Subgroups of constructible nilpotent–by–abelian groups and a generalisation of a result of Bieri-Newmann-Strebel

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Abstract *We prove a Σ-version of the Bieri-Newmann-Strebel's result that for a finitely presented group G without free subgroups of rank two $\Sigma^1(G)^c$ has no antipodal points [6]. More precisely we prove that for such a group G*

$$conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^2(G)^c$$

If G is a finitely generated nilpotent-by-abelian group we show

$$conv_{\leq 2}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^2(G,\mathbb{Z})^c$$

The latter result is used in constructing a counter example to a conjecture of H. Meinert [14] about homological properties of subgroups of constructible nilpotent-by-abelian groups.

1. Introduction.

In this paper we refine some results of R. Bieri, W. Neumann and R. Strebel and give a counter example to a H. Meinert's conjecture about finiteness properties of subgroups and higher invariants of constructible nilpotent–by–abelian groups. The counter example will show that Σ^m–Conjecture type formula (originally suggested for metabelian groups) does not hold for constructible nilpotent-by-abelian groups. A soluble group is constructible (in the sense of Baumslag and Bieri [1]) if it can be built from the trivial group using finite extensions and ascending HNN–extensions. One of the characterising properties of constrictible nilpotent–by–abelian groups is that the geometric invarint $\Sigma^1(G)^c = S(G) \setminus \Sigma^1(G)$ lies in an open half subspace [8, Thm A]. The homological geometric invariants $\{\Sigma^m(G,A)\}_{m\in\mathbb{N}}$ of a (left) $\mathbb{Z}G$–module A are defined by

$$\Sigma^m(G,A) = \{[\chi] \in S(G) \mid A \text{ is of type } FP_m \text{ over the monoid ring } \mathbb{Z}G_\chi\},$$

where $S(G)$ is the set of the equivalence classes $[\chi] = \mathbb{R}_{>0}\chi$ for non-trivial characters $\chi \in Hom(G, \mathbb{R})$ and $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. If the torsion free part of the abelianization of G has rank n we can identify $S(G)$ with the unit sphere in the euclidean space $\mathbb{R}^n$. The geometric homotopical invariants $\{\Sigma^m(G)\}_{m\in\mathbb{N}}$ are homotopical versions of the homological geometric invariants $\{\Sigma^m(G, \mathbb{Z})\}_{m\in\mathbb{N}}$ and are defined only for groups of homotopical type F_m. We omit the definition but note that by [7, Thm 6.4] $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$.

The geometric invariant $\Sigma^1(G)$ was first introduced for metabelain groups in [9] (with a definition different from the above one). There Bieri and Strebel show that a finitely generated metabelian group is finitely presented if and only if $\Sigma^1(G)^c$ does not have antipodal points. Even in the non-metabelian case one of the directions of this result holds, more precisely by [6, Thm C] if G is a finitely presented group without non–abelian free subgroups the invariant $\Sigma^1(G)^c$ has no antipodal points. The ideas introduced in the proofs of the above results can be modified to prove the following two theorems. In the case of a finitely presented abelian-by-nilpotent group G Theorem A2 is proved in [13, Thm 8.1] with methods different from ours.

Theorem A1. *If G is a finitely presented group without non-abelian free subgroups then*

$$conv_{\leq 2}(\mathbb{R}_{>0}.\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}.\Sigma^2(G)^c.$$

Theorem A2. *If G is a finitely generated nilpotent-by- abelian group*

$$conv_{\leq 2}(\mathbb{R}_{>0}.\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}.\Sigma^2(G, \mathbb{Z})^c$$

By definition for a subset X of the eucledian space $\mathbb{R}^n$ $conv_{\leq m}X$ denotes the convex hull of not more than m elements of X and we view $\mathbb{R}_{>0}\Sigma^1(G)^c$ as the subset $\{\chi \in Hom(G, \mathbb{R}) \mid [\chi] \in \Sigma^1(G)^c\}$ of $\mathbb{R}^n = Hom(G, \mathbb{R})$.

In general for finitely presented groups $\Sigma^2(G, \mathbb{Z})^c \subseteq \Sigma^2(G)^c$. Furthermore there are examples of finitely presented Artin groups where the inclusion is strict [17, Main Thm]. Still we do not know whether there is a finitely presented nilpotent-by-abelian group such that $\Sigma^2(G, \mathbb{Z})^c \neq \Sigma^2(G)^c$. Thus we cannot deduce Theorem A2 from Theorem A1 and will give separate proofs of these results.

Theorem A2 will be used in the construction of the promised counter example of [14, Conj. 13]. One of the objectives of [14] is to discuss finiteness properties of subgroups of constructible nilpotent–by–abelian groups. Obviously constructible soluble groups G are of homological type FP_∞ but the homological structure of the subgroups can be much more interesting and dificult to determine. By [7, Thm 5.1] to understand the homological structure of the subgroups containing the derived subgroup we have to calculate the homological geometric invariants $\{\Sigma^m(G,\mathbb{Z})\}_{m\in\mathbb{N}}$ and even in the case of constructible nilpotent–by–abelian groups G these invariants are not completely understood. Still by [15, Thm B] it is known that for constructible nilpotent-by-abelian groups

$$conv_{\leq m}(\mathbb{R}_{>0}.\Sigma^1(G,\mathbb{Z})^c) \subseteq \mathbb{R}_{>0}.\Sigma^m(G,\mathbb{Z})^c \subseteq conv(\mathbb{R}_{>0}.\Sigma^1(G,\mathbb{Z})^c) \qquad (1)$$

and hence

$$\Sigma^\infty(G,\mathbb{Z})^c = \cup_{m\in\mathbb{N}}\Sigma^m(G,\mathbb{Z})^c = [conv(\mathbb{R}_{>0}.\Sigma^1(G,\mathbb{Z})^c)]. \qquad (2)$$

In this paper we keep to the notations of [6] and [7] that slightly defer from the notations used by H. Meinert and R. Gehrke in whose papers $\Sigma^m(G,\mathbb{Z})$ is the relevent set in $Hom(G,\mathbb{R})$ not the projection to $S(G)$. In [14] H. Meinert shows that for a constructible nilpotent–by–abelian group G each of the following conditions implies the next one

1. $m \geq dim_{\mathbb{R}}\ span(\Sigma^1(G,\mathbb{Z})^c)$.
2. All subgroups of G of type FP_m are in fact constructible.
3. $\Sigma^m(G,\mathbb{Z})^c = \Sigma^\infty(G,\mathbb{Z})^c$.

Furthermore he conjectures that these three conditions are equivalent. But as the following theorem shows this turns wrong.

Theorem B. *There exists a constructible group G, an extension of a nilpotent of class two group N by an abelian group Q such that*

1. $3 = dim_{\mathbb{R}}\ span(\Sigma^1(G,\mathbb{Z})^c) = dim_{\mathbb{R}}\ Hom(Q,\mathbb{R})$.

2. There exists a subgroup of G of type FP_2 which is not constructible.

3. $\Sigma^2(G,\mathbb{Z})^c = \Sigma^\infty(G,\mathbb{Z})^c = [conv_{\leq m_0}(\mathbb{R}_{>0}\Sigma^1(G,\mathbb{Z})^c)]$ *where* $m_0 = min\{m \mid conv(\mathbb{R}_{>0}\Sigma^1(G,\mathbb{Z})^c) = conv_{\leq m}(\mathbb{R}_{>0}\Sigma^1(G,\mathbb{Z})^c)\} = 3$

The last part of Theorem B shows that the first inclusion in (1) can be strict and thus Σ^m–Conjecture type formula cannot hold even for nilpotent-by-abelian groups of type FP_∞ (for the definition of the Σ^m–Conjecture see [15, p.386]). It will be interesting to find a series of constructible nilpotent-by-abelian groups $\{G_m\}_{m\geq 1}$ such that

$$dim_{\mathbb{R}}\ span(\Sigma^1(G_m,\mathbb{Z})^c) - min\{t \mid \Sigma^t(G_m,\mathbb{Z})^c = \Sigma^\infty(G_m,\mathbb{Z})^c\} = m.$$

We do not know whether taking direct products of the counter example given by Theorem B will give a series with the required properties. The problem is that there is not complete understanding of how to express $\{\Sigma^m(G_1\times G_2)\}_{m\in\mathbb{N}}$ using only the geometric invariants $\{\Sigma^m(G_i)\}_{m\in\mathbb{N}}$. In [16] a direct product formula is suggested for the homological and homotopical invariants, but the homotopical version turns wrong [17, section 6]. Still it is not known whether the homological version holds. A good description of the known results in this direction can be found in [13, section 9].

Finally we note that our proof of Theorem B is based on a good description of the Schur multiplier for nilpotent groups of class 2. It is likely that a generalization of this approach should involve better understanding of the higher homology groups $H_m(N, \mathbb{Z})$ for countably generated nilpotent groups N of arbitrary nilpotency class. Very little is known for these higher homology groups except in the case when N is free nilpotent of class 2 (see [21]).

2. Preliminaries

2.1. More about the geometric invariant $\Sigma^1(G)$

In this section we review the link between valuation theory and the invariant $\Sigma_V^c(Q)$ established in [3]. By definition for a finitely generated abelian group Q and a finitely generated (left) $\mathbb{Z}Q$–module V

$$\Sigma_V(Q) = \Sigma^0(Q, V),\ \Sigma_V^c(Q) = S(Q) \setminus \Sigma_V(Q).$$

We note that for a finitely generated group G with nilpotent derived subgroup G' the map $\varphi : Hom(G/G', \mathbb{R}) \to Hom(G, \mathbb{R})$ induced by the projection $G \to G/G'$ has the property

$$\varphi(\Sigma^c_{G'/G''}(G/G')) = \Sigma^1(G)^c \tag{3}$$

i.e. in this case $\Sigma^1(G)^c$ depends only on the metabelian quotient of G. Indeed (3) is a straight corollary of the definition of the involved geometric invariants and the fact that a subset of the nilpotent group G' is a generating set if and only if it is a generating set modulo G''.

The structure of $\Sigma_V(Q)$ can be described by the real valuations of the ring $V_0 = \mathbb{Z}Q/I$ where I is the annihilator of V in $\mathbb{Z}Q$. More precisely by [3, Thm 8.1]

$$\Sigma_V^c(Q) = [\cup_{v(\mathbb{Z}) \geq 0} \Delta_V^v(Q) \setminus \{0\}] \tag{4}$$

where the union is over all non–negative valuations v of $\mathbb{Z}$ and $\Delta_V^v(Q)$ is the set of all real characters of Q that can be extended to valuations (in Bourbaki sense) of V_0 which restriction to the image of $\mathbb{Z}$ in V_0 is induced by v. In [3] $\Delta_V^v(Q)$ is described as a rationally defined polyhedron i.e. a finite union of finite intersections of closed affine subspaces of $Q \otimes_{\mathbb{Z}} \mathbb{R}$ defined by equations with rational coefficients.

2.2. The Schur multiplier for nilpotent groups

It is well known that for a group G the Schur multiplier $H_2(G, \mathbb{Z})$ is isomorphic to

$$\frac{R \cap [F, F]}{[R, F]}$$

where F is a free group and $F/R \simeq G$. In [20, Section 7, Thm M] we showed a more detailed description of the Schur multiplier for some nilpotent of class two groups. This description generalizes results from [11].

More precisely suppose G is a nilpotent of class two group with the additional property that the torsion part of the abelianization of G has finite exponent. As shown in [20,Thm M] there is a short exact sequence $0 \to A \to H_2(G, \mathbb{Z}) \to B \to 0$, where B is the kernel of the commutator map $V \wedge_{\mathbb{Z}} V \to W$ for $V = G/[G, G]$ and $W = [G, G]$ and A is the quotient of $V \otimes_{\mathbb{Z}} W$ through the additive subgroup generated by the elements of Jacoby type $v_1 \otimes [v_2, v_3] + v_2 \otimes [v_3, v_1] + v_3 \otimes [v_1, v_2]$ for $v_i \in V$ and the elements $[n] \otimes n^s$ where $[n] = nW$ runs through the torsion part of V and s is the order of $[n]$ (note $[n] \otimes n^s$ is independent of the choice of representative n for $[n]$).

We claim that this description of the Schur multiplier is invariant under any automorphism φ of G. The main part of [20, section 7.3] is devoted to the construction of a special central extension of G which implies that for some image T of $H_2(G, \mathbb{Z})$ there is a short exact sequence $1 \to A \to T \to B \to 1$. Furthermore by [11] there is an exact sequence $1 \to \gamma_3(F)/([F, R] \cap \gamma_3(F)) \to H_2(G, \mathbb{Z}) \to B \to 1$, where $\gamma_3(F)$ is the 3-rd term of the lower central series of F. Since the commutator map maps A surjectively to $\gamma_3(F)/([F, R] \cap \gamma_3(F))$ we deduce that $A \simeq \gamma_3(F)/([F, R] \cap \gamma_3(F))$ and $T \simeq H_2(G, \mathbb{Z})$. In addition the basis X of F could be chosen to be the union of free orbits under the action of the cyclic group generated by φ and so the description of the Schur multiplier is invariant under φ.

3. Proof of Theorem A2

We note that if $\Sigma^2(G, \mathbb{Z})^c = S(G)$ there is nothing to prove. So we can assume $\Sigma^2(G, \mathbb{Z}) \neq \emptyset$. By [7, Thm 5.1] G is of homological type FP_m if and only if $\Sigma^m(G,\mathbb{Z})$ is non-empty, in particular G is of type FP_2.

Let F be a free group with a finite basis $\mathcal{X}$ such that $G \simeq F/R$ for some normal subgroup R of F. Since G is of type FP_2 by [2, Prop. 1.4] the abelianization of R is finitely generated as a (left) $\mathbb{Z}G$–module, where G acts via conjugation. Let $\mathcal{R} = \{r_1, \ldots, r_m\}$ be a generating set of $R/[R,R]$ over G. Then we define Γ_1 to be the combinatorial 2–complex associated to the presentation $< \mathcal{X} \mid \mathcal{R} >$ i.e. Γ_1 has sets of vertices, edges and 2–cells respectively G, $G \times \mathcal{X}$ and $G \times \mathcal{R}$. The vertices of the edge (g,x) are g and gx and the boundary of (g,r_i) is a path at g given by spelling out the relation r_i. The group G acts freely and cocompactly on Γ_1 via left multiplication and Γ_1 is 1–acyclic.

Now we assume Theorem A2 is wrong and fix a character $\chi \in Hom(G,\mathbb{R})$ such that that

$$\chi \in (\mathbb{R}_{>0}.\Sigma^2(G,\mathbb{Z})) \cap conv_{\leq 2}(\mathbb{R}_{>0}.\Sigma^1(G)^c). \tag{5}$$

Since $[\chi] \in \Sigma^2(G,\mathbb{Z})$ the proof of [7, Thm 4.2] shows that Γ_1 can be embedded in a 2–complex Γ such that Γ is 1–acyclic, G acts cocompactly and freely on Γ and the maximal subcomplex Γ_χ of Γ contained in $h_\chi^{-1}([0,\infty))$ is 1–acyclic, where h_χ is an χ–equivariant regular height function of Γ such that the restriction of h_χ on the set G of vertices of Γ is χ i.e.

$$h_\chi : \Gamma \to \mathbb{R}$$

is a continuous function such that $h_\chi(gx) = h_\chi(x) + \chi(g)$ for $g \in G, x \in \Gamma$ and the restriction of h_χ on a cell attains its extremes on the boundary. The proof of [7, Thm 4.2] shows that the embeding of Γ_1 in Γ can be achieved by performing a finite sequence of elementary homotopy expansitions.

As $[G,G]$ acts discretely and freely on Γ the vertical maps in the following commutative diagram are covering maps, the horizontal maps are the obvious inclusions

$$\begin{array}{ccc} \Gamma_\chi & \to & \Gamma \\ \downarrow & & \downarrow \\ V = \Gamma_\chi/[G,G] & \to & W = \Gamma/[G,G] \end{array}$$

In general we do not know whether Γ_χ is 1–connected, it depends on whether $[\chi] \in \Sigma^2(G)$. Still by Hurewits theorem and the fact that Γ_χ is 1–acyclic we know that $\pi_1(\Gamma_\chi)$ is a perfect group. Then

$$[G,G] \simeq \frac{\pi_1(V)}{N} \tag{6}$$

where N is the image of $\pi_1(\Gamma_\chi)$ in $\pi_1(V)$ and hence N is a perfect group.

By assumption χ can be decomposed as a sum of two non-trivial real characters χ_1 and χ_2 such that $[\chi_1], [\chi_2] \in \Sigma^1(G)^c$. Define W_i to be the maximal subcomplex of W contained in $h_{\chi_i}^{-1}[c, +\infty)$ for some negative real number c and a χ_i–equivariant regular height function h_{χ_i} of W. We set $V_i = W_i \cap V$ and note that since $\chi = \chi_1 + \chi_2$ there exists a negative real number c_0 such that for $c \leq c_0$ the diameter of the intersection $V_1 \cap V_2$ is larger than the diameter of any 2–cell of W (remember G acts cocompactly on Γ and so on W). Then $V = V_1 \cup V_2$ is a topological decomposition of V i.e. every cell of V is contained either in V_1 or V_2. By Van Kampen theorem

$$\pi_1(V) = \pi_1(V_1) *_{\pi_1(V_1 \cap V_2)} \pi_1(V_2) \tag{7}$$

Lemma 1. *Suppose the abelianization A_i of $\pi_1(V_i)/\varphi_i(\pi_1(V_1 \cap V_2))$ is a non-trivial abelian group for $i \in \{1, 2\}$, where $\varphi_i : \pi_1(V_1 \cap V_2) \to \pi_1(V_i)$ is the map induced by the inclusion. Furthermore if one of the groups A_1 or A_2 is cyclic of order 2 the other is not. Then $\pi_1(V)/N$ contains a free subgroup of rank two.*

Proof. Let $\mu : \pi_1(V_1) *_{\pi_1(V_1 \cap V_2)} \pi_1(V_2) \to \Pi = A_1 * A_2$ be the surjective map induced from the surjections $\pi_1(V_i) \to A_i$. By the Kurosh subgroup theorem [12, Ch 7, Thm 8] the derived subgroup of $A_1 * A_2$ is a free group. As N is perfect $\mu(N)$ is a perfect subgroup of the free group $[\Pi, \Pi]$, so should be trivial. Then the homomorphism μ induces a surjection $\pi_1(V)/N \to \Pi$. By the normal form theorem for amalagamated products Π contains a non-cyclic free subgroup and so does $\pi_1(V)/N$. This completes the proof.

We remind the reader that G does not contain free non-cyclic subgroups. This together with (6) and Lemma 1 implies that for some $i_0 \in \{1, 2\}$ either $A_{i_0} = 1$ or $A_{i_0} = \mathbb{Z}_2$, say $i_0 = 1$. In addition by substituting c if necessary with a negative integer with sufficiently large absolute value we can assume $A_1 = 1$ and hence $\pi_1(V_2) \simeq \pi_1(V)$ via the inclusion of V_2 in V. Now the main idea of the proof of [9, Lemma 4.7] shows that $H_1(V_2)$ is a finitely generated module over $\mathbb{Z}[Q_\chi \cap Q_{\chi_2}]$. In particular

$$H_1(V_2) \simeq H_1(V) = \frac{\pi_1(V)}{[\pi_1(V), \pi_1(V)]} = G'/G'' \tag{8}$$

is finitely generated over $\mathbb{Z}Q_{\chi_2}$ and so

$$[\chi_2] \in \Sigma_{G'/G''}(G) = \Sigma^0(G, G'/G'') \tag{9}$$

Finally by (3) $\Sigma_{G'/G''}(G) = \Sigma^1(G)$, a contradiction with $[\chi_2] \notin \Sigma^1(G)$.

4. Proof of Theorem A1.

We assume the theorem is wrong and there exists a non-trivial real character χ of G such that $[\chi] \in \Sigma^2(G)$ and $\chi = \chi_1 + \chi_2$ for some $[\chi_1], [\chi_2] \in \Sigma^1(G)^c$. Then there exists a finite presentation $< \mathcal{X} \mid \mathcal{R} >$ of G such that the corresponding combinatorial 2–complex Γ has the property that its maximal 2-subcomplex Γ_χ contained in $h_\chi^{-1}([0,\infty))$ for a regular χ–equivariant height function h_χ is 1–connected. As in the proof of Theorem A2 we consider the embeding of $V = \Gamma_\chi/[G,G]$ into $W = \Gamma/[G,G]$ and split $W = W_1 \cup W_2$ as before i.e. W_i is a half subcomplex corresponding to the character χ_i. Furthermore we can assume that $V = V_1 \cup V_2$ is topological decomposition for $V_i = W_i \cap V$. Then by [6, Thm 5.1] as $[\chi_i] \notin \Sigma^1(G)$ the map $\pi_1(W_i) \to \pi_1(W) = [G,G]$ is not an epimorphism. Using again the Van Kampen theorem

$$[G,G] \simeq \pi_1(V) = \pi_1(V_1) *_{\pi_1(V_1 \cap V_2)} \pi_1(V_2)$$

and as $\pi_1(V)$ has no free subgroup of rank two either the map $\pi_1(V_{i_0}) \to \pi_1(V)$ is an epimorphism for some i_0 or the image of $\pi_1(V_1 \cap V_2)$ in $\pi_1(V_i)$ has index two for both $i = 1$ and $i = 2$. The later could be avoided by translating W_i's if necessary i.e. if W_i is the maximal subcomplex in $h_{\chi_i}^{-1}([c,\infty)$ we change c with a negative real number with sufficiently big absolute value. As the map $\pi_1(V) \to \pi_1(W)$ induced by the inclusion of V in W is an isomorphism (remember $\pi_1(V) \simeq [G,G]$) it follows that the composite

$$\pi_1(V_{i_0}) \to \pi_1(V) \to \pi_1(W)$$

is an epimorphism. Since the above composite factors through $\pi_1(W_{i_0}) \to \pi_1(W)$ it follows that the latter map should be an epimorphism, a contradiction.

5. Finite presentability and finite generation of some tensor products

Lemma 2. *Suppose M_0 is a finitely generated $\mathbb{Z}H$–module for some finitely generated abelian group H and T is a submonoid of H such that $\mathbb{Z}T$ is a Noetherian ring. Then M_0 is finitely generated over $\mathbb{Z}T$ if and only if for every non-negative valuation v of $\mathbb{Z}$*

$$\Delta^v_{M_0}(H) \cap \{\chi \in Hom(H,\mathbb{R}) \mid \chi(T) \geq 0\} \subseteq \{0\}$$

Proof. The proof is an obvious modification of [4, Lemma 5.1].

Now we adopt some notations from [5, section 4]. We consider a finitely generated abelian group Q and denote by I the ideal of $\mathbb{Z}[Q \times Q \times Q]$ generated by all elements

$$d_r = \prod_{1 \leq i < j \leq 3} (r^{(i)} - r^{(j)}) \text{ for } r \in \mathbb{Z}Q$$

where $r^{(i)}$ is the pure tensor $1 \otimes \ldots \otimes r \otimes \ldots \otimes 1$ with r in the i-th position and 1 elsewhere.

Lemma 3. *Suppose G is a finitely presented group, an extension of a nilpotent group of class two N by an abelian group Q and $M = N/[N, N]$. We view M as a (left) $\mathbb{Z}Q$–module, where Q acts via conjugation. If χ is a discrete character of G such that $[\chi] \in \Sigma^2(G, \mathbb{Z})$ and $\chi(N) = 0$ then $M_0 = M \otimes M \otimes M / I(M \otimes M \otimes M)$ is finitely generated over $\mathbb{Z}Q_\chi$ via the diagonal action of Q_χ.*

Proof. We want to apply Lemma 2 for $H = Q \times Q \times Q$, T the image of Q_χ under the diagonal map $\partial : Q \to Q \times Q \times Q$ and the finitely generated $\mathbb{Z}H$–module M_0. Let $v_0 \in \Delta^v_{M_0}(H)$ be a non-trivial character. Then $v_0 \in \Delta^v_{M \otimes M \otimes M}(H)$ and by the additive formula [4, Thm 4.2] $v_0 = (v_1, v_2, v_3)$ with $v_i \in \Delta^v_M(Q)$. In particular by (3) and (4)

$$\text{either } [v_i] \in \Sigma^c_M(Q) \simeq \Sigma^1(G)^c \text{ or } v_i = 0.$$

Suppose further that $v_0(T) \geq 0$. Then $v_1 + v_2 + v_3 = \lambda\chi$ for some non-negative real number λ and without loss of generality we can assume $\lambda = 0$ or 1. We note that if all v_i are pairwise different then the argument of [5, Thm 4.3, 1st case] shows that $[(v_1, v_2, v_3)] \in \Sigma_{M_0}(H)$ and by the description of Σ using Δ we have $v_0 \notin \Delta^v_{M_0}(H)$, a contradiction. If two v_i's are equal say $v_1 = v_2$ either $2v_1 + v_3 = 0$ or $2v_1 + v_3 = \chi$. By the characterization theorem of finitely presented metabelian groups [9, Thm A] and the fact that metabelian quotients of finitely presented groups which do not contain non-cyclic free subgroups are finitely presented [9, Corollary B] we see that $2v_1 + v_3 \neq 0$ if at least one of v_1 and v_3 is non-trivial. At the same time by Theorem A2 $2v_1 + v_3 \neq \chi$. Thus

$$\Delta^v_{M_0}(H) \cap \{\beta \in Hom(H, \mathbb{R}) \mid \beta(T) \geq 0\} \subseteq \{0\}$$

and Lemma 2 completes the proof.

Lemma 4. *Suppose D is a Noetherian subring of the subalgebra of S_3–invariant elements of $\mathbb{Z}[Q \times Q \times Q]$. Let M be a finitely generated $\mathbb{Z}Q$–module* *and* $\alpha : M \otimes M \otimes M \to M \otimes M \otimes M$ *be the linear map sending a pure tensor* $m_1 \otimes m_2 \otimes m_3$ *to* $m_1 \otimes m_2 \otimes m_3 - m_2 \otimes m_1 \otimes m_3 - m_3 \otimes m_1 \otimes m_2 + m_3 \otimes m_2 \otimes m_1$. *Then* $M \otimes M \otimes M$ *is finitely generated as a* D*–module if and only if the image of* α *and* $M \otimes M \otimes M/I(M \otimes M \otimes M)$ *are finitely generated over* D.

Proof. In the case of modules M over $\mathbb{F}_p Q$ the lemma is proved at the end of [18, Prop. 2]. The proof when M is a $\mathbb{Z}Q$-module is the same. The argument of [18, Prop 2] is a modification of [5, Prop 4.1].

Proposition 5. *Let* G *be a finitely presented group, an extension of a nilpotent of class two group* N *by an abelian group* Q *such that* $M = N/[N,N]$ *is torsion-free,* $[N,N] \simeq M \wedge_{\mathbb{Z}} M$ *and let* χ *be a discrete character of* G *with* $\chi(N) = 0$ *and* $[\chi] \in \Sigma^2(G, \mathbb{Z})$. *Then*

1. $M \otimes M \otimes M$ *is finitely generated over* $\mathbb{Z}Q_\chi$*;*
2. $\chi \notin conv_{\leq 3} \Delta_V^v(Q)$ *for every non–negative valuation* v *of* $\mathbb{Z}$.

Proof. We start with the observation that $\mathbb{Z}Q_\chi$ is Noetherian because χ is discrete. Let $\mathcal{R} : \ldots \to R_1 \to R_0 \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G_\chi$ with R_i finitely generated for $i \leq 2$. Since $\mathbb{Z}Q_\chi$ is a Noetherian ring the homology group $H_2(\mathcal{R} \otimes_{\mathbb{Z}N} \mathbb{Z}) \simeq H_2(N, \mathbb{Z})$ as a subquotient of a finitely generated module over $\mathbb{Z}Q_\chi$ is finitely generated over $\mathbb{Z}Q_\chi$ itself.

By the description of the Schur multiplier for nilpotent groups discussed in the preliminaries the quotient of $[N,N] \otimes_{\mathbb{Z}} M$ through the additive subgroup of elements of Jacoby type is a submodule of $H_2(N, \mathbb{Z})$ and hence is finitely generated over $\mathbb{Z}Q_\chi$. As $[N,N]$ is the exterior square of M we have that the above quotient is isomorphic to the quotient of $M \otimes M \otimes M$ through the additive subgroup J generated by the elements of Jacoby type $v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2$ and the elements $v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3$ for $v_i \in M$. It is easy to check that J is in the kernel of the map α defined in Lemma 4 and thus the image of α is finitely generated over $\mathbb{Z}Q_\chi$. By Lemma 3 the module $M_0 = M \otimes M \otimes M/I(M \otimes M \otimes M)$ is finitely generated over $\mathbb{Z}Q_\chi$ and finally by Lemma 4 $M \otimes M \otimes M$ is finitely generated over $\mathbb{Z}Q_\chi$. Once we have proved the first part of the proposition using Lemma 2 we deduce

$$\Delta^v_{M \otimes M \otimes M}(Q \times Q \times Q) \cap \{\mu \in Hom(Q \times Q \times Q, \mathbb{R}) \mid \mu(\partial(Q_\chi)) \geq 0\} \subseteq \{0\}$$

where $\partial : Q \to Q \times Q \times Q$ is the diagonal map. Then $\chi \neq v_1 + v_2 + v_3$ where $(v_1, v_2, v_3) \in \Delta^v_{M \otimes M \otimes M}(Q \times Q \times Q) = \Delta^v_M(Q) \times \Delta^v_M(Q) \times \Delta^v_M(Q)$. The latter is the additive formula for Δ from [4].

Proof of Theorem B

Step 1. We construct a group G a split extension of N by Q such that Q is free abelian of rank three, N is nilpotent of nilpotency class two, $\Sigma^1(G)^c$ contains precisely three points that lie in an open hemisphere. First we construct a $\mathbb{Z}$–torsion free $\mathbb{Z}Q$–module V with the property that $\Sigma_V^c(Q)$ contains precisely three points that lie in an open hemisphere. We define $V = V_1 \oplus V_2 \oplus V_3$, where all V_i are isomorphic to $\mathbb{Z}[\frac{1}{2}]$ as abelian groups and Q acts on V_i via the multiplicative homomorphism $\chi_i : Q \to V_i$ such that for a fixed basis $\{e_1, e_2, e_3\}$ of Q we have $\chi_i(e_j) = 2^{\partial_{i,j}}$, where $\partial_{i,j}$ is the Kroneker symbol. By the additivity of Σ^c and by (4)

$$\Sigma_V^c(Q) = \cup_{1 \leq i \leq 3}[\Delta_{V_i}^{v_0}(Q)]$$

where v_0 is the 2–adic valuation on $\mathbb{Z}$. We identify $Hom(Q, \mathbb{R})$ with $\mathbb{R}^3$ via $\mu : Hom(Q, \mathbb{R}) \to \mathbb{R}^3$ where $\chi(q) = (\mu(\chi), q)$ for every $q \in Q$ and $\chi \in Hom(Q, \mathbb{R})$. Here $Q \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^3$ is eqipped with the standard inner product $(\ ,\ )$ and e_1, e_2 and e_3 is an orthornormal basis. Then

$$\Sigma_V^c(Q) = \{[(1, 0, 0)], [(0, 1, 0)], [(0, 0, 1)]\}.$$

Define N_1 to be the nilpotent group of class two with abelianization V and derived subgroup $V \wedge_{\mathbb{Z}} V$ and set N to be the direct product of V with N_1. The action of Q on V extends to an action of Q on N_1 and by definition Q acts diagonally on $N = N_1 \times V$. Finally we set G to be the split extension of N by Q. Since $\Sigma_V^c(Q) = \Sigma_{N/[N,N]}^c(Q) \simeq \Sigma^1(G)^c$ lies in an open hemisphere of $S(G)$ it follows by [8, Thm A] that G is constructible.

Step 2. We find a subgroup H of Q of rank two such that there is a $\mathbb{Z}H$–submodule W of V that is 2–tame but not 3–tame. More precisely $\Sigma_W^c(H)$ contains no antipodal points but does not lie in an open hemisphere. We require that $W = \oplus_i(W \cap V_i)$ and $W \cap V_i \simeq \mathbb{Z}H/(J_i \cap \mathbb{Z}H)$, where $V_i = \mathbb{Z}Q/J_i$.

In general if H is a subgroup of Q the restriction map $\varphi_H : Hom(Q, \mathbb{R}) \to Hom(H, \mathbb{R})$ has the property that

$$\varphi_H(\Sigma_V^c(Q)) = \Sigma_W^c(H).$$

This can be seen through the description of Σ^c using valuations (see (4)). Now we define H to be the subgroup of Q generated by $e_1e_2^{-1}, e_2e_3^{-1}$. Then $\varphi_H(\Sigma_V^c(Q))$ contains precisely 3 characters μ_1, μ_2 and μ_3 such that $\mu_1 + \mu_2 + \mu_3 = 0$ and $\mu_i + \mu_j \neq 0$ for all i, j. Thus the image of φ_H does not contain antipodal points and does not lie in an open hemisphere.

Finally we consider the subgroup $W \rtimes H$ of G, where W embeds in the summand V of N. By the characterization theorem of finitely presented metabelian groups [9,Thm A] as $\Sigma^c_W(H)$ has no antipodal points $W \rtimes H$ is finitely presented. As $\Sigma^c_W(H)$ does not lie in an open hemisphere of $S(H)$ the group $W \rtimes H$ cannot be constructible.

Step 3. By (2) to prove the last part of the theorem it is sufficient to show that

$$\mathbb{R}_{>0}.\Sigma^2(G,\mathbb{Z})^c \supseteq conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G,\mathbb{Z})^c) \tag{10}$$

Let $[\chi]$ be a discrete element of $\Sigma^2(G,\mathbb{Z})$ such that $\chi(N) = 0$. By Proposition 5(2) and the fact that $\Sigma^1(G)^c \simeq \Sigma^c_V(Q) = [\Delta^{v_0}_V(Q)]$

$$\chi \notin conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c)$$

Thus

$$dis(\mathbb{R}_{>0}.\Sigma^2(G,\mathbb{Z})) \cap \{\chi \mid \chi(N) = 0\} \cap conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c) = \emptyset \tag{11}$$

We note that by [19, Thm C] the homomorphisms of G that are non–trivial on a normal locally polycyclic subgroup represent elements of $\Sigma^m(G)$ provided G is of type F_m. In particular this holds for $m = 1$ and hence $conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c) \mid_N = 0$. Then by (11) $dis(\mathbb{R}_{>0}.\Sigma^2(G,\mathbb{Z})) \cap conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c) = \emptyset$ and so

$$dis(conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c)) \subseteq \mathbb{R}_{>0}.\Sigma^2(G,\mathbb{Z})^c. \tag{12}$$

By [7, Thm A] $\Sigma^2(G,\mathbb{Z})^c$ is a closed subset of $S(G)$ and by the geometric description of Σ for modules over abelian groups [3, Thm 8.1, Thm A] $\Sigma^1(G)^c \simeq \Sigma^c_V(Q)$ is the projection of a rationally defined polyhedron to $S(G)$. Then the discrete points in $[conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c)]$ form a dense subset and (12) implies

$$[conv_{\leq 3}(\mathbb{R}_{>0}.\Sigma^1(G)^c)] \subseteq \overline{\Sigma^2(G,\mathbb{Z})^c} = \Sigma^2(G,\mathbb{Z})^c,$$

as required.

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