $\Delta$ will play the role of the Bieri-Strebel invariant $\Sigma^1(G)^c$. Note we establish the result only for split extensions metabelian Lie algebras. The group theoretic analogue of Theorem C is still an open problem.

**Theorem C.** *In the conditions of Theorem A if $L$ is a split extension of $A$ by $Q$ then $B$ is of type $FP_m$ if and only if $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action if and only if whenever $[v_2], \ldots, [v_{m+1}] \in \Delta(Q, A), [v_1] \in \Delta(Q, B)$, $0 = [v_1] + \ldots + [v_{m+1}]$ then all $[v_i]$ are trivial.*

## 1. Preliminaries on the invariant $\Delta$

The classification of the finitely presented Lie algebras over a field $K$ given in [B-G 1], [B-G 2] depends on the invariant $\Delta(Q, A)$, where $A$ is an abelian ideal of $L$ with abelian quotient $Q = L/A$. Let $K[Q]$ be the polynomial algebra on $n$ commuting variables where $n$ is the dimension of $Q$, so $K[Q]$ is isomorphic to the universal enveloping algebra $U(Q)$ of $Q$. By definition

$$\Delta(Q, A) = \{[\chi] \mid \chi \in Hom_K(Q, \overline{K}((t))), \chi \text{ is extendable to a ring homomorphism}$$

$$\chi' : K[Q]/Ann(A) \to \overline{K}((t))\},$$

where $[\chi] = \chi + Hom(Q, \overline{K}[[t]]) \in Hom(Q, \overline{K}((t))/Hom(Q, \overline{K}[[t]]), \overline{K}$ is the algebraic closure of $K$, $\overline{K}((t))$ is the field of fractions of $\overline{K}[[t]]$ and $Ann(A)$ is the annihilator of $V$ in $K[Q]$. The main result of [B-G 1], [B-G 2] asserts that $L$ is finitely presented as a Lie algebra if and only if the exterior square of $A$ is finitely generated over $K[Q]$ via the diagonal adjoint action if and only if whenever $[\chi_1], [\chi_2] \in \Delta(Q, A) \setminus \{0\}$ the sum $[\chi_1] + [\chi_2]$ is non-trivial i.e. $\Delta(Q, A)$ has no non-trivial antipodal elements.

## 2. Proof of Proposition 1.

This section is devoted to the proof of one of the implications of Theorem A. Our proof uses the techniques developed in [B-G 1, section 2]. As the proof is very long and technical it is split in several steps.

**Proposition 1.** *In the assumptions of Theorem A 2. implies 1.*

**Proof.** 1. Let $a_1, \ldots, a_{s_0}, y_1, \ldots, y_n$ be a generating set of $L$ such that $a_1, \ldots, a_{s_0} \in A$ and the images $x_1, \ldots, x_n$ of $y_1, \ldots, y_n$ in $Q = L/A$ form a basis of

$Q$. Furthermore for all $1 \leq j < i \leq s_0$ assume $a_{i,j} = [y_i, y_j] \in \{a_1, \ldots, a_{s_0}\} \cup \{0\}$. Let $F$ be the free Lie algebra on the generators $X_1, \ldots, X_n$ and $U(F)$ be its universal algebra. We define

$$\rho : U(F) \to U(Q)$$

to be the homomorphism of $K$-algebras sending $X_i$ to $x_i$,

$$\nu : U(Q) \to U(F)$$

the linear map sending $x_{i_1} \ldots x_{i_k}$ to $X_{i_1} \ldots X_{i_k}$ for $i_1 \leq \ldots \leq i_k$ and

$$\varphi : U(F) \to U(L)$$

the $K$-algebra homomorphism sending $X_i$ to $y_i$. Then $\rho = \tau \circ \varphi$ where

$$\tau : U(L) \to U(Q)$$

is the homomorphism of associative $K$–algebras induced by the canonical projection $L \to Q$.

The elements $X_{i_1}^{\alpha_1} \ldots X_{i_k}^{\alpha_k}$ and $x_1^{\beta_1} \ldots x_n^{\beta_n}$ of $U(F)$ and $U(Q)$ are called monomials of degree $\alpha_1 + \ldots + \alpha_k$ and $\beta_1 + \ldots + \beta_n$ respectively. If $f = f_1 \otimes f_2$ is a monomial in $U(F) \otimes U(F)$ (resp. $U(Q) \otimes U(Q)$ ) the degree of $f$ is $deg(f_1) + deg(f_2)$. For a general element $f$ of $U(F)$, $U(Q)$, $U(F) \otimes U(F)$ or $U(Q) \otimes U(Q)$ the degree $deg(f)$ is the maximal degree of the monomials in the support of $f$. By definition for a subspace $J$ of $U(F), U(Q)$, $U(F) \otimes U(F)$ or $U(Q) \otimes U(Q)$ the subspace $J_t$ is spanned by all elements of $J$ of degree at most $t$.

Note that $U(L)$ acts on $A$ via the adjoint (right) action. As $A$ is abelian this makes $A$ right $U(Q)$–module. If $f = gx_i$ is a monomial in $U(Q)$ the image of $a \in A$ under the action of $f$ denoted by $a \circ f$ is $(a \circ g) \circ x_i = [a \circ g, y_i]$ and this definition is extended by linearity for arbitrary elements of $U(Q)$. If $f \in U(L)$ we write $a \circ f$ for $a \circ \tau(f)$.

2. We adopt the notations from [B-G 1] and for an element $\lambda \in U(Q)$ write $\lambda(u)$, $\lambda(v)$ and $\lambda(d)$ for $\lambda \otimes 1, 1 \otimes \lambda \in U(Q) \otimes U(Q)$ and the image of $\lambda$ under the diagonal homomorphism

$$\delta : U(Q) \to U(Q) \otimes U(Q)$$

sending $q \in Q$ to $q \otimes 1 + 1 \otimes q$. Similarly we define for an element $\lambda \in U(F)$ elements $\lambda(U), \lambda(V)$ and $\lambda(D)$ in $U(F) \otimes U(F)$.

Now let $b_1, \ldots, b_m$ be a generating set of $B$ over $U(Q)$. We remind the reader that $a_1, \ldots, a_{s_0}$ is a generating set of $A$ as a $U(Q)$-module. Since $U(Q)$ is a Noetherian ring the annihilator ideals $Ann_{U(Q)} b_i$ and $Ann_{U(Q)} a_j$ are finitely generated over $U(Q)$ i.e.

$$Ann_{U(Q)} b_i = \sum_{t \geq 1} g_{i,t} U(Q), Ann_{U(Q)} a_j = \sum_{t \geq 1} \widetilde{g}_{j,t} U(Q)$$

$$Ann_{U(Q) \otimes U(Q)} (b_i \otimes a_j) = Ann_{U(Q)} (b_i) \otimes U(Q) + U(Q) \otimes Ann_{U(Q)} (a_j) \qquad (1)$$

We claim that for every $1 \leq r \leq m, 1 \leq s \leq s_0, 1 \leq k \leq n$ there exist elements $\phi_{rskj}, \psi_{rskj}, f_{rski}(d) \in U(Q) \otimes U(Q)$ and an integer $l$ independent of $r, s$ and $k$ such that

$$x_k(v)^{l+1} + \sum_{0 \leq i \leq l} x_k(v)^i f_{rski}(d) + \sum_{j \geq 1} g_{rj}(u) \phi_{rskj} + \sum_{j \geq 1} \widetilde{g}_{sj}(v) \psi_{rskj} = 0 \qquad (2)$$

In the case when $B = A$ formula (2) is proved in [B-G 1]. The general case can be proved using the same argument. For completeness we sketch a proof. The $U(Q)$-submodule of $B \otimes A$ generated by $\{b_r \otimes (a_s \circ x_k^j)\}_{j \geq 0}$ is finitely generated, say by $\{b_r \otimes (a_s \circ x_k^j)\}_{0 \leq j \leq l}$. Then for some $f_{rski}(d) \in U(Q) \otimes U(Q)$

$$x_k(v)^{l+1} + \sum_{i \leq l} x_k(v)^i f_{rski}(d) \in Ann_{U(Q) \otimes U(Q)} (b_r \otimes a_s)$$

Now (2) follows immediately from (1).

3. Let $\partial : \oplus_{i \leq m} e_i U(L) \to B$ be the homomorphism of $U(L)$–modules sending the generator $e_i$ of the free module $\oplus_{i \leq m} e_i U(L)$ to $b_i$. Then $B$ is finitely presented over $U(L)$ if and only if $Ker\ \partial$ is finitely generated over $U(L)$. Define

$$\widetilde{X} = \{e_i \varphi \nu(g_{ij})\}_{i,j \geq 1},$$

$$X_t = \{e_i \varphi(f_1)(a_j \circ \rho(f_2)) \mid f_1, f_2 \text{ monomials in } U(F), deg(f_1 f_2) \leq t, i \leq m, j \leq s_0\}$$

and write $V_t$ for the $U(L)$-submodule of $Ker \partial \subseteq \oplus_{i \leq m} e_i U(L)$ generated by the finite set $X_t \cup \widetilde{X}$. We aim to prove that for sufficiently big $t$

$$V_t = V_{t+1}.$$

Then $V = \cup_{m\geq 1} V_m$ is finitely generated over $U(L)$ and $Ker\ \partial / V$ is a surjective image of the quotient of $Ker\ \partial$ through the $U(L)$-submodule generated by $\cup_{t \geq 1} X_t$. This quotient is the kernel of the homomorphism of $U(Q)$–modules $\oplus_{i\leq m} e_i U(Q) \to B$ sending $e_i$ to $b_i$. The latter is finitely generated over $U(Q)$ as $U(Q)$ is Noetherian and hence $Ker\ \partial / V$ is finitely generated over $U(Q)$. Finally as $V$ is finitely generated over $U(L)$ we deduce that $Ker\ \partial$ is finitely generated over $U(L)$, as required.

**Lemma 1.1.** *If $f_1, f_2, f_3$ are monomials in $U(F)$ such that $deg(f_1 f_2 f_3) < 2t$ then*

$$e_i \varphi(f_1)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3)) \in V_t$$

**Proof.** We induct on $deg(f_1)$. If $f_1 = 1$ then $deg(f_2) < t$ or $deg(f_3) < t$, say $deg(f_3) < t$. Then $e_i(a_k \circ \rho(f_3))$ and consequently $e_i(a_k \circ \rho(f_3))(a_j \circ \rho(f_2))$ are elements of $V_t$.

If $f_1 = gY$ for some $Y = X_j$ we have

$$e_i \varphi(f_1)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3)) = e_i \varphi(g)[\varphi(Y), a_j \circ \rho(f_2)](a_k \circ \rho(f_3)) +$$

$$e_i \varphi(g)(a_j \circ \rho(f_2))[\varphi(Y), a_k \circ \rho(f_3)] + e_i \varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3))\varphi(Y) =$$

$$-e_i \varphi(g)(a_j \circ \rho(f_2 Y))(a_k \circ \rho(f_3)) - e_i \varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3 Y))$$

$$+ e_i \varphi(g)(a_j \circ \rho(f_2))(a_k \circ \rho(f_3))\varphi(Y)$$

By induction all summands are elements of $V_t$ and the proof is completed.

**Lemma 1.2.** *Let*

$$\mu : U(L) \otimes U(A) \to U(L)$$

*be the linear map sending $\lambda_1 \otimes \lambda_2$ to $\lambda_1 \lambda_2$. We consider $U(L) \otimes U(A)$ as a (right) module over $U(L) \otimes U(L)$, where the action is component wise, first component $U(L)$ acts via right multiplication and the second via the adjoint action of $L$ on $A$ i.e. for $w_1, \ldots, w_k \in A, l \in L$ the image of $w_1 \ldots w_k \in S^k A \subset U(A)$ under the action of $l$ is $(w_1 \ldots w_k) \circ l = \sum_{1 \leq i \leq k} w_1 \ldots (w_i \circ l) \ldots w_k$. We write $*$ for the described action of $U(L) \otimes U(L)$ on $U(L) \otimes U(A)$. Then*

*1. for all $\lambda \in (Ker\ \rho \otimes \rho)_{2t+1}$ we have*

$$e_i \mu((1 \otimes a_j) * (\varphi \otimes \varphi)(\lambda)) \in V_t;$$

*2. the map $\mu : U(L)\otimes U(A)\to U(L)$ is a homomorphism of $U(L)$–modules where $U(L)$ acts diagonally on the domain i.e. via the diagonal homomorphism $U(L)\to U(L)\otimes U(L)$ sending $l\in L$ to $l\otimes 1+1\otimes l$.*

**Proof.** By [B-G 1, Lemma 2.2(2)] $Ker(\rho\otimes\rho)_{2t+1}$ is spanned by $p\Delta q$, where $p, q$ are monomials in $U(F)\otimes U(F)$, $deg(pq)\leq 2t-1$ and $\Delta$ is $[X_\alpha, X_\beta]\otimes 1$ or $1\otimes [X_\alpha, X_\beta]$ for some $\alpha>\beta$. As $(U(L)\otimes A)*(\varphi\otimes\varphi)(1\otimes[X_\alpha, X_\beta])\subseteq U(L)\otimes(A\circ[y_\alpha, y_\beta])=0$ we have to consider only the case $\Delta=[X_\alpha, X_\beta]\otimes 1$. We write $p=p_1(u)p_2(v), q=q_1(u)q_2(v)$ for some monomials $p_1, p_2, q_1, q_2\in U(F)$. Then $(1\otimes a_j)*(\varphi\otimes\varphi)(p\Delta q)=\varphi(p_1[X_\alpha, X_\beta]q_1)\otimes(a_j\circ\rho(p_2q_2))$ and using $[y_\alpha, y_\beta]=a_{\alpha,\beta}\in\{a_1,\dots,a_{s_0}\}\cup\{0\}$ we get

$$\mu((1\otimes a_j)*(\varphi\otimes\varphi)(p\Delta q))=\varphi(p_1)a_{\alpha,\beta}\varphi(q_1)(a_j\circ\rho(p_2q_2))=$$

$$\varphi(p_1)[a_{\alpha,\beta},\varphi(q_1)](a_j\circ\rho(p_2q_2))+\varphi(p_1)\varphi(q_1)a_{\alpha,\beta}(a_j\circ\rho(p_2q_2))=$$

$$\varphi(p_1)(a_{\alpha,\beta}\circ\varphi(q_1))(a_j\circ\rho(p_2q_2))+\varphi(p_1q_1)a_{\alpha,\beta}(a_j\circ\rho(p_2q_2)).$$

By Lemma 1.1 both summands are in $V_t$.

The second part of the lemma follows immediately from the definition of the map $\mu$.

**Proposition 1.3.** *For sufficiently big $t$ $V_t=V_{t+1}$.*

**Proof.** Let $e_0$ be the maximal degree of the elements $f_{rski}(d), \phi_{rskj}, \psi_{rskj}$ defined in (2) for all possible $r, s, k, j, i$. We fix $t_0=max\{ln, e_0-l-1\}$, where $l$ is the positive integer used in (2), $e_0$ is the maximal degree of a monomial in (2).

Let $f_1, f_2$ be monomials in $U(F)$ with $deg(f_1f_2)=t+1\geq t_0+1$. If $f_1\neq 1$ we write $f_1=gY$ for some $Y\in\{X_1,\dots,X_n\}$. Then

$$\varphi(f_1)(a_j\circ\rho(f_2))=\varphi(g)\varphi(Y)(a_j\circ\rho(f_2))=\varphi(g)[\varphi(Y), a_j\circ\rho(f_2)]+$$

$$\varphi(g)(a_j\circ\rho(f_2))\varphi(Y)=-\varphi(g)(a_j\circ\rho(f_2Y))+\varphi(g)(a_j\circ\rho(f_2))\varphi(Y)$$

i.e. $e_i\varphi(f_1)(a_j\circ\rho(f_2))$ is in the $U(L)$-submodule generated by the elements $e_i\varphi(f)(a_j\circ\rho(f_3))$ for $deg(f)<deg(f_1), deg(ff_3)\leq deg(f_1f_2)$. Therefore to complete the proof of the proposition it is sufficient to show $e_i(a_j\circ\rho(f))\in V_t$ for all monomials $f$ in $U(F)$ with $deg(f)=t+1$.

As $t+1 \geq ln+1$ we can assume $f = X_k^{l+1} X_1^{\alpha_1} \dots X_{k-1}^{\alpha_{k-1}} X_{k+1}^{\alpha_{k+1}} \dots X_n^{\alpha_n}$ (remember $\rho(f) \in U(Q)$ and $U(Q)$ is commutative). Then (2) implies

$$(x_k^{l+1} x_1^{\alpha_1} \dots x_{k-1}^{\alpha_{k-1}} x_{k+1}^{\alpha_{k+1}} \dots x_n^{\alpha_n})(v) + \alpha + \beta + \gamma = 0, \tag{3}$$

where

$$\alpha = \sum_{i \leq l} x_k(v)^i f_{rski}(d)(x_1^{\alpha_1} \dots x_{k-1}^{\alpha_{k-1}} x_{k+1}^{\alpha_{k+1}} \dots x_n^{\alpha_n})(v),$$

$$\beta = \sum_j g_{rj}(u) \phi_{rskj}(x_1^{\alpha_1} \dots x_{k-1}^{\alpha_{k-1}} x_{k+1}^{\alpha_{k+1}} \dots x_n^{\alpha_n})(v),$$

$$\gamma = \sum_j \widetilde{g}_{sj}(v) \psi_{rskj}(x_1^{\alpha_1} \dots x_{k-1}^{\alpha_{k-1}} x_{k+1}^{\alpha_{k+1}} \dots x_n^{\alpha_n})(v).$$

The degrees of the elements involved in (3) is bounded above by

$$e_0 + \sum_{1 \leq j \neq k \leq n} \alpha_j = e_0 + deg(f) - l - 1 = e_0 + t - l \leq 2t + 1.$$

Note that $\alpha$ belongs to the $U(Q)$-submodule of $U(Q) \otimes U(Q)$ (via the diagonal action) generated by the subspace $(U(Q) \otimes U(Q))_t$. We can lift $\alpha$ to an elements $\widetilde{\alpha}$ from the $U(F)$–submodule of $U(F) \otimes U(F)$ generated by $(U(F) \otimes U(F))_t$ i.e. $(\rho \otimes \rho)(\widetilde{\alpha}) = \alpha$. We can find $\widetilde{\beta} = \sum_j (\nu g_{r,j})(U) \widetilde{\beta}_j$, $\widetilde{\gamma} = \sum_j (\nu \widetilde{g}_{s,j})(V) \widetilde{\gamma}_j$ both in $(U(F) \otimes U(F))_{2t+1}$ such that $(\rho \otimes \rho)(\widetilde{\beta}) = \beta, (\rho \otimes \rho)(\widetilde{\gamma}) = \gamma$. Then (3) implies

$$f(V) + \widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} \in Ker(\rho \otimes \rho)_{2t+1}. \tag{4}$$

Now Proposition 1.3 follows from Lemma 1.4. Indeed Lemma 1.4 together with (4) implies $e_r(a_s \circ \rho(f)) = e_r \mu((1 \otimes a_s) * (\varphi \otimes \varphi)(f(V))) \in V_t$.

**Lemma 1.4.** *For* $\lambda \in \{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}\}$ *or* $\lambda \in Ker(\rho \otimes \rho)_{2t+1}$

$$e_r \mu((1 \otimes a_s) * (\varphi \otimes \varphi)(\lambda)) \in V_t.$$

**Proof.** If $\lambda = \widetilde{\beta}$ then $e_r \mu((1 \otimes a_s) * (\varphi \otimes \varphi)(\lambda)) \in \sum_j e_r(\varphi \nu g_{rj}) U(L) \subseteq V_t$.

If $\lambda = \widetilde{\gamma}$ then $(1 \otimes a_s) * (\varphi \otimes \varphi)(\lambda) = 0$.

If $\lambda = \widetilde{\alpha}$ we use Lemma 1.2(2) to deduce $e_r \mu((1 \otimes a_s) * (\varphi \otimes \varphi)(\lambda)) \subseteq e_r \mu((1 \otimes a_s) * (U(L) \otimes U(L))_t) U(L) \subseteq V_t$.

Finally if $\lambda \in Ker(\rho \otimes \rho)_{2t+1}$ we use Lemma 1.2(1). This completes the proof of Lemma 1.4, Proposition 1.3 and Proposition 1.

## 3. Proofs of the main theorems

**Lemma 2.** *In the conditions of Theorem A if $B$ is finitely presented over $U(L)$ then $B \otimes A$ is finitely generated over $U(Q)$ via the diagonal action.*

**Proof.** Consider the following diagram with first row an exact complex of $U(L)$–modules and second row an exact complex of $U(Q)$–modules

$$\begin{array}{ccccccc} R_1 & \xrightarrow{\partial_1} & R_0 = \oplus_{i \leq m} e_i U(L) & \xrightarrow{\partial_0} & B & \to & 0 \\ & & & & \downarrow 1_B & & \\ Q_1 = B \otimes A \otimes U(A) & \xrightarrow{d_1} & Q_0 = B \otimes U(A) & \xrightarrow{d_0} & B & \to & 0 \end{array} \qquad (5)$$

where $R_0, R_1$ are free $U(L)$–modules of finite rank, $\partial_0(e_i) = b_i$, $d_0(b \otimes \lambda) = b\epsilon(\lambda)$, $\epsilon$ is the augmentation map $U(A) \to K$ and $d_1(b \otimes a \otimes \lambda) = b \otimes a\lambda$.

Let $\alpha : U(Q) \to U(L)$ be the composition $\varphi \circ \nu$, where $\varphi$ and $\nu$ are the maps defined in section 2. We fix a finite generating set $\{\sum_i e_i \lambda_{i,j}\}_j \subset \oplus_{i \leq m} e_i U(Q)$ over $U(Q)$ of the kernel of the $U(Q)$–homomorphism $\oplus_{i \leq m} e_i U(Q) \to B$ sending $e_i$ to $b_i$. Then

$$Ker\ \partial_0 = \sum_{i \leq m} (e_i A) U(L) + \sum_j \Big(\sum_{i \leq m} e_i \alpha(\lambda_{i,j})\Big) U(L)$$

and we can assume $R_1$ has a finite basis $X_1 \cup X_2$ such that $\partial_1(X_1) \subseteq \cup_{i \leq m} e_i A$, $X_2 = \{x_{2,j}\}_j, \partial_1(x_{2,j}) = \sum_{i \leq m} e_i \alpha(\lambda_{i,j})$.

Now we want to construct homomorphisms of $U(A)$–modules $\beta_i : R_i \to Q_i$ for $i = 0, 1$ that extend the identity on $B$ and commute with the differential of the diagram (5). Define $\beta_0 : R_0 \to Q_0$ by

$$\beta_0(e_i \alpha(x_1^{k_1} \dots x_n^{k_n})\lambda) = (b_i \circ (x_1^{k_1} \dots x_n^{k_n})) \otimes \lambda \text{ for } \lambda \in U(A).$$

The definition of $\beta_1$ is as follows: $\beta_1(X_2) = 0$ and for $x \in X_1, \lambda \in U(A)$ such that $\partial_1(x) = e_i a$

$$\beta_1(x\alpha(x_1^{k_1} \dots x_n^{k_n})\lambda) = ((b_i \otimes a) \circ \delta(x_1^{k_1} \dots x_n^{k_n})) \otimes \lambda,$$

where $\delta : U(Q) \to U(Q) \otimes U(Q)$ is the diagonal map and $(b_i \otimes a) \circ \delta(x_1^{k_1} \dots x_n^{k_n})$ is the image of $b_i \otimes a$ under the diagonal action of $x_1^{k_1} \dots x_n^{k_n}$.

Now we extend the rows of the diagram (5) to projective resolutions $\mathcal{R}$ and $\mathcal{Q}$ over $U(L)$ and $U(A)$ respectively and extend $\beta_0, \beta_1$, to a chain map $\beta : \mathcal{R} \to \mathcal{Q}$ of complexes over $U(A)$. The resolution $\mathcal{Q}$ is chosen in a special way. By definition it is $B \otimes \mathcal{F}$ where $\mathcal{F}$ is the "standard" resolution over $U(A)$

$$\mathcal{F}: \dots \to F_i = \wedge^i A \otimes U(A) \to F_{i-1} = \wedge^{i-1} A \otimes U(A) \to \dots \to F_0 = U(A) \to K \to 0$$

with differential

$$\partial_i(a_1 \wedge \dots \wedge a_i) = \sum_j (-1)^j (a_1 \wedge \dots \wedge \hat{a}_j \wedge \dots \wedge a_i) \otimes a_j.$$

The complex $\mathcal{F}$ is exact by [C-E, Ch 13, Thm 7.1]. Now the chain map $\beta$ induces an isomorphism between $H_i(\mathcal{R} \otimes_{U(A)} K)$ and $H_i(\mathcal{Q} \otimes_{U(A)} K) \simeq B \otimes H_i(\mathcal{F}) \simeq B \otimes \wedge^i A$ and $H_1(\mathcal{R} \otimes_{U(A)} K)$ is finitely generated over $U(Q)$. Then $B \otimes A \simeq H_1(\mathcal{Q} \otimes_{U(A)} K)$ is an $U(Q)$–module via $\beta_1$ and by the definition of $\beta_1$ the action of $U(Q)$ is the diagonal one. This completes the proof of Lemma 2.

**Lemma 3.** *If $L$ is a split extension of $A$ by $Q$ and $B$ is of homological type $FP_m$ over $U(L)$ then $B \otimes (\wedge^m A)$ is finitely generated over $U(Q)$, where $U(Q)$ acts via the diagonal homomorphism $U(Q) \to^{\otimes^{m+1}} U(Q)$ sending $q \in Q$ to $\sum_{0 \leq i \leq m} \underbrace{1 \otimes \dots \otimes 1}_{i\ times} \otimes q \otimes \underbrace{1 \otimes \dots \otimes 1}_{m-i\ times}$.*

**Proof.** Suppose

$$\mathcal{R} : \dots \to R_i \xrightarrow{\partial_i} \dots \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} B \to 0$$

is a free resolution of $B$ over $U(L)$ such that $R_i$ for $i \leq m$ is finitely generated and $\mathcal{Q} = B \otimes \mathcal{F}$ is the resolution considered in the proof of Lemma 2.

Now we construct a chain map $\alpha : \mathcal{R} \to \mathcal{Q}$ over $U(A)$ inducing identity on $B$. First $R_i = T_i \otimes_{U(A)} U(L) \simeq T_i \otimes_K U(Q)$ for some free $U(A)$–submodule $T_i$ of $R_i$. We want to define $\alpha$ in such a way that $\alpha_i(tf) = \alpha_i(t)^f$ for all $t \in T_i$, $f$ a monomial in $U(Q)$, where upper index $f$ denotes the image under the diagonal action of $f$. We proceed by induction on $i$. Suppose we have constructed $\alpha_{i-1}$, then there exists a homomorphism of $U(A)$–modules $\beta_i : R_i \to Q_i$ such that $\partial\beta_i = \alpha_{i-1}\partial$. We set $\alpha_i(tf) = \beta_i(t)^f$ for $t \in T_i, f$ a monomial in $U(Q)$. It is easy to check that $\alpha_i$ is a homomorphism of $U(A)$–modules and $\partial\alpha_i = \alpha_{i-1}\partial$. Finally $\alpha_i$ induces an isomorphism between the homology groups $H_i(\mathcal{Q} \otimes_{U(A)} K)$ and $H_i(\mathcal{R} \otimes_{U(A)} K)$. The latter is a finitely generated $U(Q)$–module for $i \leq m$ and by construction the induced by $\alpha$ action of $U(Q)$ on $H_i(\mathcal{Q} \otimes_{U(A)} K) \simeq B \otimes (\wedge^i A)$ is the diagonal one.

**Theorem 4.** *Suppose $A$ and $B$ are finitely generated $U(Q)$–modules.*

*1. $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action if and only if whenever $[v_2], \ldots, [v_{m+1}] \in \Delta(Q, A), [v_1] \in \Delta(Q, B)$ and $0 = [v_1] + \ldots + [v_{m+1}]$ we have all $[v_i]$ trivial.*

*2. If $B \otimes (\wedge^m A)$ is finitely generated over $U(Q)$ via the diagonal action then $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action.*

**Proof.** 1. We write $M$ for $B \otimes (\otimes^m A)$ and view it as a module over $\otimes^{m+1} U(Q)$. Then the diagonal embedding $\theta : U(Q) \to \otimes^{m+1} U(Q)$ induces a map

$$\theta^* : \Delta(Q^{m+1}, M) \to \Delta(Q, M)$$

By [B-G 2, Prop. 3.1] $M$ is finitely generated over $U(Q)$ via the diagonal action if and only if $(\theta^*)^{-1}(0) = 0$. As shown in [B-G 2] there is a direct product formula

$$\Delta(Q^{m+1}, M) \simeq \Delta(Q, B) \times (\Delta(Q, A))^m$$

and under this isomorphism $\theta^*$ sends $([v_1], [v_2], \ldots, [v_{m+1}])$ to $\sum_j [v_j]$. This implies immediately the first part of the theorem.

2. Now we assume the second part of the theorem is wrong and then by the first part there exist $[v_2], \ldots, [v_{m+1}] \in \Delta(Q, A)$ not all zero and $[v_1] \in \Delta(Q, B)$ such that $[v_1] + \ldots + [v_{m+1}] = 0$.

**Lemma 4.1**[G] *Suppose $\alpha_i : Q \to \overline{K}((t_i)) \simeq \overline{K}((t))$ is a linear map of vector spaces over $K$, $M$ is a finitely generated $U(Q)$-module such that $[\alpha_i] \in \Delta(Q, M)$. Then there exists a non-trivial linear map*

$$w_i : M \to \overline{K}((t_i))$$

*such that*

$$w_i(mq) = w_i(m)\alpha_i(q) \text{ for all } m \in M, q \in Q$$

We apply the above lemma for the linear maps $\alpha_i = \mu_i v_i$, where $\mu_i : \overline{K}((t)) \to \overline{K}((t_i))$ is the isomorphism of $\overline{K}$-algebras sending $t$ to $t_i$ and obtain linear maps

$$w_1 : B \to \overline{K}((t_1)), w_i : A \to \overline{K}((t_i)) \text{ for all } 2 \leq i \leq m+1$$

with the properties described in Lemma 4.1. Using the maps $w_i$ we construct another linear map

$$\varphi = w_1 \otimes w_2 \otimes \ldots \otimes w_{m+1} : B \otimes ({\otimes}^m A) \to R = \overline{K}((t_1)) \otimes \overline{K}((t_2)) \otimes \ldots \otimes \overline{K}((t_{m+1}))$$

that will play an important role in the completion of the proof of Theorem 4.

Let

$$\alpha : B \otimes ({\otimes}^m A) \to B \otimes ({\otimes}^m A)$$

be the linear map given by $\alpha(b \otimes a_1 \otimes \ldots \otimes a_m) = \sum_{\sigma \in S_m} (-1)^\sigma b \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(m)}$. As the image of $\alpha$ factors through $B \otimes (\wedge^m A)$ it is finitely generated over $U(Q)$. Note that $Im\ \alpha$ is a module over $U(Q) \otimes S$ and $\alpha$ is a homomorphism of $U(Q) \otimes S$–modules, where $S = \{\lambda \in {\otimes}^m U(Q) \mid \lambda\sigma = \lambda \text{ for all } \sigma \in S_m\}$. As ${\otimes}^m U(Q)$ is integral over $S$ the $K$-algebra ${\otimes}^{m+1} U(Q)$ is integral over $U(Q) \otimes S$ and so $V = Im\ \alpha({\otimes}^{m+1} U(Q))$ is finitely generated over $U(Q)$.

Now let $s$ be the positive integer with the properties $\varphi(V) \subseteq J^s$ and $\varphi(V) \not\subseteq J^{s+1}$, where $J$ is the ideal of $R$ generated by $t_1 - t_2, t_2 - t_3, \ldots, t_m - t_{m+1}$. Then for $v \in V$ the image of the diagonal action of $q \in Q$ on $\varphi(v)$ is $\varphi(v) \sum_i \alpha_i(q) \equiv \varphi(v) \sum_i \pi_i \alpha_i(q)$ modulo $J^{s+1}$, where $\pi_i : \overline{K}((t_i)) \to \overline{K}((t_1))$ is the isomorphism of $\overline{K}$-algebras sending $t_i$ to $t_1$. As $\sum_i [v_i] = 0$ we have $\sum_i \pi_i \alpha_i(q) \in \overline{K}[[t_1]]$ and hence $\varphi(V) + J^{s+1}/J^{s+1}$ lies in a finitely generated $\overline{K}[[t_1]]$–submodule of $J^s/J^{s+1} \simeq \overline{K}((t_1))$.

Finally we choose $v_i$ and $q \in Q$ such that $Im\ \alpha_i$ is not a subset of $\overline{K}[[t_i]]$ and $\alpha_i(q) \notin \overline{K}[[t_i]]$ and define $h = ({\otimes}^{i-1} 1) \otimes q \otimes ({\otimes}^{m-i+1} 1) \in {\otimes}^{m+1} U(Q)$. Then for $v \in V$ we have $\varphi(vh) = \varphi(v)\alpha_i(q) \equiv \varphi(v)\pi_i(\alpha_i(q))$ modulo $J^{s+1}$ and hence $\varphi(V) + J^{s+1}/J^{s+1}$ is invariant under multiplication with $f^j$ for every $j \geq 1$ where $f = \pi_i(\alpha_i(q)) \in \overline{K}((t_1)) \setminus \overline{K}[[t_1]]$. In particular $\varphi(V) + J^{s+1}/J^{s+1}$ cannot lie in a finitely generated $\overline{K}[[t_1]]$–submodule of $J^s/J^{s+1} \simeq \overline{K}((t_1))$, a contradiction.

**Theorem 5.** *If $A$ and $B$ are finitely generated $U(Q)$–modules and $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ via the diagonal action then $B$ is of type $FP_m$ over $U(L)$, where the Lie algebra $L$ is the split extension of $A$ by $Q$.*

**Proof.** The proof of Theorem 5 is based on the existence of some special long exact sequences given by Lemma 5.1.

**Lemma 5.1** *For every $k \geq 1$ the complex*

$$0 \to \wedge^k A \xrightarrow{\partial_{k,k}} \ldots \xrightarrow{\partial_{i+1,k}} \wedge^i A \otimes S^{k-i} A \xrightarrow{\partial_{i,k}} \ldots \xrightarrow{\partial_{1,k}} S^k A \to 0$$

*with differentials $\partial_{i,k}$ sending the element $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i})$ to*

$$\sum_{1 \leq j \leq i} (-1)^{i-j} (a_1 \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_i) \otimes (a_j \otimes b_1 \otimes \ldots \otimes b_{k-i})$$

*is exact.*

**Proof.** Choose a basis $A_0$ of $A$ and order it linearly. Then $\wedge^i A \otimes S^{k-i} A$ has a basis $\{(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i}) \mid a_1, \ldots, a_i, b_1, \ldots, b_{k-i} \in A_0, a_1 < \ldots < a_i, b_1 \leq \ldots \leq b_{k-i}\} = \mathcal{X}_{i,k}$. We call an element of $\mathcal{X}_{i,k}$ good if $b_1 \geq a_1$ and define by $(\wedge^i A \otimes S^{k-i} A)_{good}$ the space spanned by the good elements. A partial order on $\mathcal{X}_{i,k}$ is defined by $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i}) \leq (a'_1 \wedge \ldots \wedge a'_i) \otimes (b'_1 \otimes \ldots \otimes b'_{k-i})$ if and if $a_j \leq a'_j$ for all $j \leq i$.

**Claim 1.** $\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} + Im\ \partial_{i+1,k}$

**Proof.** We show that a non-good element $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i})$ of $\mathcal{X}_{i,k}$ can be expressed modulo the image of $\partial_{i+1,k}$ as a sum of smaller elements of $\mathcal{X}_{i,k}$. Indeed $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i}) + (-1)^{i+1} \partial_{i+1,k}(b_1 \wedge a_1 \wedge \ldots \wedge a_i) \otimes (b_2 \otimes \ldots \otimes b_{k-i})$ is a sum of elements of $\mathcal{X}_{i,k}$ smaller than $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i})$. This completes the proof of the claim.

It follows immediately from Claim 1 that

$$\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} + \partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}) \qquad (6)$$

We claim that the sum in (6) is exact and

$$\partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}) \simeq (\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}$$

For both statements it is sufficient to consider the case when $A$ is finite dimensional. In this case we define $\mu(i,k)$ to be the dimension of $(\wedge^i A \otimes S^{k-i} A)_{good}$ i.e. the number of good elements in $\mathcal{X}_{i,k}$.

**Claim 2.** $dim_K(\wedge^i A \otimes S^{k-i} A) = \mu(i,k) + \mu(i+1,k)$

**Proof.** Note that the dimension of $\wedge^i A \otimes S^{k-i} A$ is the cardinality of $\mathcal{X}_{i,k}$. It remains to show that $\mu(i+1,k)$ is the number of non-good elements in $\mathcal{X}_{i,k}$. This can be done by showing a bijection between the non-good elements in $\mathcal{X}_{i,k}$ and the good elements of $\mathcal{X}_{i+1,k}$. If $(a_1 \wedge \ldots \wedge a_i) \otimes (b_1 \otimes \ldots \otimes b_{k-i})$ is a non-good element from $\mathcal{X}_{i,k}$ then $(b_1 \wedge a_1 \wedge \ldots \otimes a_i) \otimes (b_2 \otimes \ldots \otimes b_{k-i})$ is a good element of $\mathcal{X}_{i+1,k}$. The inverse holds too and the proof of Claim 2 is completed.

Note that Claim 2 together with (6) shows that

$$\wedge^i A \otimes S^{k-i} A = (\wedge^i A \otimes S^{k-i} A)_{good} \oplus \partial_{i+1,k}((\wedge^{i+1} A \otimes S^{k-i-1} A)_{good})$$

and that the restriction of $\partial_{i+1,k}$ on $(\wedge^{i+1} A \otimes S^{k-i-1} A)_{good}$ is injective. Similarly the restriction of $\partial_{i,k}$ on $(\wedge^i A \otimes S^{k-i} A)_{good}$ is injective and hence $Im\ \partial_{i+1,k} = Ker\ \partial_{i,k}$. This completes the proof of Lemma 5.1.

Now we define $V_i$ for $i \geq 1$ to be the subspace of $\otimes^i A$ generated by the elements $\sum_{\sigma \in S_n} (-1)^\sigma\ a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$ for $a_1, \ldots, a_n \in A$. Let $W_i$ be the $U(A)$–submodule of $\otimes^{i-1} A \otimes U(A)$ generated by $V_i \subseteq (\otimes^{i-1} A) \otimes A \subset (\otimes^{i-1} A) \otimes U(A)$.

**Claim 3** *The map $\varphi_i : V_i \otimes U(A) \to W_i$ sending $v_1 \otimes \ldots \otimes v_i \otimes \lambda$ to $v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \lambda$ has kernel $W_{i+1}$.*

**Proof.** We identify $V_i$ with $\wedge^i A$ via the map sending $\sum_{\sigma \in S_n} (-1)^\sigma\ a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$ to $a_1 \wedge \ldots \wedge a_n$. Write $U(A)$ as a direct sum of the symmetric powers of $A$, the restriction of $\varphi_i$ on $\wedge^i A \otimes S^{k-i} A$ is precisely the map $\partial_{i,k}$ defined in Lemma 5.1. Then Lemma 5.1 completes the proof.

**Lemma 5.2.** *Under the assumptions of Theorem 5 for every $i \leq m$ the module $B \otimes W_i$ is of type $FP_k$ over $U(L)$ if and only if $B \otimes W_{i+1}$ is of type $FP_{k-1}$ over $U(L)$, where $U(A)$ acts on $B \otimes W_i$ via its action on the component $W_i$ and $U(Q)$ acts on $B \otimes (\otimes^{i-1} A) \otimes U(A)$ via the diagonal map $U(Q) \to \otimes^{i+1} U(Q)$ sending an element $q$ from $Q$ to $\sum_{0 \leq j \leq i} \underbrace{1 \otimes \ldots \otimes 1}_{j\ times} \otimes q \otimes \underbrace{1 \otimes \ldots \otimes 1}_{i-j\ times}$.*

**Proof.** The short exact sequence of $U(A)$–modules $0 \to W_{i+1} \to V_i \otimes U(A) \to W_i \to 0$ gives rise to a short exact sequence of $U(L)$–modules

$$0 \to B \otimes W_{i+1} \to B \otimes V_i \otimes U(A) \to B \otimes W_i \to 0 \tag{7}$$

where $U(Q)$ acts diagonally on all modules in (7). By Theorem 4(1) $B \otimes (\otimes^i A)$ is finitely generated over $U(Q)$ via the diagonal action for all $i \leq m$ and hence its submodule $B \otimes V_i$ is finitely generated over $U(Q)$. Then $(B \otimes V_i) \otimes U(A) \simeq (B \otimes V_i) \otimes_{U(Q)} U(L)$ is induced from a module of type $FP_\infty$ over $U(Q)$ and is itself of type $FP_\infty$ over $U(L)$. The dimension shifting argument [B, Prop 1.4] applied to (7) completes the proof.

Finally we are ready to complete the proof of Theorem 5. Applying Lemma 5.2 several times we obtain $B \otimes W_1$ is of type $FP_{m-1}$ over $U(L)$ if and only if $B \otimes W_m$ is of type $FP_0$ (i.e. finitely generated) over $U(L)$. Note that $B \otimes V_m$ is a generating set of $B \otimes W_m$ over $U(A)$. By assumption $B \otimes (\otimes^m A)$ is finitely generated over $U(Q)$ and so $B \otimes V_m$ is finitely generated over $U(Q)$. Finally it remains to show that $B \otimes W_1$ is of type $FP_{m-1}$ over $U(L)$ if and only if $B$ is of type $FP_m$ over $U(L)$. This follows immediately from dimension shifting argument for the short exact sequence of $U(L)$-modules

$$0 \to B \otimes W_1 \to B \otimes_K U(A) \simeq B \otimes_{U(Q)} U(L) \to B \to 0$$

induced from the short exact sequence $0 \to W_1 \to U(A) \to K \to 0$.

**Proof of Theorem A.** $1 \Leftrightarrow 2$ by Proposition 1 and Lemma 2 and $2 \Leftrightarrow 3$ by Theorem 4(1).

**Proof of Corollary B.** It is a straight corollary of Theorem A and the classification of finitely presented Lie algebras in [B-G 1], [B-G 2].

**Proof of Theorem C.** The theorem follows from Lemma 3, Theorem 4 and Theorem 5.

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