On the Approximation of Fuzzy Sets[1]

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Abstract. The purpose of this work is studying the approximation of normal fuzzy sets with compact support and the convolution $(f \bigtriangledown g)(x) = \sup\{f(x-y) \wedge g(y) : y \in X\}$ of two fuzzy sets. In particular, by using $\bigtriangledown$-convolution, a density result is proved.

1 Introduction

There exist many situations where it is necessary to approximate the arbitrary normal fuzzy sets with compact support by fuzzy sets with more convenient properties, for example, by continuous fuzzy sets or lipschitzians fuzzy sets, see for example [4], [3].

In this direction Colling and Kloeden [2] show that the normal convex fuzzy sets with compact support can be approximate by continuous fuzzy sets. In the work [6], the authors prove that the lipschitzians fuzzy sets are dense in the normal convex fuzzy sets with compact support where the level-application is continuous.

In this work, we generalize the above results in two ways:

a) We consider the normal fuzzy sets with compact support.

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b) We prove the density of lipschitzians fuzzy sets in the class considered in a).

Actually, it was only for simplicity that we derived our results in $\mathbb{R}^n$; they extend easily to the case of real reflexive, separable Banach spaces. In fact, the essential argument used is the compactness of the balls. Thus, if we consider the weak topology, the following arguments work.

The plan of the paper is as follows. In Section 2, we give the notations, definitions, preliminaries results used throughout the paper and we establishe the main result of this paper. In Section 3 we give the proof of the main result.

2 Preliminaries

Let $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{K}_c(\mathbb{R}^n)$ be, the class of all nonempty and compact subsets of $\mathbb{R}^n$, and the class of all nonempty compact and convex subsets of $\mathbb{R}^n$, respectively. The Hausdorff metric H on $\mathcal{K}(\mathbb{R}^n)$ is defined by

$$H(A,B) = \max\left\{\sup^{a\in A} d(a,B), \sup^{b\in B} d(b,A)\right\},$$

and it is known that $(\mathcal{K}(\mathbb{R}^n), H)$ is a complete and separable metric space, and $\mathcal{K}_c(\mathbb{R}^n)$ is a closed subspace of $\mathcal{K}(\mathbb{R}^n)$ (see [1], [4]). Also, by using the Minkowski sum between two sets, a linear structure of convex cone is defined on $\mathcal{K}(\mathbb{R}^n)$ by mean

$$A + B = \{a + b \ / \ a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a \ / \ a \in A\},$$

for all $A, B \in \mathcal{K}(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$.

Some properties of these above operations are (see [1], [4]):

Proposition 1 *If $A, A_1, B, B_1 \in \mathcal{K}(\mathbb{R}^n)$ then*

i) $H(\lambda A, \lambda B) = \lambda H(A,B)$, for all $\lambda \geq 0$,

ii) $H(A + B, A_1 + B_1) \leq H(A, A_1) + H(B, B_1)$,

iii) If $A, B \in \mathcal{K}_c(\mathbb{R}^n)$ then $H(A + A_1, B + A_1) = H(A,B)$.

Let F^n be the space of normal fuzzy sets with compact support, i.e., F^n consists of all $f : \mathbb{R}^n \to [0,1]$ such that:

(i) f is normal, i.e. there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 1$,

(ii) f is upper semicontinuous,

(iii) $L_0 f = \text{ supp } (f) = \mathbf{cl}\{x \in \mathbb{R} \; / \; f(x) > 0\} \in \mathcal{K}(\mathbb{R}^n)$.

For $0 < \alpha \leq 1$, denote $L_\alpha f = \{x \; / \; f(x) \geq \alpha\}$ the α-level of f. Then from (i)-(iii), it follows that $L_\alpha f \in \mathcal{K}(\mathbb{R}^n)$, for all $\alpha \in [0,1]$. Also, we can to extend H to F^n by mean

$$D(f,g) = \sup^{\alpha \in [0,1]} H(L_\alpha f, L_\alpha g),$$

and it is well known that (F^n, D) is a complete (see [4]) but not separable metric space (see [4]).

Also, we consider the following closed subspace of F^n :

$$E^n = \{f \in F^n \; / \; L_\alpha f \in K_c(\mathbb{R}^n), \; \forall \alpha \in [0,1]\}.$$

The following definition was considered in [7] in another context.

Definition 1 *Let $f, g \in F^n$ be. Then, we define the* ***sup-min convolution*** *$f \bigtriangledown g$ between f and g as*

$$(f \bigtriangledown g)(x) = \sup^{y \in \mathbb{R}^n} \{f(y) \wedge g(x-y)\}.$$

This is a variation of the classical definition of the convolution in convex analysis, see [1], [5]. Here the simbol $\wedge$ has their usual meaning, i.e.,

$$a \wedge b = \text{ minimum of } a \in [0,1] \text{ and } b \in [0,1].$$

The following result is proved analogously as in [5], [7].

Proposition 2 *If $f, g \in F^n$ then $f \bigtriangledown g \in F^n$. Moreover,*

$$L_\alpha(f \bigtriangledown g) = L_\alpha f + L_\alpha g \; , \; \forall \; \alpha \in [0,1].$$

Remark 1 *Due Proposition 2.3, by using the sup-min convolution, a linear structure is introduced on* F^n *by mean*

$$
\begin{aligned}
f \oplus g &= f \bigtriangledown g \\
(\lambda \odot f)(x) &= \begin{cases} f(x/\lambda) & if \quad \lambda \neq 0 \\ \chi_{\{\mathbf{0}\}}(x) & if \quad \lambda = 0 \end{cases}
\end{aligned}
$$

and, with these definitions, we obtain $L_\alpha(f \oplus g) = L_\alpha f + L_\alpha g$ and $L_\alpha(\lambda \odot f) = \lambda L_\alpha f$, for all $f, g \in F^n$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$ (for more details on level-sum of functions see [4]).

3 Density and sup-min convolution

In general, it is well known that the space of Lipschitzian functions is a dense subspace of the space of continuous functions with respect to the uniform metric. However, in this section we will show a different type of density. More specifically, we will prove that the space of Lipschitzian functions is a dense subspace of (F^n, D).

We recall that $g \in F^n$ is said Lipschitzian with constant $K \in \mathbb{R}^+$ if $\mid g(x) - g(y) \mid \leq K \parallel x - y \parallel$, for every $x, y \in$ supp (g).

We will denote by L^n the class of all Lipschitzian fuzzy sets $g \in F^n$.

Remark 2 *Let* $A \in \mathcal{K}(\mathbb{R}^n)$ *be. Then* $\chi_A \in F^n$*, where* χ_A *denotes the characteristic function of* A*. Moreover,* $\chi_A \in L^n$*.*

The following result is proved similarly as in [7], we give here a proof in order to make this paper self-contained.

Theorem 1 *Let* $f, g \in F^n$ *be. If* $g \in L^n$ *then* $f \bigtriangledown g \in L^n$*.*

Proof. Firstly, we observe that supp $(f \bigtriangledown g) =$ supp $(f)+$ supp (g). Writing

$$
(f \bigtriangledown g)(x) = \sup\{h_y(x) = f(y) \wedge g(x-y) \ / \ y \in \text{ supp } (f)\}
$$

then, for each $y \in$ supp (f), the function $h_y : \mathbb{R}^n \to [0,1]$ is Lipschitzian with constant K. In fact,

$$\begin{aligned} | h_y(x) - h_y(z) | &= | f(y) \wedge g(x-y) - f(y) \wedge g(z-y) | \\ &\leq | g(x-y) - g(z-y) | \\ &\leq K \parallel x - z \parallel . \end{aligned}$$

Finally, to prove that $f \bigtriangledown g$ is Lipschitzian with constant K we note that

$$h_y(z) - K \parallel x - z \parallel \leq h_y(x) \leq h_y(z) + K \parallel x - z \parallel,$$

and then, taking the supremum with respect to y, we obtain

$$(f \bigtriangledown g)(z) - K \parallel x - z \parallel \leq (f \bigtriangledown g)(x) \leq (f \bigtriangledown g)(z) + K \parallel x - z \parallel ,$$

i.e.,

$$| (f \bigtriangledown g)(x) - (f \bigtriangledown g)(z) | \leq K \parallel x - z \parallel .$$

This complete the proof.

Example 1 *Consider the function $f : \mathbb{R} \to [0,1]$ defined by*

$$f(x) = \begin{cases} \frac{1}{2} & if \quad 0 \leq x < \frac{1}{2} \\ 1 & if \quad \frac{1}{2} \leq x \leq 1 \\ 0 & if \quad x \notin [0,1]. \end{cases}$$

Then it is clear that f is not a Lipschitzian function. Nevertheless, if we consider

$$g(x) = \begin{cases} x & if \quad 0 \leq x < 1 \\ 0 & if \quad x \notin [0,1], \end{cases}$$

we have that $g \in L^n$ and,

$$L_\alpha f = \begin{cases} [0,1] & if \quad 0 \leq \alpha \leq \frac{1}{2} \\ [\frac{1}{2},1] & if \quad \frac{1}{2} < \alpha \leq 1 \end{cases} \qquad and \quad L_\alpha g = [\alpha, 1], \ \forall \alpha \in [0,1].$$

So,

$$L_\alpha(f \bigtriangledown g) = \begin{cases} [0,1] + [\alpha,1] = [\alpha,2] & if \quad 0 \leq \alpha \leq \frac{1}{2} \\ [\frac{1}{2},1] + [\alpha,1] = [\alpha + \frac{1}{2}, 2] & if \quad \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Therefore, by using the formulae $(f \bigtriangledown g)(x) = \sup\{\alpha / \ x \in L_\alpha(f \bigtriangledown g)\}$ we obtain

$$(f \bigtriangledown g)(x) = \begin{cases} x & if \quad 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & if \quad \frac{1}{2} \leq x < 1 \\ x - \frac{1}{2} & if \quad 1 \leq x < \frac{3}{2} \\ 1 & if \quad \frac{3}{2} \leq x \leq 2, \end{cases}$$

which is a Lipschitzian function with constant $K = 1$.

Our main result is the following

Theorem 2 *For each* $f \in F^n$ *there exists a sequence* $(g_p) \in L^n$ *such that* $D(g_p, f) \leq 1/p$ *for* $p = 1, 2, ...$

Proof. Let $f \in F^n$ be. Denote $B[\mathbf{0}, 1/p]$ the closed ball with radius $1/p$ centered in the origin $\mathbf{0} \in \mathbb{R}^n$. Then, because $B[\mathbf{0}, 1/p]$ is compact, we have that $\chi_{B[\mathbf{0},1/p]} \in L^n$ for each $p \in \mathbb{N}$.

Taking $g_p = f \bigtriangledown \chi_{B[\mathbf{0},1/p]}$ then, by Theorem 3.1, we have that $g_p \in L^n$, $\forall p$. Moreover, for each $\alpha \in [0,1]$,

$$\begin{aligned} H(L_\alpha g_p, L_\alpha f) &= H(L_\alpha f \bigtriangledown \chi_{B[\mathbf{0},1/p]}, L_\alpha f) \\ &= H(L_\alpha f + L_\alpha \chi_{B[\mathbf{0},1/p]}, L_\alpha f) \\ &= H(L_\alpha f + B[\mathbf{0}, 1/p], L_\alpha f + \{\mathbf{0}\}) \\ &\leq H(B[\mathbf{0}, 1/p], \{\mathbf{0}\}) \\ &= 1/p. \end{aligned}$$

So, taking supremum in $\alpha \in [0,1]$, we obtain $D(g_p, f) \leq 1/p$, for every p, and the proof is complete.

Corollary 1 (L^n, D) *is a dense subspace of* (F^n, D).

Remark 3 *In a more restricted context, Colling, Kloeden [2], by using totally different techniques, proves that continuous fuzzy sets are denses in* E^n *with respect to D-metric, where* $E^n = \{f \in F^n \;/\; L_\alpha f \in K_c(\mathbb{R}^n)\}$.

Remark 4 *In another paper, by using the multivalued Berstein polynomial, we prove that the Lipschitzian fuzzy sets are dense in the class of fuzzy sets, where the level application:*$\alpha \to L_\alpha f$ *is continuous [6].*

References

[1] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lect. Notes in Math. 580, Springer-Verlag, 1977.

[2] I. L.Colling and P.E. Kloeden, Continuous approximation of fuzzy sets, J.Fuzzy Math.**3**(2) (1995), 449-453.

[3] W. Congxin and S. Shiji, Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions, J. Information Sciences 108 (1998), 133-134.

[4] P.Diamond and P.Kloeden, Metric space of fuzzy sets:Theory and Applications, World Scientific Pub., Singapure,1994.

[5] T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N.J., 1970.

[6] M. Rojas-Medar, R.C. Bassanezi and H. Román-Flores, A generalization of the Minkowski embedding theorem and applications, Fuzzy Sets Syst.**102**(1/2) (1999), 263-269.

[7] A. Seeger and M. Volle, A convolution operation obtained by adding level sets:Classical and new results, Oper.Research **29**(2) (1995), 131-154.

[8] Y. Syan, On a convex and concave fuzzy mappings, Fuzzy Sets Syst.**103** (1999), 163-168.

[9] S. Traoré and M. Volle, On the level sum of two convex functions on Banach spaces, J.Convex Anal.**3**(1) (1996), 141-151.