Approximating Characteristics of Multiplier Operators on S^2.

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Abstract

Walsh functions on S^2 are introduced and considered. We are demonstrating different properties of such functions and establishing sharp orders of ϵ - entropy for a wide range of multiplier operators on S^2. The analysis is essentially based on martingale technique and estimates of volumes of special convex bodies which are of independent interest.

1 Introduction

The Walsh functions form an orthonormal system which has found a lot of applications in many different situations (e.g., data transmission, filtering, pattern recognition, image enhancement, etc). In the section 2 we are introducing Walsh system on S^2 and presenting different properties of such functions. In the section 3 we are establishing general upper bounds for entropy of multiplier operators. Finally, in the section 4 general lower bounds are found.

Let us remind some definitions. Let X and Y be a Banach spaces with unit balls B_X and B_Y respectively. A finite set of points $x_1, ..., x_m$ is called ϵ-net for B_X in Y if for each $x \in B_X$ there is at least one point x_k of the net such that $\|x - x_k\|_Y \leq \epsilon$. The logarithm

$$H_\epsilon = \log_2 N_\epsilon(B_X),$$

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where $N_\epsilon(B_X) := \min\{n \ : \ \{x_1, ..., x_n\}$ is an ϵ net for B_X in Y $\}$, is the entropy of the set B_X in Y.

The definition has its roots in the notion of the metric entropy of a set which Kolmogorov introduced in the 1930s (see [10]). ϵ-Entropy is connected with the complexity of the tabulation problem and information theory (see [11]).

In this paper there are several universal constants which enter into the estimates. These constants are mostly denoted by the letters $K, C, C_1, C_2, ...$ We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper. For easy of notation we will wright $a_n \gg b_n$ for two sequences, if $a_n \geq C b_n$ for $n \in \mathbb{N}$ and $a_n \asymp b_n$ if $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$ and some constants C, C_1 and C_2. Through the text $[a]$ means entire part of $a \in \mathbb{R}$.

2 Walsh Functions on S^2

In this section we define real-valued functions $\psi_n, n \in \mathbb{N}^2$, on the 2-dimensional unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ in the Euclidean space $\mathbb{R}^3$, taking only the values ± 1, which we call *Walsh functions on* S^2. We prove several properties of these functions, in particular we prove that $\{\psi_n \ : \ n \in \mathbb{N}^2\}$ is a complete orthonormal subset of the Hilbert space $L_2(S^2)$.

We define the application $\xi : D = [0, \pi] \times [0, 2\pi] \to S^2$ by

$$\xi(\theta_1, \theta_2) = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2).$$

The Lebesgue normalized measure on S^2 will be denoted by dx and the Lebesgue measure of a measurable set $A \subset S^2$ by $|A|$. If $f \in L_1(S^2)$, we have that

$$\int_{S^2} f(x)dx = \frac{1}{4\pi} \int_0^{2\pi} d\theta_2 \int_0^{\pi} f(\xi(\theta_1, \theta_2)) \sin\theta_1 d\theta_1.$$

If $k, j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $1 \leq j \leq 2^k$, we write

$$I_j^{k,2} = [(j-1)2^{-k+1}\pi, j2^{-k+1}\pi)$$

and

$$I_j^{k,1} = [a_{j-1}^k, a_j^k), 1 \leq j \leq 2^k - 1, \quad I_{2^k}^{k,1} = [a_{2^k-1}^k, a_{2^k}^k],$$

where $0 = a_0^k < a_1^k < \ldots < a_{2^k}^k = \pi$ and

$$\int_{a_{j-1}^k}^{a_j^k} \sin t dt = \int_{I_j^{k,1}} \sin t dt = 2^{-k+1}, \quad 1 \leq j \leq 2^k.$$

The Rademacher functions $r_k^{(1)} : [0, \pi] \to \mathbb{R}$ and $r_k^{(2)} : [0, 2\pi] \to \mathbb{R}$, $k \in \mathbb{N}$, are defined by

$$r_k^{(i)} = \sum_{j=1}^{2^{k+1}} (-1)^{j+1} \chi_{I_j^{k+1,i}}, \quad i = 1, 2;$$

where χ_A is the characteristic function of the set A.

Given $n \in \mathbb{N}, n \geq 1$, let $n_1, \ldots, n_k \in \mathbb{N}$ such that $n_1 > n_2 > \cdots > n_k \geq 0$ and $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$. The Walsh functions $\varphi_n^{(1)} : [0, \pi] \to \mathbb{R}$ and $\varphi_n^{(2)} : [0, 2\pi] \to \mathbb{R}$ are defined by

$$\varphi_n^{(i)}(t) = r_{n_1}^{(i)}(t) r_{n_2}^{(i)}(t) \cdots r_{n_k}^{(i)}(t), \quad i = 1, 2$$

and $\varphi_0^{(1)}(t) = 1, t \in [0, \pi]$; $\varphi_0^{(2)}(t) = 1, t \in [0, 2\pi]$. Recall that $\{\varphi_n^{(2)} : n \in \mathbb{N}\}$ is a complete orthonormal subset of $L^2([0, 2\pi])$ (see [9]).

Let $m, n, p \in \mathbb{N}, m = n + p, p \geq 1$. We have that

$$I_j^{n+1,1} = \bigcup_{l=1}^{2^p} I_{(j-1)2^p+l}^{n+p+1}$$

and hence we can write

$$r_m^{(1)}(t) = \sum_{j=1}^{2^{n+1}} \sum_{l=1}^{2^p} (-1)^{l+1} \chi_{I_{(j-1)2^p+l}^{n+p+1}}(t);$$

$$r_n^{(1)}(t) = \sum_{j=1}^{2^{n+1}} (-1)^{j+1} \sum_{l=1}^{2^p} \chi_{I_{(j-1)2^p+l}^{n+p+1}}(t).$$

Therefore

$$r_m^{(1)}(t) r_n^{(1)}(t) = \sum_{j=1}^{2^{n+1}} (-1)^{j+1} \sum_{l=1}^{2^p} (-1)^{l+1} \chi_{I_{(j-1)2^p+l}^{n+p+1}}(t)$$

and thus

$$\int_0^\pi r_m^{(1)}(t) r_n^{(1)}(t) \sin t dt = 2^{-(n+p)} \sum_{j=1}^{2^{n+1}} (-1)^{j+1} \sum_{l=1}^{2^p} (-1)^{l+1} = 0.$$

By this way we can show that, for $m, n \in \mathbb{N}, m \neq n$,

$$\int_0^\pi \varphi_m^{(1)}(t)\varphi_n^{(1)}(t)\sin t dt = 0$$

Let (Ω, H, P) be a probability space. For $f \in L^1(\Omega, H, P)$ and a sub-σ-field E of H we denote by $E[f|E]$ the conditional expectation of f with respect to E. If the σ-field E is atomic, that is, if there exists a partition $\{B_j : j \in L\}$ of Ω,$L \subset \mathbb{N}$, such that $B_j \in E$ and $P(B_j) > 0$ for all $j \in L$, then

$$E[f|E](\omega) = \sum_{j\in L}\left(\frac{1}{P(B_j)}\int_{B_j} f dP\right)\chi_{B_j}(\omega) = \sum_{j\in L}\left(\int_\Omega f\eta_j dP\right)\eta_j(\omega),$$

where

$$\eta_j = \frac{1}{(P(B_j))^{1/2}}\chi_{B_j}.$$

Thus $\{\eta_j : j \in L\}$ is an orthonormal basis of $L^2(\Omega, E)$.

Now, let $(H)_{n\geq 0}$ be an increasing sequence of sub-σ-fields of H such that H is generated by the union of the σ-fields $H_n, n \geq 0$. A martingale with respect to $(H_n)_{n\geq 0}$ is a sequence of functions $(f_n)_{n\geq 0}$ such that $f_n \in L^1(\Omega, H_n, P)$ and $f_n = E[f_{n+1}|H_n]$ for all $n \geq 0$. If $f \in L^1(\Omega, H, P)$, then $(f_n)_{n\geq 0}$ where $f_n = E[f|H_n]$ is a martingale. Given a martingale $(f_n)_{n\geq 0}$, we associate with it the sequence of differences $(d_n)_{n\geq 0}, d_0 = f_0, d_n = f_n - f_{n-1}, n \geq 1$.

Let $v = (v_n)_{n\geq 1}$ be a predictable sequence, that is, $v_n : \Omega \to \mathbb{R}$ is H_{n-1}-measurable, $n \geq 1$ and let $(d_n)_{n\geq 1}$ be the sequence of differences of a martingale $f = (f_n)_{n\geq 0}$. Then the sequence $(g_n)_{n\geq 0}$ defined by $g_n = \sum_{k=1}^n v_k d_k$ is a martingale, known as the transform of the martingale f by v.

Theorem 2.1 ([8, p.29]) *Let (Ω, H, P) and $(H_n)_{n\geq 0}$ be as above and let $1 \leq p < \infty$, $f \in L^p(\Omega, H, P)$ and $f_n = E[f|H_n], n \geq 0$. Then the sequence $(f_n)_{n\geq 0}$ converges a.e. and in the norm of $L^p(\Omega, H, P)$ to the function f.*

Theorem 2.2 ([4]) *For $1 < p < \infty$ let p^* be the maximum of p and q where $1/p + 1/q = 1$. Let $(v_k)_{k\geq 1}$ be a predictable sequence uniformly bounded in absolute value by 1 and given an integrable function f let $f_n = E[f|H_n], d_n = f_n - f_{n-1}$.*

(i) *If $1 < p < \infty$ and $f \in L^p(\Omega, H, P)$, then the series $\sum_{k=1}^{\infty} v_k d_k$ converges in $L^p(\Omega, H, P)$ and*

$$\| \sum_{k=1}^{\infty} v_k d_k \|_p \leq (p^* - 1)\|f\|_p.$$

Moreover, the constant $p^* - 1$ is the best possible.

(ii) *If $f \in L^1(\Omega, H, P)$, then for all $\lambda > 0$,*

$$\lambda P(\{\omega : \sup_{n\geq 1} | \sum_{k=1}^{n} v_k(\omega) d_k(\omega)| > \lambda\}) \leq 2\|f\|_1.$$

For $k \in \mathbb{N}$ and $x \in S^2$ we define

$$R_k^{(1)}(x) = \sum_{j=1}^{2^{k+1}} (-1)^{j+1} \chi_{\xi(I_j^{k+1,1} \times [0,2\pi))}(x),$$

$$R_k^{(2)}(x) = \sum_{j=1}^{2^{k+1}} (-1)^{j+1} \chi_{\xi([0,\pi] \times I_j^{k+1,2})}(x).$$

If $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ with $n_1 > n_2 > \cdots > n_k \geq 0$ we define for $i = 1, 2$,

$$\psi_n^{(i)}(t) = R_{n_1}^{(i)}(t) R_{n_2}^{(i)}(t) \cdots R_{n_k}^{(i)}(t), \quad \psi_0^{(i)} \equiv 1.$$

For $m = (m_1, m_2) \in \mathbb{N}^2$ we define

$$\psi_m(x) = \psi_{m_1}^{(1)}(x) \psi_{m_2}^{(2)}(x).$$

For $k \in \mathbb{N}, k \geq 1$ we define

$$G_k = \{I_j^{k-1,1} \times I_l^{k,2} : 1 \leq j \leq 2^{k-1},\ 1 \leq l \leq 2^k\};$$

$$A_k = \{\xi(G) : G \in G_k\},\ A_0 = \{S^2\};$$

$$C_k = \{0, 1, 2, \ldots, 2^{k-1} - 1\} \times \{0, 1, 2, \ldots, 2^k - 1\};$$

$$B_k = C_k \setminus C_{k-1}, k \geq 2,\ B_1 = C_1.$$

We denote by $F_k, k \geq 0$, the σ-field of subsets of S^2 generated by the partition A_k of S^2.

Let us denote by $\#F$ the cardinality of a finite set F. Then $\#C_k = \#A_k = 2^{2k-1}$ and $\#B_k = 3\ 2^{2(k-1)-1}$.

Now we will define an ordering on $\mathbb{N}^2$.

Let $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{N}^2$. If there exists $k \in \mathbb{N}$ such that $n, m \in B_k$, we define

$$n < m \Longleftrightarrow n_1 < m_1 \ \ or \ \ n_1 = m_1 \ \ and \ \ n_2 < m_2$$

If $n \in B_k$ and $m \in B_l$ for $k \neq l$, we define

$$n < m \Longleftrightarrow k < l.$$

We define an ordering $\leq$ on $\mathbb{N}^2$ by

$$n \leq m \Longleftrightarrow n = m \ \ or \ \ n < m.$$

It is easy to see that the relation $\leq$ is a total ordering on $\mathbb{N}^2$.

Lemma 2.1 *$\{\psi_n : n \in C_k\}$ is an orthonormal basis of $L^2(S^2, F_k)$, $k \geq 1$.*

Proof. Let $n = (n_1, n_2), m = (m_1, m_2) \in C_k$. It follows from Definition 1.1 that

$$\psi_n(\xi(\theta_1, \theta_2)) = \varphi^{(1)}_{n_1}(\theta_1)\varphi^{(2)}_{n_2}(\theta_2), \quad \psi_m(\xi(\theta_1, \theta_2)) = \varphi^{(1)}_{m_1}(\theta_1)\varphi^{(2)}_{m_2}(\theta_2).$$

and hence

$$\int_{S^2} \psi_n(x)\psi_m(x)dx = \frac{1}{4\pi}\int_0^{2\pi} d\theta_2 \int_0^{\pi} \psi_n(\xi(\theta_1, \theta_2))\psi_m(\xi(\theta_1, \theta_2)) \sin\theta_1 d\theta_1 =$$

$$\frac{1}{4\pi}\int_0^{\pi} \varphi^{(1)}_{n_1}(\theta_1)\varphi^{(1)}_{m_1}(\theta_1) \sin\theta_1 d\theta_1 \int_0^{2\pi} \varphi^{(2)}_{n_2}(\theta_2)\varphi^{(2)}_{m_2}(\theta_2)d\theta_2.$$

Suppose $n \neq m$. If $n_2 \neq m_2$, then

$$\int_0^{2\pi} \varphi^{(2)}_{n_2}(\theta_2)\varphi^{(2)}_{m_2}(\theta_2)d\theta_2 = 0,$$

since $\{\varphi^{(2)}_k : k \in \mathbb{N}\}$ is an orthonormal subset of $L^2([0, 2\pi])$, and if $n_1 \neq m_1$, it follows from (1.1) that

$$\int_0^{\pi} \varphi^{(1)}_{n_1}(\theta_1)\varphi^{(1)}_{m_1}(\theta_1) \sin\theta_1 d\theta_1 = 0.$$

Therefore we have that

$$\int_{S^2} \psi_n(x)\psi_m(x)dx = 0.$$

Since $\psi_n(x) = \pm 1$ for all $x \in S^2$, then

$$\int_{S^2} |\psi_n(x)|^2 dx = \int_{S^2} dx = 1.$$

It is easy to see that ψ_n is F_k-measurable for all $n \in C_k$ and thus we can conclude that $\{\psi_n : n \in C_k\}$ is an orthonormal subset of $L^2(S^2, F_k)$.

The set $\{\frac{1}{|A|^{1/2}}\chi_A : A \in A_k\}$ is an orthonormal basis of $L^2(S^2, F_k)$ and hence the dimension of $L^2(S^2, F_k)$ is $\# A_k = \# C_k$. Therefore $\{\psi_n : n \in C_k\}$ is an orthonormal basis of $L^2(S^2, F_k)$. $\blacksquare$

For $f \in L^1(S^2)$ and $n \in \mathbb{N}^2$ we define

$$S_n(f)(x) = \sum_{m \le n} c_m(f)\psi_m(x)$$

where

$$c_m(f) = \int_{S^2} f(x)\psi_m(x)dx.$$

Lemma 2.2 *Let $a_k = (2^{k-1} - 1, 2^k - 1), k \ge 1$. If $f \in L^1(S^2)$ then*

$$E[f|F_k] = S_{a_k}(f).$$

Proof. Let $f_k = E[f|F_k]$. Since $f_k \in L^2(S^2, F_k)$ and $\{\eta_A = \frac{1}{|A|^{1/2}}\chi_A : A \in A_k\}$ is an orthonormal basis of $L^2(S^2, F_k)$, then

$$f_k = \sum_{A \in A_k} \left(\int_{S^2} f_k \eta_A dx\right) \eta_A = \sum_{A \in A_k} \left(\int_{S^2} f \eta_A dx\right) \eta_A$$

and thus

$$c_m(f_k) = \sum_{A \in A_k} \left(\int_{S^2} f \eta_A dx\right) \left(\int_{S^2} \psi_m \eta_A dx\right) =$$

$$\int_{S^2} f \left(\sum_{A \in A_k} \left(\int_{S^2} \psi_m \eta_A dx\right) \eta_A\right) dx =$$

$$\int_{S^2} f \psi_m dx = c_m(f).$$

But by Lemma 1.1, $\{\psi_n : n \in C_k\}$ is an orthonormal basis of $L^2(S^2, F_k)$ and hence

$$f_k = \sum_{m \in C_k} c_m(f_k)\psi_m = \sum_{m \le a_k} c_m(f)\psi_m = S_{a_k}(f). \blacksquare$$

Theorem 2.3 *$\{\psi_n : n \in I\!N^2\}$ is an complete orthonormal subset of $L^2(S^2)$.*

Proof. Let $f \in L^2(S^2)$. By Lemma 2.2 we have that $S_{a_k}(f) = E[f|F_k], k \geq 1$. Then $(S_{a_k}(f))_{k\geq 1}$ is a martingale and by Theorem I we have that $S_{a_k}(f) \to f$ in the norm of $L^2(S^2)$, when $k \to \infty$. Suppose that f is orthonormal to ψ_m for all $m \in \mathbb{N}^2$. Then $a_m(f) = 0$ for all $m \in \mathbb{N}^2$ and hence $S_{a_k}(f) \equiv 0$. But $S_{a_k} \to f$ and thus $f \equiv 0$. ■

The next result follows immediately from Theorem 2.2 and Lemma 2.2.

Theorem 2.4 *For $1 < p < \infty$ let p^* be the maximum of p and q where $1/p + 1/q = 1$. Let $(v_k)_{k\geq 1}$ be a sequence of functions on S^2 uniformly bounded in absolute value by 1, such that v_k is F_{k-1}-measurable, $k \geq 1$, and given an integrable function $f : S^2 \to I\!R$ and $k \geq 1$, let*

$$d_k(f) = S_{a_k}(f) - S_{a_{k-1}}(f) = \sum_{m\in B_k} c_m(f)\psi_m.$$

(i) *If $1 < p < \infty$ and $f \in L^p(S^2)$, then the series $\sum_{k=1}^{\infty} v_k d_k(f)$ converges in $L^p(S^2)$ and*

$$\|\sum_{k=1}^{\infty} v_k d_k(f)\|_p \leq (p^* - 1)\|f\|_p.$$

Moreover, the constant $p^* - 1$ is the best possible.

(ii) *If $f \in L^1(S^2)$, then for all $\lambda > 0$,*

$$\lambda|\{x : \sup_{n\geq 1} |\sum_{k=1}^{n} v_k(x) d_k(f)(x)| > \lambda\}| \leq 2\|f\|_1.$$

3 Upper Bounds

Let $\psi_1, \psi_2, ...$ be a basis in a real Banach space X,

$$x = \sum_{k=1}^{\infty} x_k \psi_k$$

with unit ball $B_X = \{x | \|x\| \leq 1\}$ (we will specify then the system $\{\psi_k\}$ as the system of Walsh functions on S^2 with the fixed ordering). Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$,

$\lambda_k \neq 0$, $|\lambda_1| \geq |\lambda_2| \geq ..., \lambda_N \geq ...$, $k \in \mathbb{N}$ be a multiplier operator

$$\Lambda x = \sum_{k=1}^{\infty} \lambda_k x_k \psi_k.$$

Let us put $P_N x = \sum_{k=1}^{N} x_k \psi_k$,

$$B_N = B \cap L_N,\ \Lambda U_2 = \{y | y = \Lambda x,\ \|x\| \leq 1\},\ (\Lambda U_2)_N = \Lambda U_2 \cap L_N,$$

$$P_N B = \{z \in L_N | z = P_N x, x \in B\}\ P_N \Lambda U_2 = \{z \in L_N | z = P_N x, x \in \Lambda U_2\},$$

$$\gamma_N(\Lambda U_2) = \sup\{\|x - P_N x\| | x \in \Lambda U_2\},$$

$$\beta_N(\Lambda U_2) = \left(\sup\{\|\Lambda^{-1} x\| | \|x\| \leq 1,\ x \in L_N\right)^{-1}.$$

Using the results of the section 2 it is easy to check that

$$\beta_N = |\lambda_N|, \quad \gamma_N = |\lambda_{N+1}|.$$

It is known that (see [10] p. 280)

$$H_{\epsilon_N}(\Lambda U_2,\ L_2(S^2))$$

$$\leq -N \log_2 \beta_N + \sum_{k=1}^{N} \log_2 |\lambda_k| + N \log_2 2 \|P_n\|,$$

where $\epsilon_N = \beta_N + \gamma_N$. It is easy to verify that $C_1 = \|P_N\| = 1$ and

$$H_{2\lambda_N}(\Lambda U_2,\ L_2(S^2))$$

$$\leq -N \log_2 |\lambda_N| + \sum_{k=1}^{N} \log_2 |\lambda_k| + N. \tag{1}$$

4 Lower Bounds

The problem of estimating of the entropy from below usually splits into two parts: reduction to some finite dimensional problem in Euclidean space, and obtaining a lower estimate for volume of special body in $\mathbb{R}^n$.

As regards the first part of the problem, in many cases its solution is relatively simple; therefore the main difficulty is to obtain proper lower estimates of volumes for special convex bodies in Euclidean space which are connected with the structure of Walsh system on S^2.

4.1 The Reduction

Suppose that the multiplier sequence $\Lambda = \{\lambda_m\}$ is non-encreasing and positive. In general case we should consider a proper rearrangement, $\lambda_N^* = |\lambda_{m_N}|$, $|\lambda_{k_1}| \geq |\lambda_{k_2}| \geq ... \geq |\lambda_{k_N}| \geq ....$

We will show that for a fixed $N \in \mathbb{N}$ and $\epsilon > 0$ there are $[C/\epsilon]^N$ functions in ΛU_∞ such that for all $1 \leq m_1 \neq m_2 \leq [C/\epsilon]^N$,

$$\|f^{m_1} - f^{m_2}\|_{L_1(S^2)} \geq \lambda_N \epsilon^2/2. \tag{2}$$

Let us put $\mathcal{T}_N = lin\{\psi_1, ..., \psi_N\}$. It is easy to check that for any $t_N \in \mathcal{T}_N$

$$\|t_N\|_V = \inf_{\phi \in \mathcal{T}_N^\perp} \|t_N - \phi\|_{L_\infty(S^2)},$$

where $\phi \in \mathcal{T}_N^\perp$ means $\int_{S^2} \phi \cdot t_N dx = 0$, $\forall t_N \in \mathcal{T}_N$ and $V = J(B_1^N)^o$ (see (5) and (10) for the definition).

If there is such absolute constant $0 < C < 1$ that for all $N \in \mathbb{N}$

$$Vol_N((B_1^N)^o) \geq C^N Vol_N(B_2^N), \tag{3}$$

then the cardinality of minimal ϵ - net for $(B_\infty^N)^o$ in the Euclidean norm is $\geq (C/\epsilon)^N$. Hence there are $[C/\epsilon]^N$ polynomials $t_N^m \in J(B_1^N)^o$, $1 \leq m \leq [c/\epsilon]^N$ such that $\|t_N^{m_1} - t_N^{m_2}\|_{L_2(S^2)} \geq \epsilon$, $1 \leq m_1 \neq m_2 \leq [C/\epsilon]^N$. Let us put $f^m = \Lambda(t_N^m - \phi^m)$, $1 \leq m \leq [c/\epsilon]^N$ and $\Lambda^{1/2} = \{\lambda_k^{1/2}\}$, then $f_m \in \Lambda U_\infty$ and

$$\|f^{m_1} - f^{m_2}\|_{L_1(S^2)}$$

$$\geq \lambda_N \cdot 2^{-1} \cdot \left(\|\Lambda^{1/2}(t_N^{m_1} - t_N^{m_2})\|_{L_2(S^2)}^2 + \|\Lambda^{1/2}(\phi^{m_1} - \phi^{m_2})\|_{L_2(S^2)}^2 \right)$$

$$\geq \lambda_N \cdot \epsilon^2/2,$$

since $\phi^m \in \mathcal{T}_N^\perp$ for all $1 \leq m \leq [C/\epsilon]^N$. Now we have

$$H_{\epsilon^2 \lambda_N/2} \geq N \log_2 \left[\frac{C}{\epsilon}\right] \tag{4}$$

Let us put $\epsilon = C/2$, then (4) takes the form

$$H_{C^2 \lambda_N/8} \geq N.$$

4.2 Finite Dimensional Estimates

The approach in part of lower bounds in $\mathbb{R}^n$ makes essential use of methods and results from Geometry of Banach spaces. We shall estimate volumes of ellipsoids of F. John for a special class of convex bodies in $\mathbb{R}^n$. Let us remind that for a convex body $V \in \mathbb{R}^n$ the ellipsoid of F. John $\mathcal{E}_V$ is the ellipsoid of maximal volume contained in V.

Usually an ellipsoid which determines the Banach-Mazur distance is sufficiently far from the ellipsoid of F. John and in applications we have a very little information concerning special convex bodies in $\mathbb{R}^n$ which are connected with the structure of a fixed orthonormal system.

This is a source of fundamental difficulties which occur if we try to apply the results of the Geometry of Banach spaces to various open problems in functional spaces.

Just in a few situations it was possible to specify the ellipsoids of F. John which are connected with special orthonormal systems. For example, using shift invariance of trigonometric system it is possible to find the F. John ellipsoids.

General method of estimates of volumes of the F. John ellipsoids in the case of bounded orthonormal systems has been offered by A. Kushpel (see e.g. [6]).

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space. Furthermore, for any $\alpha, \beta \in \mathbb{R}^n$ we define $\langle \alpha, \beta \rangle = \sum_{k=1}^n \alpha_k \beta_k$. Let Vol_n be the standard n-dimensional volume of subsets in $\mathbb{R}^n$.

Let us consider the Walsh system $\{\psi_k\}$ which is orthonormal in $L_2(S^2, dx)$, where $d\nu(\tau)$ is the normalized invariant measure on S^2. Set $\Psi_n = \mathrm{span}\{\psi_1, \ldots, \psi_n\}$ and let $J : \mathbb{R}^n \to \Psi_n$ be the coordinate isomorphism that assigns to $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ the function $\psi^\alpha = \sum_{k=1}^n \alpha_k \psi_k \in \Psi_n$. The definition $\|\alpha\|^{(p)} := \|J\alpha\|_{L_p(S^2)}$ induces a norm on $\mathbb{R}^n$. Let us put

$$B_p^n := \{\alpha \in \mathbb{R}^n \mid \|J\alpha\|_{L_p(S^2)} \leq 1\}. \tag{5}$$

It is easy to see that for any $1 \leq p \leq \infty$ the set B_p^n is a convex central symmetric body in $\mathbb{R}^n$.

The following statement has been offered by A. Kushpel (see [6]).

Theorem 4.2.1 *For any $n \in \mathbb{N}$ and $1 \leq p \leq 2$*

$$Vol_n(\mathcal{E}_{B_p^n}) = Vol_n(B_2^n).$$

Proof. It is clear that for all $1 \leq p \leq 2$ and $n \in \mathbb{N}$

$$B_2^{(n)} \subset B_p^{(n)},$$

so that by the definition

$$Vol_n(\mathcal{E}_{B_p^n}) \geq Vol_n(B_2^n). \tag{6}$$

Let t_n be any polynomial, $t_n(x) = \sum_{k=1}^n \alpha_k \psi_k$, then

$$\|t_n\|_{L_1(S^2)} = \left\| \sum_{k=1}^n \alpha_k \psi_k \right\|_{L_1(S^2)}$$

$$\geq \max_{1 \leq k \leq n} |\alpha_k| = \|\alpha\|_{l_\infty^n},$$

since Walsh functions ψ_k are orthonormal on S^2 and
$|\psi_k(x)| \leq 1$ for all $1 \leq k \leq n$ and $x \in S^2$. It means that

$$B_1^n \subset Q^n, \tag{7}$$

where Q^n is n-dimensional cube. It is known that the Euclidean ball B_2^n is the unique ellipsoid of maximum volume contained in n-dimensional cube (see [5], p. 75), or

$$\mathcal{E}_{Q^n} = B_2^n. \tag{8}$$

Comparing (7) - (8) we are getting that

$$Vol_n(\mathcal{E}_{B_1^n}) \leq Vol_n(B_2^n). \tag{9}$$

¿From (6) and (9) it follows now that $Vol_n(\mathcal{E}_{B_p^n}) = Vol_n(B_2^n)$. ■

For a convex centrally symmetric body $V \subset \mathbb{R}^n$ we define the polar body V^o of V as following

$$V^o = \{x \in \mathbb{R}^n | \sup_{y \in V} |\langle x, y \rangle| \leq 1\}. \tag{10}$$

If $\|\cdot\| = \|\cdot\|_X$ is the norm on $\mathbb{R}^n$ induced by V, then V^o coincides with the unit ball of the dual space X^*.

We will need the following definition. A normed space X is said to have cotype 2 if there is some constant C sucn that

$$C \cdot Ave_{\epsilon=\pm 1} \left\| \sum \epsilon_k x_k \right\| \geq \left(\left\| \sum x_k \right\|^2 \right)^{1/2}$$

whenever $\{x_k\}$ is a finite sequence of vectors in X. The smallest such constant $C_2(X)$ is called cotype-2 constant of X. It is easy to check

$$C_2(L_1(S^2)) < 2^{1/2}, \tag{11}$$

(see e.g. [7] p. 73]).

It is known (see [3]) that if a normed n-dimensional space X with unit ball V has cotype 2 then

$$(Vol_n V)^{1/2} \geq C \cdot C_2(X) \cdot (\ln C_2(X))^4 \cdot \left(\frac{(Vol_n B_2^n)^2}{Vol_n \mathcal{E}_V} \right)^{1/n}, \tag{12}$$

where $\mathcal{E}_V$ is the ellipsoid of F. John for V, and $C_2(X)$ is the cotype 2 constant of X. Comparing Theorem 4.2.1, (11) and (12) we are getting the condition (3).

Finally, comparing (1) and (4) we are getting the following result.

Theorem 1.1 **(i)** *Let $\{\lambda_k\}_{k\in I\!N}$ be an arbitrary sequence of real numbers and $\{\lambda_k^*\}_{k\in I\!N}$ is a rearrangement of $\{|\lambda_k|\}_{k\in I\!N}$ in a nonincreasing order. There is a positive constant $C > 0$ such that for any $n \in I\!N$ and all $1 \leq p, q \leq \infty$*

$$H_{C\lambda_n^*}(\Lambda U_p, L_q(S^2)) \geq n.$$

(ii) *For any $1 \leq q \leq 2 \leq p \leq \infty$*

$$H_{2\lambda_n^*}(\Lambda U_p,\ L_q(S^2)) \leq -n \log_2 \lambda_n^* + \sum_{k=1}^{n} \log_2 \lambda_k^* + n.$$

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