Existence of Multiple Solutions for Quasilinear Systems via Fibering Method

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Abstract

Using the Fibering Method introduced by S. I. Pohozaev, we prove existence of multiple solutions for a Dirichlet problem associated to a quasilinear system involving a pair of (p,q)-Laplacian operators.

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1 Introduction

In this paper we shall study some existence and non-existence results for the following quasilinear system:

$$\left\{ \begin{array}{rcl} -\Delta_p u & = & \lambda a(x)|u|^{p-2}u + (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ \\ -\Delta_q v & = & \mu b(x)|v|^{q-2}v + (\beta+1)c(x)|u|^{\alpha+1}|v|^{\beta-1}v. \end{array} \right. \tag{1}$$

Here $\alpha, \beta, \lambda, \mu, p > 1, q > 1$ are real numbers, Δ_p and Δ_q are correspondingly the $p-$ and $q-$Laplace operators and $a(x), b(x), c(x)$ - given functions.

The system (1) will be considered in a bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition

$$u = v = 0. \tag{2}$$

Systems involving quasilinear operators of $p-$Laplacian type have been studied by various authors [2, 9]. Among other results, existence and non-existence theorems were obtained. For such purpose the method of sub-super solutions, the blow-up method and the Mountain Pass Theorem have been used (see e.g. [2, 4]).

Our main tool here is the so-called Fibering Method introduced and developed by S. I. Pohozaev in [11, 12, 13]. Its general nature enables us to prove existence and multiplicity theorems for (1),(2) in a somewhat more constructive and explicit way. The Fibering Method was applied to a single equation of $p-$Laplacian type by Drabek and Pohozaev in [3].

Dealing with existence theorems, the parameters λ and μ, appearing in (1), will be naturally related to λ_1 and μ_1, the first eigenvalue of $(-\Delta_p, W_0^{1,p})$ and $(-\Delta_q, W_0^{1,q})$ respectively. The existence and properties of the first eigenvalue of $p-$Laplacian operators, subject to homogeneous Dirichlet boundary conditions in a bounded domain, are obtained in [1, 8, 3, 5, 6].

This paper is organized as follows. In section 2 we introduce some notation, define the functions spaces that will be used throughout the paper and state our basic assumptions. For convenience of the reader we also collect some of the properties of the $p-$Laplacian eigenvalues and corresponding eigenfunctions. Section 3 contains a slight modification of the Fibering Method, adapted for vector-valued problems. The main results of this paper, that is, the existence and multiplicity theorems for the problem (1),(2) are presented in section 4. Finally, in section 5 we prove a non-existence result for classical solutions, using the celebrated Pohozaev Identity [10].

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2 The p-Laplacian operator and its eigenvalues

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 < p, q < \infty$. We define the Sobolev spaces $Y_p = W_0^{1,p}(\Omega)$ and $Y_q = W_0^{1,q}(\Omega)$ equipped with the norms

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \quad \|v\|_q = \left(\int_\Omega |\nabla v|^q dx\right)^{1/q}. \tag{3}$$

respectively. Then we denote $Y = Y_p \times Y_q$ and for $(u, v) \in Y$

$$\|(u, v)\| = \|u\|_p^p + \|v\|_q^q. \tag{4}$$

Now consider the eigenvalue equation for the $p-$Laplace operator:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{5}$$

where $a \in L^\infty(\Omega)$. The problem (5) is closely related with our main problem (1),(2). For we need the following lemma.

Lemma 1([3, 1, 8]). *There exists a number $\lambda_1 > 0$ such that:*

$$\lambda_1 = \inf \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\displaystyle\int_\Omega a(x)|u|^p dx}, \tag{6}$$

where the infimum is taken over $u \in Y_p$ such that $\int_\Omega a(x)|u|^p dx > 0$;

(*i*) *there exists a positive function $\varphi \in Y_p \cap L^\infty(\bar{\Omega})$ which is solution of* (5) *with $\lambda = \lambda_1$.*

(*ii*) *λ_1 is simple, in the sense that any two eigenfunctions, corresponding to λ_1, differ by a constant multiplier;*

(*iii*) *λ_1 is isolated, which means that there are no eigenvalues less than λ_1 and no eigenvalues in the interval $(\lambda_1, \lambda_1 + \delta)$ for some $\delta > 0$ sufficiently small.*

Note that we consider (5) in a weak sense, that is,

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega a(x)|u|^{p-2}uvdx, \\ u = 0 \text{ in } \partial\Omega \end{cases}$$

for any $v \in Y_p$.

Now we state the assumptions that we shall assume throughout this paper.

Let $\alpha, \beta, \lambda, \mu, p > 1, q > 1$ be real numbers. We shall suppose that

$$1 < p < p^*, \quad 1 < q < q^*, \tag{7}$$

$$\frac{N-p}{p}(\alpha+1) + \frac{N-q}{q}(\beta+1) < N, \tag{8}$$

where

$$p^* = Np/(N-p), \quad q^* = Nq/(N-q)$$

are the well-known critical exponents (see [9, 2]). We assume that the system (1) is super-homogeneous in the sense that

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1. \tag{9}$$

It can be seen that the latter condition is equivalent to

$$d = (\alpha+1)(\beta+1) - (\alpha-p+1)(\beta-q+1) > 0. \tag{10}$$

Moreover, since (8) is equivalent to

$$N < \frac{\alpha+\beta+2}{\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1}, \tag{11}$$

one can observe that our system is subcritical [9] which avoids non-compactness problems. See [9] for more details on this point.

Note that (8) implies

$$\alpha + 1 < p^*, \quad \beta + 1 < q^*.$$

The functions $a(x)$, $b(x)$ and $c(x)$ are supposed to be bounded in Ω:

$$a, b, c \in L^\infty(\Omega) \tag{12}$$

and

$$a(x) = a_1(x) - a_2(x); \quad a_1, a_2 \geq 0, \quad a_1(x) \not\equiv 0. \tag{13}$$

$$b(x) = b_1(x) - b_2(x); \quad b_1, b_2 \geq 0, \quad b_1(x) \not\equiv 0. \tag{14}$$

By the Sobolev inequality it can be easily seen that (7), (8) and (12) imply that the integrals

$$\int_\Omega a(x)|u|^p dx$$

and

$$\int_\Omega b(x)|v|^q dx$$

are finite for $(u, v) \in Y$. Now we can define the following functionals on Y_p and Y_q:

$$f_1(u) = \int_\Omega a(x)|u|^p dx \tag{15}$$

and

$$f_2(v) = \int_\Omega b(x)|v|^q dx. \tag{16}$$

Since a and b are bounded it is standard to check that f_1 and f_2 are weakly lower continuous. Similarly, the conditions (7), (8) and (12) imply that the functional

$$f_3(u, v) = \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1} dx \tag{17}$$

is weakly lower continuous in Y.

We shall also suppose that

$$c^+(x) \not\equiv 0. \tag{18}$$

and

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1} dx < 0 \tag{19}$$

The functions $\varphi \in Y_p$ and $\psi \in Y_q$ above are the first eigenfunctions of Δ_p and Δ_q correspondingly.

We end this section with the following

Definition (weak solution). *We say that $(u, v) \in Y$ is a weak solution of (1) if*

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla z) dx = \lambda \int_\Omega a(x)|u|^{p-2} uz dx + (\alpha + 1) \int_\Omega c(x)|u|^{\alpha-1}u|v|^{\beta+1} z dx$$

$$\int_\Omega |\nabla v|^{q-2}(\nabla v, \nabla w) dx = \mu \int_\Omega b(x)|v|^{q-2} vw dx + (\beta + 1) \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1} vw dx$$

for any $(z, w) \in Y$.

3 The Fibering Method for systems of quasilinear PDEs

The system (1) has a variational structure. Indeed, denote

$$F(x,u,v) := \frac{\lambda}{p}a(x)|u|^p + \frac{\mu}{q}b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1} \tag{20}$$

and consider

$$\mathcal{F}(x,u,v,\nabla u,\nabla v) = \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q - F(x,u,v). \tag{21}$$

Let $J : Y \to \mathbb{R}$ be defined by

$$J(u,v) := \int_\Omega \mathcal{F}(x,u,v,\nabla u,\nabla v)dx,$$

or, in a more detailed form,

$$\begin{aligned} J(u,v) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega a(x)|u|^p dx + \frac{1}{q}\int_\Omega |\nabla v|^q dx - \frac{\mu}{q}\int_\Omega b(x)|v|^q dx \\ &- \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx. \end{aligned} \tag{22}$$

Clearly the critical points of J are the weak solutions of the problem (1), (2).

The cornerstone of the Fibering method consists of the following. We express $(u,v) \in Y$ in the form

$$u = rz, \quad v = \rho w, \tag{23}$$

where the functions $z \in Y_p$, $w \in Y_q$, and r, ρ are real numbers. Since we look for non-trivial solutions we must assume that $r \neq 0$ and $\rho \neq 0$. Substituting (23) in (22) we obtain

$$\begin{aligned} J(rz,\rho w) &= \frac{|r|^p}{p}\int_\Omega |\nabla z|^p dx - \frac{\lambda |r|^p}{p}\int_\Omega a(x)|z|^p dx \\ &+ \frac{|\rho|^q}{q}\int_\Omega |\nabla w|^q dx - \frac{\mu|\rho|^q}{q}\int_\Omega b(x)|w|^q dx \\ &- |r|^{\alpha+1}|\rho|^{\beta+1}\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx. \end{aligned} \tag{24}$$

If $(u,v) \in Y$ is a critical point of J then

$$\frac{\partial J}{\partial r}(rz,\rho w) = 0 \ \text{ and } \ \frac{\partial J}{\partial \rho}(rz,\rho w) = 0. \tag{25}$$

Assuming that

$$A := \int_\Omega |\nabla z|^p dx - \lambda \int_\Omega a(x)|z|^p dx \neq 0, \tag{26}$$

$$B := \int_\Omega |\nabla w|^q dx - \lambda \int_\Omega b(x)|w|^q dx \neq 0, \tag{27}$$

$$C := \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} dx \neq 0, \tag{28}$$

we can write (24) in the following way

$$J(rz, \rho w) = \frac{|r|^p}{p} A + \frac{|\rho|^q}{q} B - |r|^{\alpha+1}|\rho|^{\beta+1} C. \tag{29}$$

The conditions (25) are equivalent to

$$\frac{\partial J}{\partial r} = 0 \Leftrightarrow |r|^{p-2} r A - (\alpha+1)|r|^{\alpha-1} r |\rho|^{\beta+1} C = 0,$$

$$\frac{\partial J}{\partial \rho} = 0 \Leftrightarrow |\rho|^{q-2} \rho B - (\beta+1)|r|^{\alpha+1} |\rho|^{\beta-1} \rho C = 0,$$

that is,

$$\left\{ \begin{array}{rcl} |r|^{p-2} A - (\alpha+1)|r|^{\alpha-1}|\rho|^{\beta+1} C & = & 0 \\ |\rho|^{q-2} B - (\beta+1)|r|^{\alpha+1}|\rho|^{\beta-1} C & = & 0. \end{array} \right. \tag{30}$$

Resolving the system (30) we obtain as an intermediate step that

$$|r|^{p-\alpha-1} = |\rho|^{\beta+1} C(\alpha+1)/A.$$

Hence A and C must have the same sign. Analogously

$$|\rho|^{q-\beta-1} = |r|^{\alpha+1} C(\beta+1)/B$$

and B and C must also have the same sign. Thus A, B and C must have the same sign! Note that the conditions (26), (27) and (28) have been essentially used. Hence the solution of (30) is given by

$$|r| = \left(\frac{(\alpha+1)^{\beta-q+1}|B|^{\beta+1}}{(\beta+1)^{\beta+1}|C|^q|A|^{\beta-q+1}} \right)^{1/d}, \tag{31}$$

$$|\rho| = \left(\frac{(\beta+1)^{\alpha-p+1}|A|^{\alpha+1}}{(\alpha+1)^{\alpha+1}|C|^p|B|^{\alpha-p+1}} \right)^{1/d}, \tag{32}$$

where $d > 0$ is given in (9).

The fact that A, B, C must have the simultaneously the same sign leads us to consider two cases. In the next sections we shall assume that

$$A > 0, \quad B > 0, \quad C > 0 \tag{33}$$

or

$$A < 0, \quad B < 0, \quad C < 0. \tag{34}$$

Thus, in both cases (33) and (34), the functions $r = r(z, w)$ and $\rho = \rho(z, w)$ are well defined. Now we insert the expressions for $r = r(z, w)$ and $\rho = \rho(z, w)$, determined by (31) and (32), into (29). In this way we obtain a functional

$$I(z, w) = J(r(z, w)z, \rho(z, w)w) \tag{35}$$

given by

$$\begin{aligned} I(z,w) \quad &= \quad K \left| \int_\Omega |\nabla z|^p dx - \lambda \int_\Omega a(x)|z|^p dx \right|^{(\alpha+1)q/d} \\ &\times \quad \frac{\left| \int_\Omega |\nabla w|^q dx - \mu \int_\Omega b(x)|w|^q dx \right|^{(\beta+1)p/d}}{\left| \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} dx \right|^{pq/d}}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} K \quad &= \quad \Big(\tfrac{(\alpha+1)^{(\beta-q+1)p/d}}{p(\beta+1)^{(\beta+1)p/d}} + \tfrac{(\beta+1)^{(\alpha-p+1)q/d}}{q(\alpha+1)^{(\alpha+1)q/d}} \\ &- \quad \tfrac{1}{(\alpha+1)^{(\alpha+1)q/d}(\beta+1)^{(\beta+1)p/d}} \Big)\, sign \left(\textstyle\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1} dx \right). \end{aligned}$$

Therefore, provided z and w satisfy (33) or (34), we have

$$\left. \frac{\partial J}{\partial r} \right|_{r=r(z,w), \rho=\rho(z,w)} = 0 \tag{37}$$

and

$$\left. \frac{\partial J}{\partial \rho} \right|_{r=r(z,w), \rho=\rho(z,w)} = 0. \tag{38}$$

Next we introduce the following notation: for any functional $f : Y \to \mathbb{R}$ we denote by

$$f'(z, w)(h_1, h_2)$$

the Gatêaux derivative of f at $(z, , w) \in Y$ in direction of $(h_1, h_2) \in Y$.

Let

$$E_1(z) = \int_\Omega |\nabla z|^p dx - \lambda \int_\Omega a(x)|z|^p dx, \tag{39}$$

$$E_2(w) = \int_\Omega |\nabla w|^q dx - \mu \int_\Omega b(x)|w|^q dx, \tag{40}$$

and

$$E_i^{(1)}(z,w)(h_1,h_2) = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0,\sigma=0} E_i(z+\varepsilon h_1, w+\sigma h_2),$$

$$E_i^{(2)}(z,w)(h_1,h_2) = \frac{\partial}{\partial \sigma}\Big|_{\varepsilon=0,\sigma=0} E_i(z+\varepsilon h_1, w+\sigma h_2),$$

$$I^{(1)}(z,w)(h_1,h_2) = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0,\sigma=0} I(z+\varepsilon h_1, w+\sigma h_2),$$

$$I^{(2)}(z,w)(h_1,h_2) = \frac{\partial}{\partial \sigma}\Big|_{\varepsilon=0,\sigma=0} I(z+\varepsilon h_1, w+\sigma h_2).$$

It is easy to see that the following lemma holds. We omit the straightforward details.

Lemma 2. *(1) The functional I is homogeneous of degree 0, that is, for every $z \in Y_p$, $w \in Y_q$ such that $\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx \neq 0$ and every $t \neq 0$ we have*

$$I(tz, tw) = I(z,w).$$

(2) I is even and

$$I'(z,w)(z,w) = 0.$$

Remark 1. If $(z,w) \in Y$ is a critical point of I, by well-known properties of $p-$Laplace Dirichlet integral (see [7]) it follows that $(|z|,|w|) \in Y$ is also a critical point of I.

The next two lemmas are direct consequences of the results proved in [11, 12, 13].

Lemma 3. *Let (z,w) be a critical point of I, which satisfies* (33) *or* (34)*. Then the function (u,v) defined by*

$$u(x) = rz(x), \quad v(x) = \rho w(x)$$

where $r \neq 0$ and $\rho \neq 0$ are determined by (31) *and* (32)*, is a critical pont of J.*

Proof. Since (z,w) is a critical point of I we have

$$I'(z,w)(h_1,h_2) = (I^{(1)}(z,w)(h_1,h_2), I^{(2)}(z,w)(h_1,h_2)) = 0.$$

Therefore, since

$$\left.\frac{\partial J}{\partial r}\right|_{r=r(z,w),\rho=\rho(z,w)} = \left.\frac{\partial J}{\partial \rho}\right|_{r=r(z,w),\rho=\rho(z,w)} = 0.$$

(see (37) and (38)), by the chain rule we have

$$\begin{aligned} 0 &= I^{(1)}(z,w)(h_1,h_2) \\ &= r(z,w)J^{(1)}(rz,\rho w)(h_1,h_2) + \left.\frac{\partial J}{\partial r}\right|_{r=r(z,w),\rho=\rho(z,w)} \frac{\partial r}{\partial z} \\ &+ \left.\frac{\partial J}{\partial \rho}\right|_{r=r(z,w),\rho=\rho(z,w)} \frac{\partial \rho}{\partial z} \\ &= r(z,w)J^{(1)}(rz,\rho w)(h_1,h_2). \end{aligned}$$

Thus $J^{(1)}(u,v)=0$. Analogously $J^{(2)}(u,v)=0$ and therefore $J'(u,v)=0$.

Lemma 4. *Let E_1 and E_2 be defined by* (39) *and* (40)*. Consider*

$$E_1(z,w)=c_1 \quad \text{and} \quad E_2(z,w)=c_2,$$

where $c_i \in \mathbb{R}$, $(i=1,2)$. Suppose that

$$\det\begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} \neq 0 \quad \text{if} \quad E_1(z,w)=c_1 \quad \text{and} \quad E_2(z,w)=c_2, \tag{41}$$

Then every critical point of I with the conditions $E_1(z,w)=c_1$ and $E_2(z,w)=c_2$ is a critical point of I.

Proof. Let (z,w) be a conditional critical point of I. By the Euler Theorem there exist $m_1,m_2 \in \mathbb{R}$ such that

$$I'(z,w) = m_1E_1'(z,w) + m_2E_2'(z,w). \tag{42}$$

Since by Lemma 2 we have $I'(z,w)(z,w)=0$, by (42) we obtain:

$$m_1E_1^{(1)} + m_2E_2^{(1)} = 0,$$
$$m_1E_1^{(2)} + m_2E_2^{(2)} = 0.$$

Now by (41) we have

$$\det\begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} \neq 0,$$

Therefore $m_1=m_2=0$. Thus $I'(z,w)=0$, that is, (z,w) is a critical point of I.

4 Existence and multiplicity results

Our first aim is to prove the existence of a critical point of I with appropriate constraints. This in turn will be an actual critical point of I and hence a critical point of J - a weak solution of (1). We have already pointed out that the existence and multiplicity results are in connection with the first eigenvalues λ_1 and μ_1 of the p and $q-$Laplacian respectively. We distinguish the following six cases:

$$\begin{array}{ll}
(1) & 0 \le \lambda < \lambda_1, \quad 0 \le \mu < \mu_1, \\
(2) & 0 \le \lambda < \lambda_1, \quad \mu = \mu_1, \\
(3) & 0 \le \lambda < \lambda_1, \quad \mu > \mu_1, \\
(4) & \lambda = \lambda_1, \quad \mu = \mu_1, \\
(5) & \lambda = \lambda_1, \quad \mu > \mu_1, \\
(6) & \lambda > \lambda_1, \quad \mu > \mu_1.
\end{array}$$

The rest three possible cases can be treated analogously. In order not to increase the volume of the paper, we shall not present details for the cases (2), (3) and (5) merely pointing out that the methods of the next subsections carry over to these cases.

4.1 Existence theorem for $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$

The form of the functional J suggests that we consider

$$E_1(z) = 1 \quad \text{and} \quad E_2(w) = 1. \tag{43}$$

as the constraints in Lemma 4. Indeed, we calculate

$$E_1^{(1)} = pE_1(z) = pA,$$

$$E_1^{(2)} = E_2^{(1)} = 0,$$

$$E_2^{(2)} = qE_2(w) = qB.$$

Therefore

$$\det \begin{pmatrix} E_1^{(1)} & E_2^{(1)} \\ E_1^{(2)} & E_2^{(2)} \end{pmatrix} = pqAB > 0,$$

and the conditions of Lemma 4 are fulfilled. Moreover, since we are assuming (43), the inequalities (33) hold, that is, $1 = E_1 = A > 0$, $1 = E_2 = B > 0$ and

$$C = \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0.$$

Further, the functional I becomes

$$I(z,w) = K\frac{1}{\left(\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx\right)^{pq/d}}. \tag{44}$$

The main result in this subsection is the following

Theorem 1. *Suppose that* $(7)-(18)$ *hold and that, in addition,* $\lambda \in [0,\lambda_1)$, $\mu \in [0,\mu_1)$. *Then the problem* $(1),(2)$ *has at least two positive weak solutions* $(u_i, v_i) \in Y$, $i = 1,2$.

The proof of this theorem will be a consequence of the next two propositions.

Proposition 1. *Suppose that the conditions* $(7)-(18)$ *hold and that, in addition,* $\lambda \in [0,\lambda_1)$, $\mu \in [0,\mu_1)$. *Then the problem* $(1),(2)$ *has at least one positive weak solution* $(u_1, v_1) \in Y$.

Proof. The formulas (39) and (40) suggest to consider an auxiliary problem: find a maximizer (z^*, w^*) of

$$0 < M_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \mid E_1(z) = 1 \ \text{ and } \ E_2(w) = 1\right\}. \tag{45}$$

We claim that the problem (45) has a solution. Indeed, the sets

$$X_\lambda = \{z \in Y_p \mid E_1(z) = 1\},$$

and

$$X_\mu = \{w \in Y_q \mid E_2(w) = 1\},$$

are non-empty. By Lemma 1 we have that for any $z \in X_\lambda$:

$$\|z\|_p^p = \lambda\int_\Omega a(x)|z|^p dx + 1 \le \frac{\lambda}{\lambda_1}\|z\|_p^p + 1,$$

that is,

$$\|z\|_p^p \le \frac{\lambda_1}{\lambda_1 - \lambda}$$

and analogously

$$\|w\|_q^q \le \frac{\mu_1}{\mu_1 - \mu}.$$

Since $0 \le \lambda < \lambda_1$ and $0 \le \mu < \lambda_1$, we have

$$\|(z,w)\| = \|z\|_p^p + \|w\|_q^q \le \frac{\lambda_1}{\lambda_1 - \lambda} + \frac{\mu_1}{\mu_1 - \mu}.$$

Therefore a maximizing sequence (z_n, w_n) for (45) is bounded in Y. Thus we can suppose that (z_n, w_n) converges weakly in Y to some (z^*, w^*). By (17)

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx = M_{\lambda,\mu} > 0.$$

In particular $z^* \not\equiv 0$ and $w^* \not\equiv 0$.

The weakly lower semicontinuity of the corresponding norms, (7), (8) and $E_1(z_n) = 1$, $E_2(w_n) = 1$ imply that

$$E_1(z^*) \leq 1, \quad E_1(w^*) \leq 1.$$

since

$$\|z^*\|_p^p \leq \liminf_{n\to\infty} \|z_n\|_p^p,$$

$$\|w^*\|_q^q \leq \liminf_{n\to\infty} \|w_n\|_q^q,$$

$$\int_\Omega a(x)|z^*|^p dx = \lim_{n\to\infty} \int_\Omega a(x)|z_n|^p dx,$$

$$\int_\Omega b(x)|w^*|^q dx = \lim_{n\to\infty} \int_\Omega b(x)|w_n|^p dx.$$

If $E_1(z^*) < 1$ then there exists a number $t_1 > 1$ such that $E_1(t_1 z^*) = 1$ and hence $t_1 z^* \in X_\lambda$. If $E_2(w^*) < 1$ then there exists a number $t_2 > 1$ such that $E_2(t_2 w^*) = 1$ and hence $t_2 w^* \in X_\mu$. Therefore

$$\begin{aligned} \int_\Omega c(x)|t_1 z^*|^{\alpha+1}|t_2 w^*|^{\beta+1}dx &= t_1^{\alpha+1} t_2^{\beta+1} \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx \\ &= t_1^{\alpha+1} t_2^{\beta+1} M_{\lambda,\mu} \\ &> M_{\lambda,\mu} = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0\right\}, \end{aligned}$$

a contradiction. Thus $E_1(z^*) = 1$ or $E_2(w^*) = 1$. If $E_1(z^*) = 1$, $E_2(w^*) < 1$ or $E_1(z^*) < 1$, $E_2(w^*) = 1$ we can obtain another contradiction. Hence $(z^*, w^*) \in X_\lambda \times X_\mu$ is a solution of (45). By Lemma 4 it follows that (z^*, w^*) is a critical point of I. By Remark 1 we may assume $z^* \geq 0$ and $w^* \geq 0$. Thus, by Lemma 3, $(u_1 = r_1 z^*, v_1 = \rho_1 w^*)$ is a critical point of J. Therefore $(u, v) \in Y$ is a non-negative weak solution of (1), (2). Using the same arguments as in [3] we deduce that $u_1 > 0$, $v_1 > 0$ in Ω. This completes the proof.

Remark 2. In the scalar case it is known that weak solutions of

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + b(x)|u|^{q-2}u$$

belong to $C^{1,\nu}_{loc}(\Omega)$ for some ν (see [3]). Since our system is subcritical (see (11)), we expect that a similar result holds for (1). The regularity problem for weak solutions of quasilinear variational elliptic systems will be studied elsewhere.

Proposition 2. *Suppose that* (7) − (18) *hold and that, in addition,* $\lambda \in [0, \lambda_1)$, $\mu \in [0, \mu_1)$. *Then the problem* (1), (2) *has another positive weak solution* $(u_2, v_2) \in Y$.

Proof. Consider the following:

$$0 < \hat{M}_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \mid E_1(z) + E_2(w) = 1\right\}. \qquad (46)$$

Then the set

$$X_{\lambda,\mu} = \{(z, w) \in Y \mid E_1(z) + E_2(w) = 1\}$$

is not empty. By $E_1(z) + E_2(w) = 1$ and Lemma 1, for any $(z, w) \in X_{\lambda,\mu}$ we have

$$\|z\|_p^p + \|w\|_q^q \le 1 + \frac{\lambda}{\lambda_1}\|z\|_p^p + \frac{\mu}{\mu_1}\|w\|_q^q,$$

that is,

$$\frac{\lambda_1 - \lambda}{\lambda_1}\|z\|_p^p + \frac{\mu_1 - \mu}{\mu_1}\|w\|_q^q \le 1.$$

Since each of the summands above is strictly positive (recall that $\lambda < \lambda_1$, $\mu < \mu_1$), the latter inequality implies

$$\|z\|_p^p \le \frac{\lambda_1}{\lambda_1 - \lambda}$$

and

$$\|w\|_q^q \le \frac{\mu_1}{\mu_1 - \mu}.$$

Therefore $\|(z, w)\|$ is bounded. Hence we may suppose that a maximizing sequence (z_n, w_n) for (46) is bounded in Y. Thus we can assume that (z_n, w_n) converges weakly in Y to some (z^*, w^*). By (17) it follows that

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx = \hat{M}_{\lambda,\mu} > 0.$$

In particular $z^* \not\equiv 0$ and $w^* \not\equiv 0$.

The weakly lower semicontinuity of the corresponding norms, (7), (8) and $E_1(z_n) + E_2(w_n) = 1$ imply that

$$E_1(z^*) + E_1(w^*) \le 1,$$

that is

$$(\|z^*\|_p^p - \lambda \int_\Omega a(x)|z^*|^p dx) + (\|w^*\|_q^q - \mu_1 \int_\Omega b(x)|w^*|^q dx) \leq 1.$$

Since $\lambda < \lambda_1$, $\mu < \mu_1$ both summands above are positive. Hence

$$0 < E_1(z^*) + E_2(w^*) \leq 1.$$

We claim that actually

$$E_1(z^*) + E_2(w^*) = 1.$$

Indeed, if $E_1(z^*) + E_2(w^*) < 1$ there exists $t > 1$ be such that

$$t(E_1(z^*) + E_2(w^*)) = 1.$$

Then $(t^{1/p}z^*, t^{1/q}w^*) \in X_{\lambda,\mu}$ and

$$\begin{aligned}
\int_\Omega c(x)|t^{1/p}z^*|^{\alpha+1}|t^{1/q}w^*|^{\beta+1}dx &= t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx \\
&= t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \hat{M}_{\lambda,\mu} \\
&> \hat{M}_{\lambda,\mu} = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \mid \right. \\
&\qquad \left. E_1(z) + E_2(w) = 1\right\},
\end{aligned}$$

a contradiction (note that we have used (9)). Therefore we have proved the claim. Hence $(z^*, w^*) \in X_{\lambda,\mu}$ is a solution of (46). By an analogue of Lemma 4 for one constraint of type $E(z,w) = const$, (z^*, w^*) is a critical point of I. Indeed, since in our case $E(z,w) = E_1(z)+E_2(w) = 1$ the condition $E'(z,w)(z,w) \neq 0$ if $E(z,w) = 1$ is easily verified. The rest of the proof is the same as that of Proposition 1.

Proof of Theorem 1. It remains to show that the solutions found in Propositions 1 and 2 are distinct. The proof is by contradiction. Suppose that $(u_1, v_1) = (u_2, v_2)$. By the proofs of Propositions 1 and 2 it follows that

$$\frac{E_1(u_1)}{r_1^p} = \frac{E_2(v_1)}{\rho_1^q} = 1$$

and

$$\frac{E_1(u_2)}{r_2^p} + \frac{E_2(v_2)}{\rho_2^q} = 1,$$

where r_i, ρ_i, $i = 1,2$ are determined by (31) and (32), with z_i^*, w_i^*, $i = 1,2$. These relations imply that if the solutions are not distinct then there exists a number $m > 1$ such that

$$r_1^p = \frac{r_2^p}{m}, \quad \rho_1^q = \frac{\rho_2^q}{m'}, \quad \frac{1}{m} + \frac{1}{m'} = 1.$$

By (31) and (32) we have

$$r_1 = (c_1 C^{-q})^{1/d}, \quad \rho_1 = (c_2 C^{-p})^{1/d},$$

$$r_2 = (c_1 C^{-q}\frac{(1-s)^{\beta+1}}{s^{\beta+1-q}})^{1/d}, \quad \rho_2 = (c_2 C^{-p}\frac{s^{\alpha+1}}{(1-s)^{\alpha+1-p}})^{1/d},$$

where we have introduced a parameter $s = E_1(z_2^*)$. We note that the exact values of c_1 and c_2 are not important for the proof. Since $s \in (0,1)$, it is easy to show that the conditions $m > 1$ and $m' > 1$ are equivalent to

$$s^{\beta+1-q} < (1-s)^{\beta+1}$$

and

$$s^{\alpha+1} > (1-s)^{\alpha+1-p}.$$

From the last two inequalities, which simultaneously hold for certain $s \in (0,1)$, we obtain that

$$s^d > 1,$$

where $d > 0$ is given by (10). This is impossible for $s \in (0,1)$. Thus we have reached a contradiction. This concludes the proof.

4.2 The eigenvalue case $\lambda = \lambda_1$, $\mu = \mu_1$

We consider the problem (46) with $\lambda = \lambda_1$ and $\mu = \mu_1$. In this case the corresponding set $X_{\lambda,\mu}$ is not bounded in Y. Therefore we need to impose an additional condition on our data. Henceforth we shall suppose that the condition (19) is fulfilled.

Theorem 2. *Suppose that* (7) – (19) *hold and* $\lambda = \lambda_1$, $\mu = \mu_1$. *Then the problem* (1), (2) *has at least one positive weak solution* $(u, v) \in Y$.

Proof. The arguments of the proof of this theorem would be the same as those of Proposition 2 if we can prove that the problem (46) with $\lambda = \lambda_1$, $\mu = \mu_1$ has a solution.

Let (z_n, w_n) be a maximizing sequence such that

$$E_1(z_n) + E_2(w_n) = 1, \ \int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}dx = \hat{m}_n \to \hat{M}_{\lambda_1,\mu_1} > 0.$$

Suppose that $\|(z_n, w_n)\| \to \infty$ and put

$$s_n = \frac{z_n}{\|(z_n, w_n)\|^{1/p}}, \quad t_n = \frac{w_n}{\|(z_n, w_n)\|^{1/q}}, \quad \|(s_n, t_n)\| = 1.$$

Then

$$\|(z_n, w_n)\| \, [\, (\|s_n\|_p^p - \lambda_1 \int_\Omega a(x)|s_n|^p dx) + (\|t_n\|_p^p - \mu_1 \int_\Omega b(x)|t_n|^q dx) \,] = 1.$$

Therefore

$$\|s_n\|_p^p - \lambda_1 \int_\Omega a(x)|s_n|^p dx + \|t_n\|_q^q - \mu_1 \int_\Omega b(x)|t_n|^q dx = \frac{1}{\|(z_n, w_n)\|} \to 0, \quad n \to \infty.$$

Hence

$$\begin{aligned} & \|(s_n, t_n)\| - \lambda_1 \int_\Omega a(x)|s_n|^p dx - \mu_1 \int_\Omega b(x)|t_n|^p dx \\ = & \ \frac{1}{\|(z_n, w_n)\|} \to 0. \end{aligned} \tag{47}$$

and thus

$$\lim_{n\to\infty} \left[\lambda_1 \int_\Omega a(x)|s_n|^p dx + \mu_1 \int_\Omega b(x)|t_n|^p dx \right] = 1,$$

since $\|(s_n, t_n)\| = 1$. We may assume that (s_n, t_n) converges weakly in Y to some (s^*, t^*). Thus

$$\lambda_1 \int_\Omega a(x)|s^*|^p dx + \mu_1 \int_\Omega b(x)|t^*|^p dx = 1,$$

which implies that $(s^*, t^*) \not\equiv (0, 0)$. Furthermore

$$\|(s^*, t^*)\| \le \liminf_{n\to\infty} \|(s_n, t_n)\| = 1.$$

Now from (47) we deduce that

$$(\|s^*\|_p^p - \lambda_1 \int_\Omega a(x)|s^*|^p dx) + (\|t^*\|_q^q - \mu_1 \int_\Omega b(x)|t^*|^q dx) = 0.$$

The variational properties of the first eigenvalue of the p and $q-$Laplacian imply that both summands in the above relation are non-negative. Hence both are zero, which means, by Lemma 1, that

$$s^* = c_1 \varphi, \quad t^* = c_2 \psi.$$

Since

$$\begin{aligned} \int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1} dx &= \|(z_n, w_n)\|^{\frac{\alpha+1}{p} + \frac{\beta+1}{q}} \int_\Omega c(x)|s_n|^{\alpha+1}|t_n|^{\beta+1} dx \\ &= \hat{m}_n \to \hat{M}_{\lambda_1, \mu_1} > 0, \end{aligned}$$

we conclude that

$$\int_\Omega c(x)|s^*|^{\alpha+1}|t^*|^{\beta+1}dx \geq 0,$$

and therefore

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx \geq 0,$$

which contradicts (19). Thus we can assume that (z_n, w_n) is bounded and

$$\lim_{n\to\infty}(z_n, w_n) = (z^*, w^*)$$

weakly in Y. Then

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}dx \to \int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx = M_{\lambda_1,\mu_1} > 0.$$

This means that $z^* \neq 0$ and $w^* \neq 0$. Furthermore

$$0 \leq E_1(z^*) + E_2(w^*) \leq 1.$$

We claim that

$$0 < E_1(z^*) + E_2(w^*) \leq 1.$$

Indeed, first suppose that

$$0 = E_1(z^*) + E_2(w^*),$$

that is

$$0 = (\|z^*\|_p^p - \mu_1 \int_\Omega a(x)|z^*|^p dx) + (\|w^*\|_q^q - \mu_1 \int_\Omega b(x)|w^*|^q dx).$$

Therefore by Lemma 1 we know that

$$z^* = k_1\varphi, \quad w^* = k_2\psi,$$

for some $k_1, k_2 \neq 0$, and then

$$\int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx = |k|_1^{\alpha+1}|k|_2^{\beta+1}\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx = \hat{M}_{\lambda_1,\mu_1} > 0,$$

which is a contradiction since (19) holds.

Next, suppose that

$$0 < E_1(z^*) + E_2(w^*) < 1.$$

Then we can find $t > 1$ such that

$$t(E_1(z^*) + E_2(w^*)) = 1.$$

Further

$$\int_\Omega c(x)|t^{1/p} z^*|^{\alpha+1}|t^{1/q} w^*|^{\beta+1}dx = t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx$$

$$= t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\hat{M}_{\lambda,\mu}$$

$$> \hat{M}_{\lambda,\mu} = \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \mid E_1(z) + E_2(w) = 1\right\},$$

another contradiction.

In this way we have proved that

$$E_1(z^*) + E_2(w^*) = 1,$$

and therefore (z^*, w^*) is a maximizer of the problem (46) with $\lambda = \lambda_1$, $\mu = \mu_1$. The rest of the proof is the same as that of the Proposition 1. This completes the proof.

4.3 Existence of three distinct solutions for $\lambda > \lambda_1$, $\mu > \mu_1$

Theorem 3. *Suppose that* (7) – (19) *hold,* $\lambda > \lambda_1$ *and* $\mu > \mu_1$. *Then there exist* $\delta > 0$ *and* $\sigma > 0$ *such that for* $\lambda \in (\lambda_1, \lambda_1 + \delta)$, $\mu \in (\mu_1, \mu_1 + \sigma)$ *the problem* (1), (2) *has at least three positive weak solutions in* Y.

The proof of the above theorem will be a consequence of several lemmas.

To begin with, we define

$$M_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \quad | \quad E_1(z) = 1 \;\text{ and }\; E_2(w) = 1\right\}, \tag{48}$$

and

$$\tilde{M}_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \quad | \quad E_1(z) \le 1 \;\text{ and }\; E_2(w) \le 1\right\}. \tag{49}$$

Lemma 5. *The problems* (48) *and* (49) *are equivalent.*

Proof. Since $c^+ \not\equiv 0$ (see (18)) any maximizer of (48) is a maximizer of (49). Suppose for a moment that $(z, w) \in Y$ is a maximizer of (49) and $E_1(z) < 1$ or $E_2(w) < 1$. For instance, let $E_1(z) < 1$. Therefore there exists $k > 1$ such that $E_1(z) = 1$. Then

$$\int_\Omega c(x)|kz|^{\alpha+1}|w|^{\beta+1}dx = k^{\alpha+1}\int_\Omega c(x)|z^*|^{\alpha+1}|w^*|^{\beta+1}dx = k^{\alpha+1}\tilde{M}_{\lambda,\mu}, > \tilde{M}_{\lambda,\mu} \tag{50}$$

which is a contradiction. Thus $E_1(z) = E_2(w) = 1$. Therefore any maximizer of (49) is a maximizer of (48).

Lemma 6. *Let* (7) - (19) *hold. Then there exist* $\delta_1 > 0$ *and* $\varepsilon_1 > 0$ *such that for any* $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ *and* $\mu \in (\mu_1, \mu_1 + \varepsilon_1)$ *the problem* (47) *has a non-trivial solution* $(z_1, w_1) \in Y$.

Proof. From Lemma 5 we shall deduce the existence of $\delta_1 > 0$ and $\varepsilon_1 > 0$ corresponding to the problem (49). Suppose that the claim is not true, that is, there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that the problem (49) with $\lambda = \lambda_s = \lambda_1 + \delta_s$ and $\mu = \mu_s = \mu_1 + \varepsilon_s$ does not have solution. Fix an integer s and consider (49) with λ_s and μ_s. Denoting by (z_n^s, w_n^s) the corresponding maximizing sequence, we have

$$\lim_{n\to\infty} \int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx = \tilde{M}_{\lambda_s,\mu_s} > 0,$$

$$E_1(z_n^s) \leq 1,$$

and

$$E_2(w_n^s) \leq 1.$$

If (z_n^s, w_n^s) would be bounded, we may assume that it converges weakly in Y to some (z_0^s, w_0^s), when $n \to \infty$. Then

$$\int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx \to \int_\Omega c(x)|z_0^s|^{\alpha+1}|w_0^s|^{\beta+1}dx = \tilde{M}_{\lambda_s,\mu_s} > 0,$$

$$\int_\Omega |\nabla z_0^s|^p dx - \lambda_s \int_\Omega a(x)|z_0^s|^p dx \leq 1,$$

$$\int_\Omega |\nabla w_0^s|^q dx - \mu_s \int_\Omega b(x)|w_0^s|^q dx \leq 1.$$

Therefore (z_0^s, w_0^s) is a solution of (49) - a contradiction. Thus we may consider (z_n^s, w_n^s) to be unbounded. Let

$$(h_n^s, t_n^s) = \frac{(z_n^s, w_n^s)}{\|(z_n^s, w_n^s)\|}.$$

Since $\|(h_n^s, t_n^s)\| = 1$ we may assume that

$$\lim_{n\to\infty}(h_n^s, t_n^s) = (h_0^s, t_0^s)$$

weakly in Y. Then

$$\int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx = \|(z_n^s, w_n^s)\|^{\alpha+\beta+2}\int_\Omega c(x)|h_n^s|^{\alpha+1}|t_n^s|^{\beta+1}dx \to \tilde{M}_{\lambda_s,\mu_s} > 0,$$

therefore

$$\int_\Omega c(x)|h_0^s|^{\alpha+1}|t_0^s|^{\beta+1}dx \geq 0. \tag{51}$$

From the inequality $E_1(z_n^s) \leq 1$, that is,

$$\|(z_n^s, w_n^s)\|^p(\|h_n^s\|_p^p - \lambda_s\int_\Omega a(x)|h_n^s|^p dx) \leq 1$$

it follows that

$$\|h_n^s\|_p^p - \lambda_s\int_\Omega a(x)|h_n^s|^p dx \leq \frac{1}{\|(z_n^s, w_n^s)\|^p}.$$

By letting $n \to \infty$ we get

$$\|h_0^s\|_p^p - \lambda_s\int_\Omega a(x)|h_0^s|^p dx \leq 0. \tag{52}$$

On the other hand, summing up

$$\lambda_s\int_\Omega a(x)|h_n^s|^p dx \geq \|h_n^s\|_p^p - \frac{1}{\|(z_n^s, w_n^s)\|^p},$$

and

$$\mu_s\int_\Omega b(x)|t_n^s|^q dx \geq \|t_n^s\|_q^q - \frac{1}{\|(z_n^s, w_n^s)\|^q},$$

and letting $n \to \infty$, we obtain

$$\lambda_s\int_\Omega a(x)|h_0^s|^p dx + \mu_s\int_\Omega b(x)|t_0^s|^q dx \geq 1. \tag{53}$$

Clearly $\|(h_0^s, t_0^s)\| \leq 1$. This allows us to suppose that (h_0^s, t_0^s) converges weakly in Y to some (h_0, t_0). Letting $s \to \infty$ in (53), we get that

$$\lambda_1\int_\Omega a(x)|h_0|^p dx + \mu_1\int_\Omega b(x)|t_0|^q dx \geq 1.$$

Hence $(h_0, t_0) \not\equiv (0,0)$. Next, from the inequality (52) we obtain

$$0 \leq \|h_0\|_p^p - \lambda_s\int_\Omega a(x)|h_0|^p dx \leq 0.$$

The latter and the Lemma 1 imply that $h_0 = l\varphi$, $l \neq 0$. Starting with $E_2(w_n^s) \leq 1$ we can obtain $t_0 = k\psi$, $k \neq 0$, in a similar way. Then by (51) we get that

$$\int_\Omega c(x)|h_0|^{\alpha+1}|t_0|^{\beta+1}dx \geq 0,$$

and thus

$$|l|^{\alpha+1}|k|^{\beta+1}\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx \geq 0.$$

This contradicts our assumption (19).

Therefore there exist $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for any $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ and $\mu \in (\mu_1, \mu_1 + \varepsilon_1)$ the problem (49) has a solution $(z_1, w_1) \in Y$. By Lemma 5 $(z_1, w_1) \in Y$ is a solution of (48).

Lemma 7. *The set*

$$W^- = \left\{(z, w) \in Y \mid \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx = -1\right\}$$

is not empty and $m_{\lambda,\mu} < 0$, $\lambda > \lambda_1$, $\mu > \mu_1$, *where*

$$m_{\lambda,\mu} = \inf\left\{E_1(z) + E_2(w) \mid \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx = -1\right\}. \tag{54}$$

Proof. Set $z = \varphi$ and $w = \psi$. Then by (19) we have

$$\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx = \int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx < 0.$$

Therefore there exists $k \in \mathbb{R}$ such that

$$\int_\Omega c(x)|k\varphi|^{\alpha+1}|\psi|^{\beta+1}dx = -1,$$

and hence $(k\varphi, \psi) \in W^-$.

Since $\lambda > \lambda_1$ and $\mu > \mu_1$, we have

$$E_1(k\varphi) = |k|^p(\lambda_1 - \lambda)\int_\Omega a(x)|\varphi|^p dx < 0,$$

and

$$E_2(l\psi) = |l|^q(\mu_1 - \mu)\int_\Omega b(x)|\psi|^q dx < 0.$$

These inequalities imply that $m_{\lambda,\mu} < 0$.

Lemma 8. *Assume that* (7) - (19) *hold. Then there exist* $\delta_2 > 0$ *and* $\varepsilon_2 > 0$ *such that for any* $\lambda \in (\lambda_1, \lambda_1 + \delta_2)$ *and* $\mu \in (\mu_1, \mu_1 + \varepsilon_2)$ *the problem* (54) *has a non-trivial solution* $(z_2, w_2) \in Y$ *satisfying* $E_1(z_2) + E_2(w_2) < 0$.

Proof. The proof is by contradiction and it is analogous to that of Lemma 6.

Assume that the opposite assertion holds. Then there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that the problem (54) with $\lambda = \lambda_s = \lambda_1 + \delta_s$ and

$\mu = \mu_s = \mu_1 + \varepsilon_s$ does not have solution. Fix an integer s and consider (54) with λ_s and μ_s. Denote by (z_n^s, w_n^s) the corresponding maximizing sequence:

$$\int_\Omega c(x)|z_n|^{\alpha+1}|w_n|^{\beta+1}dx = -1,$$

$$\int_\Omega |\nabla z_n^s|^p dx - \lambda_s \int_\Omega a(x)|z_n^s|^p dx + \int_\Omega |\nabla w_n^s|^q dx - \mu_s \int_\Omega b(x)|w_n^s|^q dx \to m_{\lambda_s,\mu_s} < 0.$$

If (z_n^s, w_n^s) would be bounded, we can obtain as before that, there exists a solution (z_0^s, w_0^s) of (54):

$$\int_\Omega c(x)|z_0^s|^{\alpha+1}|w_0^s|^{\beta+1}dx = -1$$

and

$$\int_\Omega |\nabla z_0^s|^p dx - \lambda_s \int_\Omega a(x)|z_0^s|^p dx + \int_\Omega |\nabla w_0^s|^q dx - \mu_s \int_\Omega b(x)|w_0^s|^q dx = m_{\lambda_s,\mu_s} < 0,$$

which is a contradiction. Thus we may assume that (z_n^s, w_n^s) is unbounded. With the same notation as in Lemma 6, it follows that

$$\int_\Omega c(x)|h_n^s|^{\alpha+1}|t_n^s|^{\beta+1}dx = -\frac{1}{\|(z_n^s, w_n^s)\|^{\alpha+\beta+2}} \to 0.$$

Since the functional f_3 (see (17)) is lower weakly continuous we obtain

$$\int_\Omega c(x)|h_0^s|^{\alpha+1}|t_0^s|^{\beta+1}dx = 0. \tag{55}$$

Analogously to previous proofs, (55) enables us to conclude that

$$\int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1}dx = 0.$$

This contradicts (19). The fact that $E_1(z_2) + E_2(w_2) < 0$ follows from Lemma 7.

Lemma 9. *Let (7) - (19) hold. Then there exist $\delta_3 > 0$ and $\varepsilon_3 > 0$ such that for any $\lambda \in (\lambda_1, \lambda_1 + \delta_3)$ and $\mu \in (\mu_1, \mu_1 + \varepsilon_3)$ the problem (47) has another non-trivial solution $(z_3, w_3) \in Y$.*

Proof. Set

$$N_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \quad | \quad E_1(z) + E_2(w) = 1\right\}. \tag{56}$$

and

$$\hat{N}_{\lambda,\mu} := \sup\left\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx > 0 \quad | \quad E_1(z) + E_2(w) \le 1\right\}. \tag{57}$$

Following the argument of Lemma 5 it is easy to prove that the problems (56) and (57) are equivalent. (See the end of the proof of Proposition 2.) Therefore we shall deduce the existence of $\delta_3 > 0$ and $\varepsilon_3 > 0$ corresponding to the problem (57). Suppose that this is not true, that is, there exist sequences $\delta_s \to 0$, $\delta_s > 0$, and $\varepsilon_s \to 0$, $\varepsilon_s > 0$, such that the problem (57) with $\lambda = \lambda_s = \lambda_1 + \delta_s$ and $\mu = \mu_s = \mu_1 + \varepsilon_s$ does not have solution. Fix an integer s and consider (57) with λ_s and μ_s. Denoting by (z_n^s, w_n^s) the corresponding maximizing sequence, we have

$$\lim_{n\to\infty} \int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx = \hat{N}_{\lambda_s,\mu_s} > 0,$$

$$E_1(z_n^s) + E_2(w_n^s) \leq 1.$$

If (z_n^s, w_n^s) would be bounded, we may assume that it converges weakly in Y to some (z_0^s, w_0^s), when $n \to \infty$. Then

$$\int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx \to \int_\Omega c(x)|z_0^s|^{\alpha+1}|w_0^s|^{\beta+1}dx = \hat{N}_{\lambda_s,\mu_s} > 0,$$

$$E_1(z_0^s) + E_2(w_0^s) \leq 1.$$

Therefore (z_0^s, w_0^s) is a solution of (57) - a contradiction. Thus we may consider (z_n^s, w_n^s) to be unbounded. Let

$$h_n^s = \frac{z_n^s}{\|(z_n^s, w_n^s)\|^{1/p}}, \quad t_n = \frac{w_n^s}{\|(z_n^s, w_n^s)\|^{1/q}}, \quad \|(h_n^s, t_n^s)\| = 1.$$

Thus we may assume that

$$\lim_{n\to\infty}(h_n^s, t_n^s) = (h_0^s, t_0^s)$$

weakly in Y. Then

$$\int_\Omega c(x)|z_n^s|^{\alpha+1}|w_n^s|^{\beta+1}dx = \|(z_n^s, w_n^s)\|^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}} \int_\Omega c(x)|h_n^s|^{\alpha+1}|t_n^s|^{\beta+1}dx \to \hat{N}_{\lambda_s,\mu_s} > 0,$$

therefore

$$\int_\Omega c(x)|h_0^s|^{\alpha+1}|t_0^s|^{\beta+1}dx \geq 0. \tag{58}$$

From the inequality $E_1(z_n^s) + E_2(w_n^s) \leq 1$, that is,

$$\|(z_n^s, w_n^s)\| \; [\; (\|h_n^s\|_p^p - \lambda_s \int_\Omega a(x)|h_n^s|^p dx) + \|t_n^s\|_q^q - \mu_s \int_\Omega b(x)|t_n^s|^q dx) \;] \leq 1$$

it follows that

$$\|h_n^s\|_p^p - \lambda_s \int_\Omega a(x)|h_n^s|^p dx + \|t_n^s\|_q^q - \mu_s \int_\Omega b(x)|t_n^s|^q dx \leq \frac{1}{\|(z_n^s, w_n^s)\|}. \tag{59}$$

By letting $n \to \infty$ we get

$$(\|h_0^s\|_p^p - \lambda_s \int_\Omega a(x)|h_0^s|^p dx) + (\|t_0^s\|_q^q - \mu_s \int_\Omega b(x)|t_0^s|^q dx) \leq 0. \tag{60}$$

On the other hand we can obtain from (59) that

$$\lambda_s \int_\Omega a(x)|h_0^s|^p dx + \mu_s \int_\Omega b(x)|t_0^s|^q dx \geq 1. \tag{61}$$

Clearly $\|(h_0^s, t_0^s)\| \leq 1$. This allows us to suppose that (h_0^s, t_0^s) converges weakly in Y to some (h_0, t_0). Letting $s \to \infty$ in (61), it follows that

$$\lambda_1 \int_\Omega a(x)|h_0|^p dx + \mu_1 \int_\Omega b(x)|t_0|^q dx \geq 1.$$

Hence $(h_0, t_0) \not\equiv (0, 0)$.

Now from (60), by letting $s \to \infty$, we infer

$$(\|h_0\|_p^p - \lambda_1 \int_\Omega a(x)|h_0|^p dx) + (\|t_0\|_q^q - \mu_1 \int_\Omega b(x)|t_0|^q dx) \leq 0.$$

By the definition of λ_1 and μ_1 both summands above are non-negative. Therefore

$$\|h_0\|_p^p - \lambda_s \int_\Omega a(x)|h_0|^p dx = 0$$

and

$$\|t_0\|_q^q - \mu_s \int_\Omega b(x)|t_0|^q dx = 0.$$

The last two equalities and Lemma 1 imply that $h_0 = l\varphi$, $l \neq 0$ and $t_0 = k\psi$, $k \neq 0$. Then by (58), letting $s \to \infty$, we get that

$$\int_\Omega c(x)|h_0|^{\alpha+1}|t_0|^{\beta+1} dx \geq 0,$$

and thus

$$|l|^{\alpha+1}|k|^{\beta+1} \int_\Omega c(x)|\varphi|^{\alpha+1}|\psi|^{\beta+1} dx \geq 0,$$

a contradiction to (19). This completes the proof.

Proof of Theorem 3. Let δ_1, ε_1, $(z_1, w_1) \in Y$, δ_2, ε_2, $(z_2, w_2) \in Y$ and δ_2, ε_2, $(z_2, w_2) \in Y$ be as in Lemmas 6, 8, 9 respectively. Denote $\delta = \min(\delta_1, \delta_2, \delta_3)$ and $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Now we substitute (z_i, w_i), $i = 1, 2, 3$, in (31) and (32). In this way we obtain three pairs of positive numbers: (r_i, ρ_i), $i = 1, 2, 3$. Set

$$u_i = r_i z_i, \quad v_i = \rho_i w_i, \quad i = 1, 2, 3.$$

By Lemma 3, (u_1, v_1), (u_2, v_2) and (u_3, v_3) are weak solutions of (1),(2). By Lemma 6 it follows that

$$\frac{E_1(u_1)}{r_1^p} = E_1(z_1) = 1,$$

and

$$\frac{E_2(v_1)}{\rho_1^q} = E_2(w_1) = 1.$$

Thus

$$(u_1, v_1) \in S = \{(u, v) \mid \frac{E_1(u_1)}{r_1^p} = 1 \text{ and } \frac{E_2(v_1)}{\rho_1^q} = 1\}.$$

On the other hand, by Lemma 8 we have

$$\frac{E_1(u_2)}{|r_2|^p} + \frac{E_2(v_2)}{|\rho_2|^q} = E_1(z_2) + E_2(w_2) < 0.$$

Hence at least one of $E_1(u_2)$ and $E_2(v_2)$ is negative. Therefore (u_2, v_2) does not belong to S. We conclude that (u_1, v_1) and (u_2, v_2) are distinct. Similarly (u_2, v_2) and (u_3, v_3) are distinct. An argument analogous to that in the proof of Theorem 1 shows that (u_1, v_1) and (u_3, v_3) are distinct too. The rest of the proof is the same as that of Theorem 1. This completes the proof of Theorem 3.

5 A non-existence result of classical solutions

In this section we shall establish a non-existence result of classical solutions for a potential system associated to $(p, q)-$Laplacian operators. However, it is clear that 'the considered solutions are classical' does not seem to be a natural hypothesis for this kind of problem. Indeed, the natural class to consider should be the class of weak solutions.

Our argument, which is based on an earlier result by Pohozaev [10] (see also [14, 6]), enables only to consider classical solutions. We should mention that in the scalar case, Guedda and Veron [6] proved a Pohozaev type identity for weak solutions of the problem

$$\begin{cases} -div\,(|\nabla u|^{p-2}\nabla u) \;\; = \;\; f(u, v) \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

under some suitable growth assumption on f. We are confident that a Pohozaev type identity for weak solutions of potential systems associated to $p-$Laplacian operators still holds if the potential does not growth very fast. However, in the present paper we shall not consider this kind of generalization.

Let $\Omega \subset \mathbb{R}$ be a smooth bounded domain. Consider the following quasilinear potential system

$$\begin{cases} -div\,(|\nabla u|^{p-2}\nabla u) & = & \frac{\partial F}{\partial u}(x,u,v) \text{ in } \Omega \\ -div\,(|\nabla v|^{q-2}\nabla v) & = & \frac{\partial F}{\partial v}(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{62}$$

where $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R})$. Let $(u,v) \in (C^2(\Omega) \cap C^0(\bar{\Omega}))^2$ be a classical solution of (62). Then the Pohozaev Identity ([10]) for (62) can be written in the form

$$\begin{aligned} \frac{N-p}{p}\int_\Omega |\nabla u|^p dx \;+\; & \frac{N-q}{q}\int_\Omega |\nabla v|^q dx - N\int_\Omega F(x,u,v)dx - \int_\Omega D_x F(x,u,v)dx \\ & = -\left(1-\frac{1}{p}\right)\int_{\partial\Omega}|\nabla u|^p (x,\nu)dx - \left(1-\frac{1}{q}\right)\int_{\partial\Omega}|\nabla v|^q (x,\nu)dx. \end{aligned} \tag{63}$$

Now we are ready to prove the next

Theorem 4. *Suppose that Ω is strictly-starshaped with respect to the origin. Let $a,b,c \in C^1(\bar{\Omega})$ and $(u,v) \in (C^2(\Omega) \cap C^0(\bar{\Omega}))^2$ be a solution of (1), (2). Suppose that the assumptions in section 2 hold. In addition, assume that for any $\gamma, \sigma \in \mathbb{R}$ the following inequalities hold*

$$\frac{N-p}{p} + \gamma \geq 0,$$

$$\frac{N-q}{q} + \sigma \geq 0,$$

and for $x \in \Omega$ we have

$$\left(\frac{\lambda N}{p} - \gamma\lambda\right) a(x) - \frac{\lambda}{p}(\nabla a(x), x) \geq 0,$$

$$\left(\frac{\mu N}{q} - \sigma\mu\right) b(x) - \frac{\mu}{q}(\nabla b(x), x) \geq 0,$$

$$-Nc(x) - N\nabla(c(x),x) - ((\alpha+1)\gamma + (\beta+1)\sigma)c(x) \geq 0.$$

Then $u = v = 0$.

Proof. Multiplying the first equation of (1) by γu and integrating by parts we get

$$\gamma \int_\Omega |\nabla u|^p dx = \gamma\lambda \int_\Omega a(x)|u|^p dx + \gamma(\alpha+1) \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1} dx. \qquad (64)$$

Similarly

$$\sigma \int_\Omega |\nabla v|^q dx = \sigma\mu \int_\Omega b(x)|v|^q dx + \sigma(\beta+1) \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1} dx. \qquad (65)$$

Now we recall that the potential F is given by (20). Then substitute (20) into (63). Further, sum up the obtained identity with (64) and (65). Then the resulting identity, the inequalities given in the theorem, and the fact that Ω is strictly-starshaped imply that $u = v = 0$.

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