Large deviations in random media of zero mean asymmetric zero range processes.

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Abstract

We consider an asymmetric zero range process with zero mean in infinite volume with random jump rates starting from equilibrium. We investigate large deviations from hydrodynamical limit of the empirical distribution of particles and prove an upper and lower bound for a large deviation principle. Our main argument is based on a super-exponential estimate in infinite volume. We adapt a method developed by Kipnis & al. (1989) and Benois & al. (1995).

Keywords: Asymmetric zero range process, Hydrodynamical limit, Large deviations, Random media.

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1 Introduction

The zero range process is one of the simplest particle systems that has been successfully investigated recently in random or inhomogeneous media (cf. for instance Benjamini & al. (1996), Evans (1996), Krug-Ferrari (1996), Landim (1996), Gielis & al. (1998), Bahadoran (1998), Seppäläinen-Krug (1999), Koukkous (1999), Andjel & al. (2000)) .

We describe informally the evolution of the process as follows. On the d-dimensional lattice $\mathbb{Z}^d$, we consider a sequence of random variables $p = (p_x)_{x\in\mathbb{Z}^d}$ (*called an environment*) in $[a_0, a_1]$ (where $0 < a_0 \le a_1 < \infty$). At any $x \in \mathbb{Z}^d$, each particle at site x jumps to the left or to the right with a rate depending only on the total number of particles at site x before

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the jump. The random media state therefore that the jump rate is accelerated or decelerated by the value p_x of p at site x.

Benjamini & al. (1996) have studied the asymmetric version of a zero-range process in infinite volume when the environment is an i.i.d. sequence of random variables (with $a_1 = 1$) and have proved the asymptotic hydrodynamical behavior of the system. Koukkous (1999) showed the hydrodynamical limit in the symmetric case for a stationary and ergodic environment whose marginal law is absolutely continuous with respect to the Lebesgue measure. He proved that the empirical measure of particles converges in probability to the weak solution of a non-linear diffusion equation which does not depend on the environment p and he generalized in this way some results of Benjamini & al. (1996).

The equilibrium fluctuations (Central limit results for the density field) were studied in G. Gielis & al. (1998). They proved that the density field converges weakly to a generalized Ornstein-Uhlenbeck process.

Recently, Andjel & al. (2000) showed the convergence to the maximal invariant measure for an asymmetric zero range process with constant rate in inhomogeneous and random media in dimension 1 starting from an upper-critical non-equilibrium measure.

In this spirit of hydrodynamical behavior investigation, a natural open question can be formulated as follows: From the hydrodynamical limit of the empirical measure with some continuous density $\mu(\cdot)$ (with respect to Lebesgue measure) and given an event Γ for which $\mu \notin \bar{\Gamma}$, how to control the "*deviant*" behavior of the system inside Γ ? This is the subject of large deviation principles (LDP) related to hydrodynamical limit of the empirical measure.

In this paper, we investigate a d-dimensional zero mean asymmetric zero-range process in random media. In the deterministic case of the environment, the LDP results have been treated by many authors among which Landim (1992), Benois (1996) and Benois & al. (1995). In this last article an upper and a lower bound of LDP in infinite volume of empirical density are proven when the process starts from equilibrium.

The crucial ingredient of the proofs focuses on the so-called super-exponential estimate: it consists in approximating, by rigorous functions of the density field, the correlation field obtained in the computation of some exponential martingales related to the jumps of particles (see Kipnis & al. (1989) and Donsker-Varadhan (1989)). Once we prove this result, the LDP result (and also the hydrodynamical limit) of the empirical measure is obtained by standard arguments.

In a random media, the difficulty in adapting standard arguments relies on the absence of translation invariance of the invariant measures for the process. For this reason, our approach uses essentially both results of Koukkous (1999) and Benois & al. (1995).

The paper is organized as follows: We introduce the notations and assumptions used through the paper and state the main results in Section 2. Section 3 is devoted to the proof of the super-exponential estimate. In the last section we give a proof of an upper bound of LDP result. We omit the proof of lower bound since, once one has proven the upper bound, it is similar to the arguments given in Benois & al. (1995) without major modifications.

2 Notation and results

Let $0 < a_0 \leq a_1 < \infty$ and consider a sequence of random variables $\{p_x,\ x \in \mathbb{Z}^d\}$ on $[a_0, a_1]$ distributed according to a stationary and ergodic measure m, such that its one-dimensional marginal law is absolutely continuous with respect to the Lebesgue measure. We assume that $m\{p : a_0 \leq p_0 \leq a_1\} = 1$ and for every $\varepsilon > 0$, $m\{p : p_0 \in [a_0, a_0 + \varepsilon)\} m\{p : p_0 \in (a_1 - \varepsilon, a_1]\} > 0$.

We denote by $\mathbb{X}_d := \mathbb{N}^{\mathbb{Z}^d}$ the configuration space and by Greek letters η and ξ its elements. As usual $\eta(x)$ stands for the total number of particles at site x for the configuration η. For each environment p, we are interested in the Markov process $(\eta_t)_{t\geq 0}$ on $\mathbb{X}_d$ whose generator is defined by

$$(\mathcal{L}_p f)(\eta) = \sum_{x,y \in \mathbb{Z}^d} p_x g(\eta(x)) T(x,y) [f(\eta^{x,y}) - f(\eta)], \tag{1}$$

where $f : \mathbb{X}_d \to \mathbb{R}$ is a cylinder function, that is f only depends on η through a finite number of coordinates. $T(\cdot,\cdot)$ is a transition probability on $\mathbb{Z}^d$. The function g is positive and vanishes at 0: $g(0) = 0 < g(k)$ for all $k \geq 1$. In the previous formula, $\eta^{x,y}(z)$ is the configuration obtained from η when a particle jumps from x to y:

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ \eta(x) - 1 & \text{if } z = x \\ \eta(y) + 1 & \text{if } z = y \end{cases} .$$

For every non-negative real φ we denote by ν_φ^p the product measure on $\mathbb{X}_d$ whose marginals are defined by

$$\nu_\varphi^p\{\eta : \eta(x) = k\} = \frac{1}{Z(\varphi p_x^{-1})} \frac{(\varphi p_x^{-1})^k}{g(k)!}, \quad \text{for all } k \geq 0,$$

where $g(k)! = g(1)g(2)...g(k)$ if $k > 0$ and $g(0)! = 1$. Those measures (see Benjamini & al. (1996)) are invariant for the process. In this formula, $Z : \mathbb{R}_+ \to \mathbb{R}_+$ is the partition function

$$Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!}.$$

Let φ^* be the radius of convergence of $Z(\cdot)$; we assume that

$$\lim_{\varphi \uparrow \varphi^*} Z(\varphi) = +\infty. \tag{2}$$

Denote by $\nu_\varphi(\cdot) := \nu_\varphi^1(\cdot)$ the invariant measure of the process $(\eta_t)_{t\geq 0}$ when m is the Dirac measure concentrated on the set $\{p : p_x = 1, x \in \mathbb{Z}^d\}$ (see Andjel (1982)). We define $M : [0, \varphi^*) \to \mathbb{R}_+$ by $M(\varphi) = \nu_\varphi[\eta(0)]$, the expected number of particles at 0 with respect to ν_φ.

A simple computation shows that $M(\varphi) = \varphi \partial_\varphi \log Z(\varphi)$ and from assumption (2) we check that M is an increasing, continuous, one-to-one function from $[0, \varphi^*)$ to $\mathbb{R}_+$.

We define the "density" of particles (*i.e.* the expected number of particles at 0) with respect to the random media by the continuous and increasing function $R : [0, a_0\varphi^*) \to \mathbb{R}^+$ such that

$$R(\varphi) = m[M(\varphi p_0^{-1})]$$

and in order to ensure the existence of an invariant measure for any given value of the density, we assume that

$$\lim_{\varphi \uparrow a_0\varphi^*} R(\varphi) = \infty. \tag{3}$$

Under this assumption the function R is one to one from $[0, a_0\varphi^*)$ to $\mathbb{R}_+$. We denote by Φ its inverse (which is also a continuous increasing bijection).

For a density $\rho > 0$ we write

$$\bar{\nu}_\rho^p = \nu_{\Phi(\rho)}^p$$

We easily check that

$$\Phi(\rho) = \bar{\nu}_\rho^p \Big[p_x g(\eta(x)) \Big] \quad \text{for all} \quad x \in \mathbb{Z}^d.$$

In the following we state all the hypotheses assumed throughout this paper.

[H1] The transition probability $T(\cdot, \cdot)$ on $\mathbb{Z}^d$ is a zero-mean irreducible translation invariant probability with finite range. That is

$$T(x, y) = T(0, y - x) =: T(y - x),$$
$$\text{there exists a constant } A > 0 \text{ such that } T(x) = 0 \text{ if } |x| \geq A$$
$$\text{and} \quad \sum_{x \in \mathbb{Z}^d} x\, T(x) = 0.$$

[H2] The rate function g has bounded variation:

$$g^* = \sup_k |g(k+1) - g(k)| < \infty.$$

Under the hypotheses [H1] and [H2], Andjel (1982) has proven the existence of a unique Markov process with corresponding generator defined by (1) in deterministic environment (*i.e.* $p \equiv 1$). His proof applies also in our case.

Let $(\sigma_{ij})_{\{1 \le i,j \le d\}}$ be a symmetric nonnegative definite matrix defined by the covariance matrix of the transition probability $T(\cdot)$:

$$\sigma_{ij} = \sum_{y \in \mathbb{Z}^d} y_i y_j T(y) \quad \text{where} \quad y = (y_1, \cdots, y_d)$$

[H3] In order to avoid the degenerate case of the hydrodynamic equation, we assume $(\sigma_{ij})_{\{1 \le i,j \le d\}}$ to be a positive definite matrix. That is there exists $\kappa > 0$ such that

$$\sum_{i,j} \sigma_{ij} x_i x_j \ge \kappa \sum_i x_i^2, \quad \text{for all} \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

[H4] To ensure some finite exponential moments of $\eta(x)$ under the measures ν_φ^p we shall assume that there exists a convex and increasing function $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that
(i) $\omega(0) = 0$,
(ii) $\lim_{x \to \infty}(\frac{\omega(x)}{x}) = \infty$ and
(iii) for all density φ there exists a positive constant $\theta := \theta(\varphi)$ such that

$$\nu_\varphi \Big[\exp\{\theta \omega(\eta(0))\} \Big] < \infty.$$

This last assumption ensures also that $Z(\cdot)$ has infinite radius of convergence. It holds for exemple if $g(k+1) - g(k) \ge g_0^*$ for some constant g_0^* and k sufficiently large.

We will denote by ω^* the Legendre transform of ω given by:

$$\omega^*(x) = \sup_{\alpha > 0}\{\alpha x - \omega(\alpha)\}. \tag{4}$$

In the next paragraphs, we define the state space of the process and its topology. Denote by $\mathcal{C}(\mathbb{R}^d)$ (resp. $\mathcal{C}_K(\mathbb{R}^d)$) the space of continuous (resp. with compact support) functions on $\mathbb{R}^d$ with classic uniform norm. Let $\mathcal{M}_+$ denote the space of positive Radon measures on $\mathbb{R}^d$ with the weak topology induced by $\mathcal{C}_K(\mathbb{R}^d)$ via $\langle \pi, H \rangle = \int H \, \mathrm{d}\pi$ for $H \in \mathcal{C}_K(\mathbb{R}^d)$ and $\pi \in \mathcal{M}_+$.

We fix a positive time parameter $\mathcal{T} > 0$. For each realization of the environment p and all fixed positive density ρ, $\mathbb{P}_{\rho,p}^N$ will denote the probability measure on the path space

$\mathcal{D}([0,\mathcal{T}],\mathbb{X}_d)$ corresponding to the Markov process $(\eta_t)_{t\in[0,\mathcal{T}]}$ with generator $N^2\mathcal{L}_p$ starting from the measure $\bar{\nu}_\rho^p$. By $\mathbb{E}_{\rho,p}^N$ we denote the expectation under $\mathbb{P}_{\rho,p}^N$.

Let $\pi_\cdot^N$ be the empirical measure defined on $\mathcal{D}([0,\mathcal{T}],\mathcal{M}_+)$ by

$$\pi_t^N(du) = \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_t(x)\delta_{x/N}(du),$$

for $0\le t\le\mathcal{T}$. Let $Q_{\rho,p}^N$ denote the measure on the path space $D([0,\mathcal{T}],\mathcal{M}_+)$ associated to the process $\pi_\cdot^N$ with generator $N^2\mathcal{L}_p$ starting from $\bar{\nu}_\rho^p$.

To investigate the large deviations of the empirical measure, we shall consider some small perturbations of the zero range process as mentionned earlier. For this, we will need the following notation.

Let $\mathcal{C}_K^{l,k}([0,\mathcal{T}]\times\mathbb{R}^d)$ denote the space of compact support functions with $l\in\mathbb{N}$ continuous derivatives in time and $k\in\mathbb{N}$ continuous derivatives in space. Let $\mathcal{C}_\rho(\mathbb{R}^d)$ be the set defined by

$$\mathcal{C}_\rho(\mathbb{R}^d) = \mathcal{C}(\mathbb{R}^d)\cap\{u:\mathbb{R}^d\to\mathbb{R}^+;\quad u(x)=\rho \text{ for } \ |x| \text{ sufficiently large}\}.$$

For a fixed γ in $\mathcal{C}_\rho(\mathbb{R}^d)$ and for some smooth function H in $\mathcal{C}_K^{1,2}([0,\mathcal{T}]\times\mathbb{R}^d)$ we consider the Markov process generated by

$$N^2(\mathcal{L}_{N,t}^{p,H}f)(\eta) = N^2\sum_{x,y\in\mathbb{Z}^d}p_x g(\eta(x))T(y)e^{\{H(t,\frac{x+y}{N})-H(t,\frac{x}{N})\}}[f(\eta^{x,x+y})-f(\eta)],$$

where f is a cylinder function. Let $\bar{\nu}_{\gamma,N}^p$ be the initial product measure of this process with marginals

$$\bar{\nu}_{\gamma,N}^p\{\eta,\eta(x)=k\} = \bar{\nu}_{\gamma(x/N)}^p\{\eta,\eta(x)=k\}$$

for all $x\in\mathbb{Z}^d$ and $k\in\mathbb{N}$. We therefore denote by $\mathbb{P}_{\gamma,N}^{p,H}$ and $Q_{\gamma,N}^{p,H}$ the small perturbations of $\mathbb{P}_{\rho,p}^N$ and $Q_{\rho,p}^N$ respectively.

For any path $\pi\in\mathcal{D}([0,\mathcal{T}],\mathcal{M}_+)$, denote by u_t the Radon-Nikodym derivative of π with respect to the Lebesgue measure λ: $u_t=\frac{\mathrm{d}\pi_t}{\mathrm{d}\lambda}$. Let $\mathcal{A}=\mathcal{A}(\rho)$ be the space path of $\pi\in\mathcal{D}([0,\mathcal{T}],\mathcal{M}_+)$ such that u_t is the solution of the PDE

(E)
$$\begin{cases} \partial_t u & = (\sigma/2)\triangle(\Phi(u)) - \sum_{i=1}^d \partial_{x_i}(\Phi(u)\partial_{x_i}H) \\ u(0,\cdot) & = \gamma(\cdot) \quad . \end{cases}$$

for some $\gamma\in\mathcal{C}_\rho(\mathbb{R}^d)$ and some $H\in\mathcal{C}_K^{1,3}([0,\mathcal{T}]\times\mathbb{R}^d)$. $\triangle$ stands the Laplacian operator.

The following notation is devoted to the definition of the rate functional of the large deviation principle for $(\pi_\cdot^N)_{0\le t\le\mathcal{T}}$.

For $H \in \mathcal{C}_K^{1,2}([0,\mathcal{T}] \times \mathbb{R}^d)$, we define $\mathcal{J}_H \ : \mathcal{D}([0,\mathcal{T}],\mathcal{M}_+) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{J}_H(\pi) = \mathcal{J}_H^1(\pi) - \mathcal{J}_H^2(\pi)$$

where

$$\mathcal{J}_H^1(\pi) = \Big\langle u_\mathcal{T}, H_\mathcal{T} \Big\rangle - \Big\langle u_0, H_0 \Big\rangle - \int_0^\mathcal{T} \Big\langle u_t, \partial_t H_t \Big\rangle \ \mathrm{d}t,$$

$$\mathcal{J}_H^2(\pi) = \frac{\sigma}{2} \int_0^\mathcal{T} \Big\langle \Phi(u_t), \sum_{i=1}^d \Big(\partial_{x_i}^2 H_t + (\partial_{x_i} H_t)^2 \Big) \Big\rangle \ \mathrm{d}t,$$

such that $\mathcal{J}_H(\cdot) = \infty$ outside $\mathcal{D}([0,\mathcal{T}],\mathcal{M}_+)$ or if π_t is not absolutely continuous with respect to the Lebesgue measure λ for some $0 \leq t \leq \mathcal{T}$.

We are now ready to define the part of the large deviations rate function, $\mathcal{I}_0(\cdot) : \mathcal{D}([0,\mathcal{T}],\mathcal{M}_+) \to [0,\infty]$ coming from the stochastic evolution:

$$\mathcal{I}_0(\pi) = \sup_{H \in \mathcal{C}_K^{1,2}([0,\mathcal{T}] \times \mathbb{R}^d)} \mathcal{J}_H(\pi).$$

The other part of the large deviations rate function coincides with the behaviour of deviations coming from the initial state. Let $h(\cdot|\rho)$ be the entropy defined for a positive function $\gamma : \mathbb{R}^d \to \mathbb{R}^+$ by

$$h(\gamma|\rho) = \int_{\mathbb{R}^d} \left\{ \gamma(x) \log\left(\frac{\Phi(\gamma(x))}{\Phi(\rho)} \right) - \mathbb{E}_m \left[\log\left(\frac{Z(\Phi(\gamma(x))p_0^{-1})}{Z(\Phi(\rho)p_0^{-1})} \right) \right] \right\} \ \mathrm{d}x.$$

Thus, the rate function of the large deviation principle is defined for a density $\rho > 0$ by

$$\mathcal{I}_\rho(\pi) = \mathcal{I}_0(\pi) + h(u_0|\rho).$$

From now on, for each $x \in \mathbb{Z}^d$, we denote by $\eta^l(x)$ the mean density of particles in a box of length $(2l+1)$ centered at x :

$$\eta^l(x) = \frac{1}{(2l+1)^d} \sum_{|y-x| \leq l} \eta(y).$$

For each cylinder function $\psi : \mathbb{X}_d \to \mathbb{R}$, we define

$$\tilde{\psi}(\rho) := m\Big[\nu_{\Phi(\rho)}^p(\psi) \Big] \ , \tag{5}$$

and we say that Ψ is a Lipschitz function if

$$\exists k_0 \in \mathbb{N} \text{ and } c_0 > 0 \text{ such that } \quad \Big| \Psi(\eta) - \Psi(\xi) \Big| \leq c_0 \sum_{|x| \leq k_0} \Big| \eta(x) - \xi(x) \Big|,$$

for all η and ξ in $\mathbb{X}_d$.

Denote by τ_x the shift operator defined by $\tau_x \psi(\eta(\cdot)) = \psi(\tau_x \eta(\cdot))$ where $\tau_x \eta(y) = \eta(x+y)$.

We can now state our results:

Theorem 2.1 *Let Ψ be a cylinder Lipschitz function and $H \in \mathcal{C}_K^{0,2}([0,\mathcal{T}] \times \mathbb{R}^d)$. Under hypotheses [H1] to [H4], for all $\delta > 0$ we have*

$$\overline{\lim_{\varepsilon \to 0}} \ \overline{\lim_{N \to \infty}} \ \frac{1}{N^d} \log \mathbb{P}_{\rho,p}^N \left[\left| \int_0^{\mathcal{T}} W_{N,\varepsilon}^{H,\Psi}(t,\eta_t) \ \mathrm{d}t \right| > \delta \right] = -\infty \tag{6}$$

m-almost surely, where

$$W_{N,\varepsilon}^{H,\Psi}(t,\eta) = \frac{1}{N^d} \sum_x H(t, x/N) \Big[\tau_x \Psi(\eta) - \tilde{\Psi}(\eta^{\varepsilon N}(x)) \Big].$$

This theorem, called the super-exponential estimate, will be a crucial argument in the proof of the following large deviations principle:

Theorem 2.2 *Under hypotheses [H1] to [H4], for every closed subset $\mathcal{C}$ and every open subset $\mathcal{O}$ of $\mathcal{D}([0,\mathcal{T}], \mathcal{M}_+)$, we have*

$$\limsup_{N \to \infty} \frac{1}{N^d} \log Q_{\rho,p}^N(\mathcal{C}) \leq - \inf_{\pi \in \mathcal{C}} \mathcal{I}_\rho(\pi)$$

and

$$\liminf_{N \to \infty} \ \frac{1}{N^d} \log Q_{\rho,p}^N(\mathcal{O}) \geq - \inf_{\pi \in \mathcal{O} \cap \mathcal{A}} \mathcal{I}_\rho(\pi)$$

m-almost surely.

Remarks:

Before starting to prove our results, we would like to mention some remarks and claims that we will use and whose proofs will be omitted. For complete details the reader is refered to Kipnis-Landim's book (1999) and Benois & al. (1995).

[R1] From Lemma I.3.5 of Kipnis-Landim's book (1999), the function defined by $\varphi \longrightarrow \nu_\varphi$ for $\varphi > 0$, is an increasing function (see also the proof of lemma 4.3 in Benois & al. (1995)). Therefore, assumption [H4] implies that for a fixed environment p defined in the beginning of the last section, for all $x \in \mathbb{Z}^d$ and $\varphi > 0$, there exists $\theta := \theta(x,\varphi) > 0$ such that

$$\nu_\varphi^p \Big[\exp\{\theta \omega(\eta(x))\} \Big] < \infty \quad m\text{-almost surely.}$$

[R2] Assumption [H4] ensures that the function ω^* defined by (4) is also a continuous convex function such that $\omega^*(0) = 0$.

[R3] A simple computation shows that from the second condition in [H4], for every $\varepsilon > 0$ the function $\omega^{-1}(r) - \varepsilon r$ is negative for each $r \geq C_2(\varepsilon)$, for some constant $C_2(\varepsilon)$ dependent only on ε.

[R4] By definition of ω in [H4], the function defined on $\mathbb{R}_+^*$ by $\Omega(r) = \frac{\omega(r)}{r}$ is an increasing function.

[R5] For each cylinder Lipschitz function $\Psi(\cdot)$, the function $\tilde{\Psi}(\cdot)$ given by (5) is also a Lipschitz function (see Lemma I.3.6 of Kipnis-Landim (1999)). Moreover one can check that $\tilde{\Psi}(k) \le Ck$ for all $k \in \mathbb{Z}$ for some constant C.

The strategy adopted to prove our results is similar to the one presented in Benois & al. (1995). However, we use some arguments developed in Koukkous (1999) in order to overcome the failed translation invariance propriety of the invariant measure of zero range process in random media. We thus detail only the main differences.

From now on, to keep the notation simple, we will restrict our study to the one-dimensional case. The reader can extend the proofs to any dimension without more difficulty.

3 Proof of Theorem 2.1

Let G be a positive continuous function on $\mathbb{R}$ defined by

$$G(x) = \sup_{y \in [x-1, x+1]} \max\Big\{ |H(y)|, |\partial_y H(y)|, |\partial_y^2 H(y)| \Big\}. \tag{7}$$

We have

$$\begin{aligned} &\mathbb{P}_{\rho,p}^N \Big[\int_0^{\mathcal{T}} W_{N,\varepsilon}^{H,\Psi}(t,\eta_t)\ \mathrm{d}t > \delta \Big] \\ &\qquad \le \mathbb{P}_{\rho,p}^N \Big[\int_0^{\mathcal{T}} \Big\{ W_{N,\varepsilon}^{H,\Psi}(t,\eta_t)\ \mathrm{d}t - \frac{\beta}{N} \sum_x G\Big(\frac{x}{N}\Big) \omega(\eta_t(x)) \Big\}\ \mathrm{d}t > \delta/2 \Big] \\ &\qquad\qquad + \mathbb{P}_{\rho,p}^N \Big[\int_0^{\mathcal{T}} \frac{\beta}{N} \sum_x G\Big(\frac{x}{N}\Big) \omega(\eta_t(x))\ \mathrm{d}t > \delta/2 \Big]. \end{aligned} \tag{8}$$

for every $\beta > 0$.

By Tchebycheff exponential inequality the first term in the left hand side in (8) is bounded above by

$$\exp\{-N\theta\delta/2\} \mathbb{E}_{\rho,p}^N \Big[\exp \theta \int_0^{\mathcal{T}} \Big\{ N W_{N,\varepsilon}^{H,\Psi}(t,\eta_t) - \beta \sum_x G\Big(\frac{x}{N}\Big) \omega(\eta_t(x)) \Big\}\ \mathrm{d}t \Big]$$

for every $\theta > 0$.

Therefore, we have to prove two Lemmas:

Lemma 3.1 *For every $G \in \mathcal{C}_K(\mathbb{R})$,*

$$\overline{\lim_{A \to \infty}}\ \overline{\lim_{N \to \infty}} \frac{1}{N} \log \mathbb{P}_{\rho,p}^N \Big[\int_0^{\mathcal{T}} \frac{1}{N} \sum_x G(x/N) \omega(\eta_t(x))\ \mathrm{d}t > A \Big] = -\infty \tag{9}$$

m-almost surely.

Lemma 3.2 *For any $\theta > 0$ and $\beta > 0$*

$$\overline{\lim_{\varepsilon \to 0}} \; \overline{\lim_{N \to \infty}} \; \frac{1}{N} \log \mathbb{E}^N_{\rho,p} \left[\exp \theta \int_0^{\mathcal{T}} \left\{ W^{H,\Psi}_{N,\varepsilon}(t, \eta_t) - \beta \sum_x G\Big(\frac{x}{N}\Big) \omega(\eta_t(x)) \right\} \, \mathrm{d}t \right] = 0. \tag{10}$$

m-almost surely.

Proof of Lemma 3.1.

Using respectively Tchebycheff exponential inequality and Jensen inequality, we show that for every positive constant θ, the logarithmic term in (9) is bounded above by

$$-\theta A N + \log \mathbb{E}^N_{\rho,p} \left[\frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \exp \Big\{ \sum_x \theta \mathcal{T} G(x/N) \omega(\eta_t(x)) \Big\} \, \mathrm{d}t \right].$$

¿From the begining of [R1] and since the product measure $\bar{\nu}^p_\rho$ is invariant for the process and $p_x \in [a_0, a_1]$, a simple computation shows that the right hand side term in (9) is bounded above by

$$\overline{\lim_{A \to \infty}} \; \overline{\lim_{N \to \infty}} \; \inf_{\theta > 0} \left\{ -\theta A + \frac{1}{N} \sum_x \log \nu_{\Phi(\rho) a_0^{-1}} \left[\exp \Big\{ \theta \mathcal{T} G(x/N) \omega(\eta(0)) \Big\} \right] \right\}. \tag{11}$$

Let $B > 0$ be such that

$$\mathrm{supp} G \subset [-B, B].$$

¿From [**H4**], there exists $\theta_0 > 0$ such that

$$\nu_{\Phi(\rho) a_0^{-1}} \left[\exp \Big\{ \theta_0 \mathcal{T} \|G\|_\infty \omega(\eta(0)) \Big\} \right] < \infty.$$

The lemma is proved in fact that (11) is bounded above by

$$\overline{\lim_{A \to \infty}} \left\{ -\theta_0 A + (2B+1) \log \nu_{\Phi(\rho) a_0^{-1}} \left[e^{\{\theta_0 \mathcal{T} \|G\|_\infty \omega(\eta(0))\}} \right] \right\}.$$

Proof of Lemma 3.2.

Let

$$V(\eta) = \theta \Big\{ N W^{H,\Psi}_{N,\varepsilon}(0, \eta) - \beta \sum_x G\Big(\frac{x}{N}\Big) \omega(\eta(x)) \Big\}.$$

Let $\mathcal{L}^p_V$ be the generator $N^2 \mathcal{L}_p + V$ and $\mathcal{L}^{p,*}_V$ its adjoint operator, which is equal to $N^2 \mathcal{L}^*_p + V$. If we denote by $S^{V,p}_t$ the semigroup associated to the generator $\mathcal{L}^p_V$, by the Feyman-Kac formula the expectation in the lemma is equal to

$$\langle S^{V,p}_{\mathcal{T}} 1, 1 \rangle \leq \langle S^{V,p}_{\mathcal{T}} 1, S^{V,p}_{\mathcal{T}} 1 \rangle^{(1/2)}.$$

Now, if we denote by λ_V the largest eigenvalue of the self-adjoint operator $\mathcal{L}_V^p + \mathcal{L}_V^{p,*}$,

$$\partial_t \langle S_t^{V,p}1, S_t^{V,p}1 \rangle = \langle (\mathcal{L}_V^p + \mathcal{L}_V^{p,*}) S_t^{V,p}1, S_t^{V,p}1 \rangle \le \lambda_V \langle S_t^{V,p}1, S_t^{V,p}1 \rangle.$$

By Gronwall's lemma we show that

$$\langle S_{\mathcal{T}}^{V,p}1, S_{\mathcal{T}}^{V,p}1 \rangle \le \exp\Big\{ \mathcal{T}\lambda_V \Big\}. \tag{12}$$

Recall that we did not assume $T(\cdot)$ to be symmetric and therefore $\nu^p_{\Phi(\rho)}$ can be non-reversible for the process. However, at this level, our study is dealing with the reversible generator $N^2(\mathcal{L}_p + \mathcal{L}_p^*)$. Thus we can assume the generator $\mathcal{L}_p$ to be reversible and $T(\cdot)$ given by $T(x) = (1/2)\mathbf{1}_{\{|x|=1\}}$.

Let

$$I^p_{x,x+1}(f) = \frac{1}{2}\int p_x g(\eta(x)) \left[\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right]^2 \bar\nu^p_\rho(\,\mathrm{d}\eta),$$

and $D_p(\cdot)$ the Dirichlet form given by

$$D_p(f) = \sum_x I^p_{x,x+1}(f).$$

Using the variational formula for the largest eigenvalue of a self-adjoint operator (see appendix A3.1 of Kipnis-Landim (1999)), from (12) we reduce the proof of the lemma to show that for every positive θ

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_f \Big\{ \int \theta \Big[W_{N,\varepsilon}^{H,\Psi}(\eta) - \frac{\beta}{N}\sum_x G\Big(\frac{x}{N}\Big)\omega(\eta(x)) \Big] f(\eta)\bar\nu^p_\rho(\,\mathrm{d}\eta) - N D_p(f) \Big\} \le 0.$$

The supremum is taken over all positive densities functions with respect to $\bar\nu^p_\rho$.

We use now some computations from Benois & al. (1995) and Kipnis & al. (1989). Let

$$W_l^\Psi(\eta) = \frac{1}{2l+1}\sum_{|y|\le l} \Psi(\eta(y)) - \tilde\Psi(\eta^l(0))$$

In this way, we can rewrite the term

$$W_{N,\varepsilon}^{H,\Psi}(\eta) - \frac{\beta}{N}\sum_x G\Big(\frac{x}{N}\Big)\omega(\eta(x))$$

as

$$\frac{1}{N}\sum_x \Big\{ H\Big(\frac{x}{N}\Big)\Big[\tau_x\Psi(\eta) - \frac{1}{2l+1}\sum_{|y-x|\le l}\tau_y\Psi(\eta)\Big] - \frac{\beta}{3}G\Big(\frac{x}{N}\Big)\omega(\eta(x))\Big\}$$
$$+\frac{1}{N}\sum_x \Big\{ H\Big(\frac{x}{N}\Big)\tau_x W_l^\Psi(\eta) - \frac{\beta}{3}G\Big(\frac{x}{N}\Big)\omega(\eta(x))\Big\}$$

$$+\frac{1}{N}\sum_x\left\{H\Big(\frac{x}{N}\Big)\Big[\tilde\Psi(\eta^l(x))-\tilde\Psi(\eta^{\varepsilon N}(x))\Big]-\frac{\beta}{3}G\Big(\frac{x}{N}\Big)\omega(\eta(x))\right\}.$$

¿From the assumption on Ψ, we chek easily that there exist $C(\Psi,p)$ such that for all $x\in\mathbf{Z}$ $\Psi(\eta(x))\le C(\Psi,p)\eta(x)$. Then from the definitions of $\omega^*(\cdot)$ and $G(\cdot)$ (cf. (4) and (7)), the first term in the last expression is bounded above by

$$\frac{1}{N}\sum_x\left\{\Big|\frac{1}{2l+1}\sum_{|y-x|\le l}H\Big(\frac{y}{N}\Big)-H\Big(\frac{x}{N}\Big)\Big|\Psi(\eta(x))-\frac{\beta}{3}G\Big(\frac{x}{N}\Big)\omega(\eta(x))\right\}$$

$$\le\frac{\beta}{3N}\sum_x G\Big(\frac{x}{N}\Big)\left\{\frac{3\mathcal{C}(\Psi,p)l}{\beta N}\eta(x)-\omega(\eta(x))\right\}$$

$$\le\omega^*\left\{\frac{3\mathcal{C}(\Psi,p)l}{\beta N}\right\}\frac{\beta\|G\|_\infty}{3}$$

This last term vanishes as $N\uparrow\infty$ since $\omega^*(\cdot)$ is continuous and $\omega^*(0)=0$.

Now, to achieve the proof of the lemma 3.2, we shall prove:

Lemma 3.3 *For any $b>0$*

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}}\ \sup_f$$

$$\left\{\frac{1}{N}\sum_x\int\Big[H\Big(\frac{x}{N}\Big)\tau_xW_l^\Psi(\eta)-\beta G\Big(\frac{x}{N}\Big)\omega(\eta(x))\Big]f(\eta)\ \mathrm{d}\bar\nu_\rho^p(\ \mathrm{d}\eta)-bND_p(f)\right\}\le 0 \tag{13}$$

m-almost surely. The supremum is taken over all positive densities functions with respect to $\bar\nu_\rho^p$.

And, thanks to remarks [R5], we have to prove that:

Lemma 3.4 *For any $b>0$*

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_f$$

$$\left\{\frac{1}{N}\sum_x\int\Big[H\Big(\frac{x}{N}\Big)\Big|\eta^{\varepsilon N}(x)-\eta^l(x)\Big|-\beta G\Big(\frac{x}{N}\Big)\omega(\eta(x))\Big]f(\eta)\ \mathrm{d}\bar\nu_\rho^p(\ \mathrm{d}\eta)-bND_p(f)\right\}\le 0 \tag{14}$$

m-almost surely. The supremum is taken over all positive densities functions with respect to $\bar\nu_\rho^p$.

Proof of Lemma 3.3.

Using the convexity of ω and definition of G, we check that

$$\begin{aligned}\frac{1}{N}\sum_x\Big|H\Big(\frac{x}{N}\Big)\Big|\omega(\eta^l(x)) &\le \frac{1}{N}\sum_x\Big|H\Big(\frac{x}{N}\Big)\Big|\frac{1}{2l+1}\sum_{|y-x|\le l}\omega(\eta(y))\\ &= \frac{1}{N}\sum_x\omega(\eta(x))\frac{1}{2l+1}\sum_{|y-x|\le l}\Big|H(y/N)\Big|\\ &\le \frac{1}{N}\sum_x\omega(\eta(x))G\Big(\frac{x}{N}\Big) \end{aligned}\tag{15}$$

At the beginning, we introduce some notations in order to deal in our study of (13) with the boxes of length $(2l+1)$. Indeed, the term

$$H\Big(\frac{x}{N}\Big)\tau_x W_l^{\Psi}(\eta) - \beta\Big|H\Big(\frac{x}{N}\Big)\Big|\omega(\eta^l(x))$$

depends on η only through $\eta(x-l)\cdots\eta(x+l)$. Thus we may restrict the integral to microscopic blocks. Denote by $\Lambda_l = \{-l\cdots l\}$ the box of length $(2l+1)$ centered at the origin. For a fixed $z \in \mathbf{Z}$, we denote by $\Lambda_{z,l}$ the box $z+\Lambda_l$, by $\mathbf{X}^l$ the configuration space $\mathbf{N}^{\Lambda_l}$, by $\bar{\nu}^p_{\rho,z,l}$ the product measure $\bar{\nu}^{\theta_z p}_{\rho}$ restricted to $\mathbf{X}^l$, by $f_{z,l}$ the density, with respect to $\bar{\nu}^p_{\rho,z,l}$, of the marginal of the measure $f(\eta)\bar{\nu}^{\theta_z p}_{\rho}(d\eta)$ on $\mathbf{X}^l$ and by $D^p_{\rho,z,l}(h)$ the Dirichlet form on $\mathbf{X}^l$ given by

$$D^p_{\rho,z,l}(h) = \sum_{\substack{|x-y|=1\\ x,y\in\Lambda_{z,l}}} \int p_x g(\eta(x))\left[\sqrt{h(\eta^{x,y})} - \sqrt{h(\eta)}\right]^2 \bar{\nu}^p_{\rho,z,l}(\,\mathrm{d}\eta).$$

Thus, from (15) and since the Dirichlet form is convex (by Schwarz inequality), the supremum in the lemma is bounded above by the supremum over all positive densities f (with respect to $\bar{\nu}^p_{\rho}$) of the term

$$\frac{1}{N}\sum_x\Big\{\int\Big[H\Big(\frac{x}{N}\Big)W_l^{\Psi}(\eta) - \beta\Big|H\Big(\frac{x}{N}\Big)\Big|\omega(\eta^l(0))\Big]f_{x,l}\bar{\nu}^p_{\rho,x,l}(\,\mathrm{d}\eta) - \frac{bN^2}{C(l)}D^p_{\rho,x,l}(f_{x,l})\Big\} \qquad (16)$$

As in the proof of lemma 3.1 of Koukkous (1999) we may now characterize the sites x where the environment degenerates (behaves badly).

Fix $\delta > 0$, $\alpha > 0$ and $n \in \mathbf{N}$ sufficiently large such that $\frac{a_1-a_0}{n} < \delta$. For $0 \le j \le n-2$, let $I_j^{\delta} = [\beta_j, \beta_{j+1}[$ where $\beta_j \in [a_0, a_1]$ is such that

$$\beta_j = a_0 + (a_1 - a_0)\Big(\frac{j}{n}\Big).$$

Let $I^{\delta}_{n-1} = [\beta_{n-1}, a_1]$ and notice that, for $0 \le j \le n-1$, we have $|\beta_{j+1} - \beta_j| < \delta$.

Fix $k < l$ and $L = [\frac{2l+1}{2k+1}]$. We now subdivide Λ_l into L disjoint cubes of length $(2k+1)$; let $B_1, \cdots, B_L$ be such that

$$B_i \subseteq \Lambda_l, \qquad B_i \cap B_j = \emptyset \ \text{ for } i \neq j \ \text{ and } \qquad B_i = x_i + \Lambda_k \ \text{ for some } x_i \in \mathbf{Z}.$$

We take $B_1 = \Lambda_k$ and let $B_0 = \Lambda_l - \cup_{j=1}^{L} B_j$. Finally we define $B_j(x) = x + B_j$ for $0 \le j \le L$ and $x \in \mathbf{Z}$.

For $x \in \mathbf{Z}$, $n \in \mathbf{N}$, $0 \le j \le n-1$ and $1 \le i \le L$, $N^{l,k,\delta}_{x,j,i}(p)$ is the average number of sites y in $B_i(x)$ such that $p_y \in I^{\delta}_j$:

$$N^{l,k,\delta}_{x,j,i}(p) = \frac{1}{(2k+1)}\sum_{z\in B_i(x)} \mathbf{1}_{\{p_z \in I^{\delta}_j\}}.$$

For $\alpha > 0$, we let

$$A^{l,k,\delta}_{x,i,\alpha} = \Big\{p, \quad \Big|N^{l,k,\delta}_{x,j,i}(p) - m(I^\delta_j)\Big| \leq \alpha \ \text{ for all } j, \ 0 \leq j \leq n-1\Big\}.$$

To keep notation simple, we denote $A^{l,k,\delta}_{0,1,\alpha}$ by $A^{l,k,\delta}_\alpha$. Let

$$A^{l,k,\delta}_{x,\alpha} = \Big\{p, \quad \frac{1}{L}\sum_{i=1}^{L} \mathbf{1}_{\{p\in A^{l,k,\delta}_{x,i,\alpha}\}} \geq 1-\alpha\Big\}.$$

¿From the definition of ω^* and the property of $\Psi(\cdot)$ and $\tilde{\Psi}(\cdot)$ given in the remarks [R5], a simple computation shows that the integral term in (16) is bounded by

$$C_1 = \beta \|H\|_\infty \omega^*\Big(\frac{2C(\Psi,p)}{\beta}\Big). \tag{17}$$

Therefore, the supremum over all positive densities f (with respect to $\bar{\nu}^p_\rho$) of the term (16) is bounded above by

$$\frac{1}{N}\sum_x \sup_{p\in A^{l,k,\delta}_{0,\alpha}} \sup_{h\in\mathcal{B}^l_p} \Big\{ \int \Big[H\Big(\frac{x}{N}\Big)W^\Psi_l(\eta) - \beta\Big|H\Big(\frac{x}{N}\Big)\Big|\omega(\eta^l(0))\Big]h(\eta)\bar{\nu}^p_{\rho,0,l}(\,\mathrm{d}\eta) - \frac{bN^2}{C(l)}D^p_{\rho,0,l}(h)\Big\}$$

$$+ C_1\frac{1}{N}\sum_x \mathbf{1}_{\{p\notin A^{l,k,\delta}_{x,\alpha}\}} \tag{18}$$

where $\mathcal{B}^l_p$ is the set of positive density functions with respect to $\bar{\nu}^p_{\rho,0,l}$.
By ergodicity and stationary of the environment law, the second term converges m-almost surely, as $N\uparrow\infty$, to

$$C_1 m\Big\{p \notin A^{l,k,\delta}_{0,\alpha}\Big\}.$$

Again the ergodicity of m ensures that this expression vanishes as $l\uparrow\infty$ and $k\uparrow\infty$ afterwards. Now, let us turn to the first term in (18). If we denote

$$\mathbf{E}^p_h[f] = \int h(\eta)f(\eta)\ \mathrm{d}\bar{\nu}^p_{\rho,0,l}(\eta),$$

the integral term in (18) is bounded above by

$$2C(\Psi)\Big|H\Big(\frac{x}{N}\Big)\Big|\Big\{\mathbf{E}^p_h\Big[\eta^l(0)\Big] - \frac{\beta}{2C(\Psi)}\mathbf{E}^p_h\Big[\omega(\eta^l(0))\Big]\Big\}.$$

Recall that ω is a convex and increasing function. Thus, by Jensen's inequality, the last expression is bounded above by

$$2C(\Psi)\Big|H\Big(\frac{x}{N}\Big)\Big|\Big\{\omega^{-1}\Big[\mathbf{E}^p_h\Big[\omega(\eta^l(0))\Big]\Big] - \frac{\beta}{2C(\Psi)}\mathbf{E}^p_h\Big[\omega(\eta^l(0))\Big]\Big\}.$$

¿From the remarks [R3], we claim that there exists a finite constant $C_2 = C_2(\beta, C(\Psi))$ such that the integral term in (18) is negative if $\mathbf{E}_h^p\Big[\eta^l(0)\Big] \geq C_2$. Let $B > 0$ be such that $suppH \subset [-B, B]$, then from (17) and the last claim, we check that the first term in (18) is bounded above by

$$(2B+1)\|H\|_\infty \sup_{p\in A_{0,\alpha}^{l,k,\delta}} \sup_{f\in \mathcal{B}_l^p(\frac{2C_1C(l)}{bN^2}, C_2)} \left| \int W_l^\Psi(\eta) f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta)\right|$$

where $\mathcal{B}_l^p(a,b)$ is defined for positive constant a and b by

$$\mathcal{B}_l^p(a,b) = \left\{ f \in \mathcal{B}_l^p : D_{\rho,0,l}^p(f) \leq a \text{ and } \mathbf{E}_f^p\Big[\omega(\eta^l(0))\Big] \leq b \right\}.$$

The weak topology of the set of probability measures on $\mathbf{X}^l$ ensures that, by definition, $\mathcal{B}_l^p(\frac{2C_1C(l)}{bN^2}, C_2)$ is one of its compact subsets. Therefore, by the lower semi-continuity of the Dirichlet form, we know that

$$\begin{aligned} &\overline{\lim_{N\to\infty}} \sup_{p\in A_{0,\alpha}^{l,k,\delta}} \sup_{f\in \mathcal{B}_l^p(\frac{2C_1C(l)}{bN^2}, C_2)} \left| \int W_l^\Psi(\eta) f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta)\right| \\ &\qquad \leq \sup_{p\in A_{0,\alpha}^{l,k,\delta}} \sup_{f\in \mathcal{B}_l^p(0, C_2)} \left| \int W_l^\Psi(\eta) f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta)\right|. \end{aligned} \tag{19}$$

¿From the assumption on Ψ (and $\tilde{\Psi}$), for every positive constant C_3, the term in absolute value is bounded above by

$$2C(\Psi)\int \mathbf{1}_{\{\eta^l(0)\geq C_3\}} \eta^l(0) f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta) + \left| \int W_l^\Psi(\eta) \mathbf{1}_{\{\eta^l(0)\leq C_3\}} f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta)\right|.$$

By remarks [R4], the first term in the last expression is bounded above by

$$\begin{aligned} 2C(\Psi)\Big(\frac{C_3}{\omega(C_3)}\Big) \int \omega(\eta^l(0)) f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta) &= 2C(\Psi)\Big(\frac{C_3}{\omega(C_3)}\Big) \mathbf{E}_f^p\Big[\omega(\eta^l(0))\Big] \\ &\leq 2C_2C(\Psi)\Big(\frac{C_3}{\omega(C_3)}\Big) \end{aligned}$$

for all $f \in \mathcal{B}_l^p(0, C_2)$. From (H4), this last term vanishes as $C_3 \uparrow \infty$. At this point, we achieve by proving that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{l\to\infty}} \sup_{p\in A_{0,\alpha}^{l,k,\delta}} \sup_{f\in \mathcal{B}_l^p(0, C_2)} \left| \int W_l^\Psi(\eta) \mathbf{1}_{\{\eta^l(0)\leq C_3\}} f(\eta) \bar{\nu}_{\rho,0,l}^p(\,\mathrm{d}\eta)\right| \leq \mathcal{C}(\delta, \alpha) \tag{20}$$

where $\mathcal{C}(\delta,\alpha)$ vanishes as $\alpha \downarrow 0$ and $\delta \downarrow 0$ afterwards. We omit this proof since it is developed in the proof of lemma 3.1 in Koukkous (1999).

Proof of Lemma 3.4.

First of all, we approximate (replace) the average over a small macroscopic box by an average over large microscopic boxes. More precisely, for N sufficiently large we check that

$$\begin{aligned}
&\frac{1}{N}\sum_x \Big|H\Big(\frac{x}{N}\Big)\Big|\Big|\eta^{\varepsilon N}(x) - \eta^l(x)\Big| \\
&\quad\le \frac{1}{N}\sum_x \Big|H\Big(\frac{x}{N}\Big)\Big|| \frac{1}{(2\varepsilon N+1)} \sum_{2l+1<|y|\le \varepsilon N} |\eta^l(x) - \eta^l(x+y)| + \mathcal{O}\Big(\frac{l}{\varepsilon N}\Big)\sum_x G\Big(\frac{x}{N}\Big)\eta(x) \\
&\quad\le \frac{1}{N}\sum_x \Big|H\Big(\frac{x}{N}\Big)\Big| \frac{1}{(2\varepsilon N+1)} \sum_{2l+1<|y|\le \varepsilon N} |\eta^l(x) - \eta^l(x+y)| + \frac{\beta}{N}\sum_x G\Big(\frac{x}{N}\Big)\omega(\eta(x))
\end{aligned}$$

Define

$$\omega_l(\eta,\xi,x,z) = \Big(\omega(\eta^l(x)) + \omega(\xi^l(z))\Big)$$

$$W_A^l(\eta,\xi,x,z) = |\eta^l(x) - \xi^l(y)|\mathbf{1}_{\{\eta^l(x)\vee\xi^l(z)\le A\}}$$

and to keep notation simple, we denote $W_A^l(\eta,\xi,0,0)$ by $W_A^l(\eta,\xi)$ and $\omega_l(\eta,\xi,0,0)$ by $\omega_l(\eta,\xi)$.

As in the previous proof, we introduce an indicator function and in the same way as in (15), we reduce our proof to show that, for every positive constant A

$$\begin{aligned}
\overline{\lim_{l\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_f \Bigg\{ \frac{1}{N}\sum_x \Big|H\Big(\frac{x}{N}\Big)\Big| \frac{1}{(2\varepsilon N+1)} \sum_{2l+1<|y|\le\varepsilon N} \qquad\qquad (21)\\
\int \Big[W_A^l(\eta,\eta,x,x+y) - \beta\omega_l(\eta,\eta,x,x+y)\Big] f(\eta)\bar\nu_\rho(\,\mathrm{d}\eta) - bND_p(f)\Bigg\} \le 0
\end{aligned}$$

¿From the definition of

$$\frac{1}{(2\varepsilon N+1)}\sum_{2l+1<|y|\le\varepsilon N} W_A^l(\eta,\eta,x,x+y)$$

and since $\eta^l(x)$ and $\eta^l(x+y)$ depend on the configuration η only through its values on the set

$$\Lambda_{x,y,l} := \Lambda_{x,l} \cup \Big(y + \Lambda_{x,l}\Big),$$

we shall replace f by its conditional expectation with respect to the σ-algebra generated by $\{\eta(z); z\in\Lambda_{x,y,l}\}$. Some notation are necessary. For all $y\in\mathbf{Z}$, we define the shift operator $\theta_y(\cdot)$ on environments by $(\theta_y p)(x) = p(x+y)$.

For fixed integer l and environments p and q, we denote by $\tilde{\mathbf{X}}^l$ the configuration space $\mathbf{N}^{\Lambda_l}\times\mathbf{N}^{\Lambda_l}$, by $\bar\nu^{p,q}_{\rho,x,l}$ the product measure $\bar\nu_\rho^{\theta_x p}\ \otimes \bar\nu_\rho^{\theta_x q}$ restricted to $\tilde{\mathbf{X}}^l$, and by $f^p_{x,y,l}$ the conditional expectation of f with respect to the σ-algebra generated by $\{\eta(z); z \in \Lambda_{x,y,l}\}$. Thus the supremum in (21) is bounded above by

$$\sup_f \left\{ \frac{1}{N} \sum_x \left| H\left(\frac{x}{N}\right) \right| \frac{1}{(2\varepsilon N+1)} \sum_{2l+1<|y|\leq \varepsilon N} \int \left[\tau_x W_A^l(\xi_1,\xi_2) - \beta\omega_l(\xi_1,\xi_2) \right] f_{x,y,l}^p(\xi_1,\xi_2) \bar{\nu}_{\rho,x,l}^{p,\theta_y p}(\,\mathrm{d}\xi) - bND^p(f) \right\}.$$

Let us turn now to the Dirichlet form of $f_{x,y,l}^p$ into microscopic boxes $\Lambda_{x,y,l}$. Let $D_l^{p,q}(h)$ be

$$D_l^{p,q}(h) = I_{l,1}^{p,q}(h) + I_{l,2}^{p,q}(h) + \sum_{\substack{z,z'\in\Lambda_l \\ |z-z'|=1}} I_{z,z',1}^{p,q}(h) + \sum_{\substack{z,z'\in\Lambda_l \\ |z-z'|=1}} I_{z,z',2}^{p,q}(h)$$

where, for each $z, z' \in \Lambda_l$, such that $|z-z'| = 1$,

$$I_{z,z',1}^{p,q}(h) = 1/2 \int p_z g(\xi_1(z)) \Big[\sqrt{h(\xi_1^{z,z'},\xi_2)} - \sqrt{h(\xi_1,\xi_2)} \Big]^2 \bar{\nu}_{\rho,0,l}^{p,q}(d\xi),$$

$$I_{z,z',2}^{p,q}(h) = 1/2 \int q_z g(\xi_2(z)) \Big[\sqrt{h(\xi_1,\xi_2^{z,z'})} - \sqrt{h(\xi_1,\xi_2)} \Big]^2 \bar{\nu}_{\rho,0,l}^{p,q}(d\xi),$$

$$I_{l,1}^{p,q}(h) = 1/2 \int p_0 g(\xi_1(0)) \Big[\sqrt{h(\xi_1^{0,-},\xi_2^{0,+})} - \sqrt{h(\xi_1,\xi_2)} \Big]^2 \bar{\nu}_{\rho,0,l}^{p,q}(d\xi),$$

$$I_{l,2}^{p,q}(h) = 1/2 \int q_0 g(\xi_2(0)) \Big[\sqrt{h(\xi_1^{0,+},\xi_2^{0,-})} - \sqrt{h(\xi_1,\xi_2)} \Big]^2 \bar{\nu}_{\rho,0,l}^{p,q}(d\xi).$$

The configurations $\xi^{0,\pm}(\cdot)$ are defined by

$$\xi^{0,\pm}(z) = \begin{cases} \xi(z) & \text{if } z \neq 0 \\ \xi(0) \pm 1 & \text{if } z = 0. \end{cases}$$

We claim that

$$\frac{1}{N} \sum_x \frac{1}{(2\varepsilon N+1)} \sum_{2l+1<|y|\leq \varepsilon N} D_l^{p,\theta_y p}\left(f_{x,y,l}^p \right) \leq C(l)\varepsilon^2 N D^p(f). \tag{22}$$

The proof of the claim is omitted. We can see Lemma 4.3 of Koukkous (1999) for several details.

¿From the same notation in the proof of Lemma 3.3, we separate the sites where the environment behaves badly and repeat the computation in the begining of (17). Using (22) and introducing the indicator function of the environements afterwards, our lemma is a consequence of the following results

Lemma 3.5

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \frac{1}{(2\varepsilon N+1)}\sum_{2l+1<|y|\leq\varepsilon N}\frac{1}{N}\sum_{x}\Big[\mathbf{1}_{\{\theta_x p\notin A^{l,k,\delta}_{0,\alpha}\}}+\mathbf{1}_{\{\theta_{x+y} p\notin A^{l,k,\delta}_{0,\alpha}\}}\Big]=0$$

m almost surely.

Lemma 3.6 *For positive constants a and b, let*

$$\mathcal{B}^{p,q}_l(a,b)=\Big\{h\geq 0,\mathbf{E}_{\bar{\nu}^{p,q}_{\rho,0,l}}[h]=1,D^{p,q}_l(h)\leq a,\mathbf{E}^{p,q}_h\Big[\omega_l(\xi_1,\xi_2)\Big]\leq b\Big\}$$

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{l\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_{p,q\in A^{l,k,\delta}_{0,\alpha}}\ \sup_{h\in\mathcal{B}^{p,q}_l\left(\frac{(2\varepsilon N+1)}{bN^2}C_1,C_2\right)}\mathbf{E}^{p,q}_h\Big(W^l_A(\xi_1,\xi_2)\Big)\leq\mathcal{C}(\delta,\alpha)\tag{23}$$

where $\mathcal{C}(\delta,\alpha)$ vanishes as $\alpha\downarrow 0$ and $\delta\downarrow 0$ afterwards.

The lemma 3.5 is trivially proved using the ergodicity and stationarity of m. (see Koukkous (1999)).
Since $\mathcal{B}^{p,q}_l\left(\frac{(2\varepsilon N+1)}{bN^2}C_1,C_2\right)$ is a compact subset of the probability measures set on $\mathbf{X}^l\times\mathbf{X}^l$ endowed with the weak topology, by the lower semi-continuity of the Dirichlet form, to prove (23) it is enough to prove that

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{\alpha\to 0}}\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{l\to\infty}}\ \sup_{p,q\in A^{l,k,\delta}_{0,\alpha}}\ \sup_{h\in\mathcal{B}^{p,q}_l(0,C_2)}\mathbf{E}^{p,q}_h\Big(W^l_A(\xi_1,\xi_2)\Big)=0.$$

which is proved in Koukkous (1999) (see the proof of lemma 4.2 at formula (23)).

4 Proof of Theorem 2.2

The proof of lower bound presented in Benois & al. (1995) is easily adapted for our case using some computations already developed in the previous proof of super-exponential estimate and some arguments presented in the below upper bound's proof. We therefore omit details for the reader.

Let $H\in\mathcal{C}^{1,2}_K([0,\mathcal{T}]\times\mathbf{R})$ and $\gamma\in\mathcal{C}_\rho(\mathbf{R})$. From Girsanov formula, the Radon-Nikodym derivative of $\mathbf{P}^{p,H}_{\gamma,N}$ with respect to $\mathbf{P}^N_{\rho,p}$ is given by

$$\exp N\Big\{\mathcal{J}^1_H(\pi^N_t)+h^{p,N}_\gamma(\pi^N_0|\rho)-N\int_0^t\sum_{x,y}p_x g(\eta_s(x))T(y)\Big[e^{\{H(t,\frac{x+y}{N})-H(t,\frac{x}{N})\}}\ -\ 1\Big]\,\mathrm{d}s\Big\}\tag{24}$$

where $h^{p,N}_\gamma(\cdot|\rho) : \mathcal{M}_+ \to \mathbf{R}$ is defined by

$$h^{p,N}_\gamma(\mu|\rho) = \left\langle \mu, \log\left(\frac{\Phi(\gamma(\cdot))}{\Phi(\rho)}\right)\right\rangle - \frac{1}{N}\sum_x \log\left[\frac{Z(\Phi(\gamma(x/N))p_x^{-1})}{Z(\Phi(\rho)p_x^{-1})}\right].$$

Upper bound :

The proof is dealing only with a fixed compact subset $\mathcal{C}$ of $\mathcal{D}([0,\mathcal{T}],\mathcal{M}_+)$. To extend this result to a closed subset, we need exponential tightness for $Q^N_{\rho,p}$. It is easily obtained thanks to the proof presented in Benois (1996) (see also Lemma V.1.5 in Kipnis-Landim (1999)).

For every $q > 1$,

$$Q^N_{\rho,p}(\mathcal{C}) = \mathbf{E}^N_{\rho,p}\left[\left(\frac{\mathrm{d}\mathbf{P}^N_{\rho,p}}{\mathrm{d}\mathbf{P}^{p,H}_{\gamma,N}}\right)^{1/q}\left(\frac{\mathrm{d}\mathbf{P}^{p,H}_{\gamma,N}}{\mathrm{d}\mathbf{P}^N_{\rho,p}}\right)^{1/q}\mathbf{1}_{\{\pi^N\in\mathcal{C}\}}\right].$$

Let ϑ_ε be the approximation of identity defined by $(2\varepsilon)^{-1}\mathbf{1}_{[-\varepsilon,\varepsilon]}(x)$ and $*$ the classic convolution product.

For $0 \le s \le \mathcal{T}$, let

$$u^{p,H}_{\varepsilon,N}(\eta_s) = \frac{\sigma}{2N}\sum_k \{\partial_x^2 H(s,k/N) + [\partial_x H(s,k/N)]^2\}\{p_k g(\eta_s(k)) - \Phi(\eta_s^{\varepsilon N}(k))\}$$

and

$$u^p_{N,H}(\eta_s) = \frac{1}{N}\sum_k p_k g(\eta_s(k))\Bigg\{\sum_j T(j)N^2\left[e^{\{H(t,\frac{k+j}{N})-H(t,\frac{k}{N})\}} - \mathbf{1}\right]$$
$$-\frac{\sigma}{2}\left\{\partial_x^2 H(s,k/N) + (\partial_x H(s,k/N))^2\right\}\Bigg\}$$

¿From (24), a simple computation shows that $\left(\mathrm{d}\mathbf{P}^N_{\rho,p}/\ \mathrm{d}\mathbf{P}^{p,H}_{\gamma,N}\right)$ is bounded above by

$$\exp N\left\{ -\mathcal{J}^1_H(\pi^N_\mathcal{T}) + \mathcal{J}^2_H(\pi^N * \vartheta_\varepsilon) - h^{p,N}_\gamma(\pi^N_0|\rho) + \int_0^\mathcal{T}\left\{u^{p,H}_{\varepsilon,N}(\eta_s) + u^p_{N,H}(\eta_s)\right\}\,\mathrm{d}s\right\}$$

Thus, $\frac{1}{N}\log Q^N_{\rho,p}(\mathcal{C})$ is bounded above by

$$\frac{1}{q}\sup_{\pi\in\mathcal{C}}\left\{ -\mathcal{J}^1_H(\pi^N_\mathcal{T}) + \mathcal{J}^2_H(\pi^N * \vartheta_\varepsilon) - h^{p,N}_\gamma(\pi^N_0|\rho)\right\} \tag{25}$$
$$+\frac{1}{N}\log \mathbf{E}^N_{\rho,p}\left[\left(\frac{\mathrm{d}\mathbf{P}^{p,H}_{\gamma,N}}{\mathrm{d}\mathbf{P}^N_{\rho,p}}\right)^{1/q}\exp\left\{\frac{N}{q}\int_0^\mathcal{T}\left(u^{p,H}_{\varepsilon,N}(\eta_s) + u^p_{N,H}(\eta_s)\right)\,\mathrm{d}s\right\}\right]$$

Let $\bar{H}$ be a real continuous function with the same support as $\sup_t |H_t|$, such that it bounds above $\sup_{0\le t\le \mathcal{T}}[|\partial_x^2 H_t| + (\partial_x H_t)^2 + |H_t|]$.

Let $C_0 \in \mathbf{N}$ such that $suppH \subset [0,\mathcal{T}]\times[-(C_0-1),(C_0+1)]$. Using Hölder's inequality, we show that, for $q' \in \mathbf{R}$ such that $(1/q)+(1/q')=1$, the second term in (25) is bounded above by

$$\begin{aligned}\frac{1}{3Nq'}\log \mathbf{E}^N_{\rho,p}\Big[\exp\Big\{\frac{3Nq'}{q}\Big(\int_0^{\mathcal{T}} u^{p,H}_{\varepsilon,N}(\eta_s)\ \mathrm{d}s - \int_0^{\mathcal{T}}\frac{\alpha}{N}\sum_k \bar{H}(\frac{k}{N})\omega(\eta_s(k))\ \mathrm{d}s\Big)\Big\}\Big]\\ +\frac{1}{3Nq'}\log \mathbf{E}^N_{\rho,p}\Big[\exp\Big\{\frac{3Nq'}{q}\int_0^{\mathcal{T}} u^{p}_{N,H}(\eta_s)\ \mathrm{d}s\Big\}\Big] \qquad (26)\\ +\frac{1}{3Nq'}\log \mathbf{E}^N_{\rho,p}\Big[\exp\Big\{\frac{3Nq'}{q}\Big(\frac{\alpha}{N}\int_0^{\mathcal{T}}\sum_k \bar{H}(\frac{k}{N})\omega(\eta_s(k))\ \mathrm{d}s\Big)\Big\}\Big]\end{aligned}$$

Using similar arguments as in the proof of lemma 3.2 (see (11)), we check that the last term in (26) is bounded above by

$$R_1(\alpha,q,H) = \frac{2C_0}{3q}\log \nu_{\Phi(\rho)a_0^{-1}}\Big[e^{\{\frac{3\alpha q'\mathcal{T}}{q}\|\bar{H}\|_\infty \omega(\eta(0))\}}\Big]$$

which vanishes as $\alpha \downarrow 0$ for each fixed q and H thanks to assumption [H4].

¿From assumption [H2], we check that $g(k) \le g^* k$ for all $k\in\mathbf{Z}$ and therefore $\Phi(\rho)\le g^*\rho$. Thus, we repeat the same argument as above, a simple computation shows that the second term in (26) is bounded above by

$$R_2(q,H,N) = \frac{2C_0}{3q'}\log \nu_{\Phi(\rho)a_0^{-1}}\Big[e^{\{\frac{\beta}{N}\eta(0)\}}\Big]$$

where $\beta=\beta(\mathcal{T},g^*,H,a_1,q,\sigma)$.

For each fixed q and H, it is easy to see that $R_2(q,H,N)$ vanishes as $N\uparrow\infty$.

Let us turn to the first term in (26) and denote $R_3(\alpha,q,H,\varepsilon)$ its limit when $N\uparrow\infty$. A similar computation as in the proof of the super-exponential estimate (see lemma 3.2 and its proof), gives that

$$\lim_{\varepsilon\to 0} R_3(\alpha,q,H,\varepsilon) = 0$$

for all $\alpha>0$, $q>1$ and smooth function H.

In the other hand notice that by a simple computation and from the ergodicity and stationarity of m, we prove that $h^{p,N}_\gamma(\pi_0^N|\rho)$ converges (uniformly in $\pi\in\mathcal{C}$) to $h(\gamma|\rho)$ when $N\uparrow\infty$.

We therefore proved that $\overline{\lim}_{N\to\infty}(1/N)\log Q^N_{\rho,p}(\mathcal{C})$ is bounded above by

$$\inf_{H,\gamma,q,\alpha,\varepsilon}\Big\{\frac{1}{q}\sup_{\pi\in\mathcal{C}}\Big\{-\mathcal{J}^1_H(\pi)+\mathcal{J}^2_H(\pi*\vartheta_\varepsilon)-h(\gamma|\rho)\Big\}+R_3(\alpha,q,H,\varepsilon)+R_1(\alpha,q,H)\Big\}$$

where the infimum is taken over all $H \in \mathcal{C}_K^{1,2}([0,\mathcal{T}]\times\mathbf{R})$, $\gamma \in \mathcal{C}_\rho(\mathbf{R})$, $q>1$, $\alpha>0$ and $\varepsilon>0$.

At this level, using the continuity of $\mathcal{J}_H^2(\cdot * \vartheta_\varepsilon)$ for every fixed H and $\varepsilon>0$, the compacity of $\mathcal{C}$ and the arguments developed in (Kipnis & al. (1989)) to permute the supremum and infimum, we check that this last expression is bounded above by

$$-\inf_{\pi\in\mathcal{C}} \sup_{H,\gamma,q,\alpha,\varepsilon} \left\{ \frac{1}{q}\left\{ -\mathcal{J}_H^1(\pi) + \mathcal{J}_H^2(\pi * \vartheta_\varepsilon) - h(\gamma|\rho)\right\} + R_3(\alpha,q,H,\varepsilon) + R_1(\alpha,q,H)\right\}$$

We conclude therefore our proof by letting $\varepsilon \downarrow 0$. $\alpha \downarrow 0$ and $q \downarrow 1$.

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