The compression semigroup of a cone is connected

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Abstract

Let $W \subset \mathbb{R}^n$ be a pointed and generating cone and denote by $S(W)$ the semigroup of matrices with positive determinant leaving W invariant. The purpose of this paper is to prove that $S(W)$ is path connected. This result has the following consequence: Semigroups with nonempty interior in the group $\mathrm{Sl}(n, \mathbb{R})$ are classified into types, each type being labelled by a flag manifold. The semigroups whose type is given by the projective space $\mathbb{P}^{n-1}$ form one of the classes. It is proved here that the semigroups in $\mathrm{Sl}(n, \mathbb{R})$ leaving invariant a pointed and generating cone are the only maximal connected in the class of $\mathbb{P}^{n-1}$.

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1 Introduction

Let W be a convex cone in $\mathbb{R}^n$ and form its compression semigroup of matrices

$$S(W) = \{g \in \mathrm{Gl}^+(n, \mathbb{R}) : gW \subset W\},$$

where $\mathrm{Gl}^+(n, \mathbb{R})$ stands for the group of real matrices having positive determinant. The purpose of this paper is to prove that $S(W)$ is connected if mild conditions on W are assumed. Precisely, recall that W is said to a be pointed cone in case $\pm v \in W$ implies $v = 0$. Also, W is generating if $\mathbb{R}^n = W + (-W)$, or equivalently, if $\mathrm{int} W \neq \emptyset$, where int stands for the interior of a set with respect to the standard topology of $\mathbb{R}^n$.

Theorem 1 *If W is pointed and generating then $S(W)$ is path connected.*

Clearly, $S(W)$ is a closed subsemigroup of $\mathrm{Gl}^+(n, \mathbb{R})$. Moreover, it is known – and easy to prove – that in case W is pointed and generating, $S(W)$ has nonempty interior in $\mathrm{Gl}^+(n, \mathbb{R})$, taken with its standard topology (c.f. Proposition 4 below).

Apart from being fruitful examples of semigroups in Lie groups the interest in the semigroups $S(W)$ stays in the fact that they form (in essence) a class of maximal semigroups in the special linear group $\mathrm{Sl}(n, \mathbb{R})$. In order to discuss this we note first that the identity matrix 1 as well as the scalar matrices $\lambda \cdot 1$, $\lambda > 0$, are in $S(W)$. Analogously, a matrix $g \in S(W)$ if and only if $(\det g)^{1/n} g \in S(W)$. Therefore if we consider the compression semigroup

$$S_W = S(W) \cap \mathrm{Sl}(n, \mathbb{R}) = \{g \in \mathrm{Sl}(n, \mathbb{R}) : gW \subset W\},$$

it follows that $S(W) = \mathbb{R}^+ \cdot S_W$ and S_W is the image of $S(W)$ under the continuous map $g \mapsto (\det g)^{1/n} g$. Hence if one of the semigroups $S(W)$ or S_W is connected, the same happens to the other. In what follows we take advantage of the theory of semigroups in semi-simple Lie groups and work within $\mathrm{Sl}(n, \mathbb{R})$. The proof of Theorem 1 will be accomplished by showing that S_W is connected.

To see the connection between S_W and maximal semigroups in $\mathrm{Sl}(n, \mathbb{R})$ let $[W]$ be the subset of the projective space $\mathbb{P}^{n-1}$ underlying W, that is, $[W]$ is the subset of lines in $\mathbb{R}^n$ contained in $W \cup -W$. Put

$$S[W] = \{g \in \mathrm{Sl}(n, \mathbb{R}) : g[W] \subset [W]\}.$$

It was proved in [10], Theorem 6.12, that $S[W]$ is a maximal semigroup of $\mathrm{Sl}(n, \mathbb{R})$ (see also [9], for more details about maximal semigroups). Clearly $g \in S[W]$ if

and only if $g \in S_W$ or $gW \subset -W$. It is rather easy to prove the existence of $g \in S[W]$ such that $gW \subset -W$ (see Lemma 11, below), so that S_W is not a maximal semigroup. However, by proceeding like in the proof that $S[W]$ is connected we get that S_W is a *maximal connected* semigroup in the sense that if $S_W \subset T$ with T a connected subsemigroup of $\mathrm{Sl}(n, \mathbb{R})$ then either $T = S_W$ or $T = \mathrm{Sl}(n, \mathbb{R})$.

Corollary 2 S_W *is maximal connected in* $\mathrm{Sl}(n, \mathbb{R})$.

There is a converse to this corollary, ensuring that a semigroup in a certain class of maximal connected subsemigroups of $\mathrm{Sl}(n, \mathbb{R})$ must be S_W for some pointed and generating cone W. This is the class of semigroups whose type is the projective space $\mathbb{P}^{n-1}$. We refer the reader to [8] and [9] for the definition of the type of a semigroup and in particular of the type of S_W (see also [7] for a discussion specific to semigroups in $\mathrm{Sl}(n, \mathbb{R})$). It was observed in [8], Example 4.10, that if a semigroup is connected and of type $\mathbb{P}^{n-1}$ then it is contained in S_W for some pointed generating cone $W \subset \mathbb{R}^n$. Therefore we get from the fact that S_W is connected the following characterization of the maximal connected semigroups of the projective space type:

Corollary 3 *Let* $\mathcal{C}$ *stand for the class of semigroups* $S \subset \mathrm{Sl}(n, \mathbb{R})$, *with* $\mathrm{int} S \neq \emptyset$, *which are maximal connected of type* $\mathbb{P}^{n-1}$. *Then*

$$\mathcal{C} = \left\{ S_W : W \subset \mathbb{R}^n \text{ is a pointed generating cone} \right\}.$$

Finally we mention that the semigroup of all matrices leaving invariant a cone W – without any determinantal restriction – is trivially a convex cone in the space of matrices, and hence connected. Our results, however, refer to the semigroups $S(W)$ and S_W which are far from being convex cones.

2 S_W is connected

In this section we prove the main result of this paper, namely Theorem 1. From now on we let $W \subset \mathbb{R}^n$ stand for a pointed generating convex cone. As before denote by S_W the semigroup of matrices in $\mathrm{Sl}(n, \mathbb{R})$ leaving W invariant.

We refer the reader to Hilgert, Hofmann and Lawson [2] for the general theory of semigroups. In particular, the concept of Lie wedge $\mathcal{L}(S)$ of a semigroup $S \subset \mathrm{Sl}(n, \mathbb{R})$ is defined by

$$\mathcal{L}(S) = \left\{ X \in \mathfrak{sl}(n, \mathbb{R}) : \exp(tX) \in \mathrm{cl} S \text{ for all } t \geq 0 \right\}$$

where $\mathfrak{sl}(n, \mathbb{R})$ is the Lie algebra of trace zero $n \times n$-matrices. In what follows we denote by S_{inf} the semigroup generated by $\mathcal{L}(S_W)$, namely

$$S_{\mathrm{inf}} = \langle \exp(\mathcal{L}(S_W))) \rangle.$$

Since S_W is closed, it follows that S_{inf} is a subsemigroup of S_W. Furthermore, being generated by one-parameter semigroups S_{inf} contains the identity and is path connected. It is a consequence of the next statement that $\mathcal{L}(S)$ is a generating cone in $\mathfrak{sl}(n, \mathbb{R})$, implying that S_{inf} has nonempty interior in $\mathrm{Sl}(n, \mathbb{R})$ and that the interior of S_W is dense in S_W, i.e., $S_W = \mathrm{cl}(\mathrm{int} S_W)$.

Proposition 4 *Suppose that $V \subset \mathbb{R}^n$ is a codimension one subspace with $V \cap W = \{0\}$. Take a basis*
$$\beta = \{f_1, \ldots, f_n\}$$
of $\mathbb{R}^n$ such that $f_1 \in W$ and $\{f_2, \ldots, f_n\} \subset V$. Let $H \in \mathfrak{sl}(n, \mathbb{R})$ be such that its matrix with respect to β is

$$H = \mathrm{diag}\{n - 1, -1, \ldots, -1\}.$$

Then $H \in \mathcal{L}(S_W)$. Moreover, if $f_1 \in \mathrm{int}\, W$ then $H \in \mathrm{int} \mathcal{L}(S_W)$ so that $\exp(tH) \subset \mathrm{int} S_W$ for all $t > 0$.

Proof: Take $x \in W$, $x \neq 0$. Since $V \cap W = \{0\}$, it follows that $\mathbf{B} = (f_1 + V) \cap W$ is a cone basis of W in the affine subspace $f_1 + V$. Hence up to multiplication by a positive scalar we have
$$x = f_1 + a_2 f_2 + \cdots + a_n f_n.$$
Therefore,

$$
\begin{aligned}
Hx &= (n - 1) f_1 - (a_2 f_2 + \cdots + a_n f_n) \\
&= n f_1 - x,
\end{aligned}
$$

that is, $x + Hx = nf_1 \in W$. By the invariance theorem for cones (see [2], Theorem I.5.27), it follows that $\exp tH \in S_W$ for all $t \geq 0$, which means that $H \in \mathcal{L}(S_W)$.

Now, assume that $f_1 \in \mathrm{int} W$. Note that the cone basis $\mathbf{B}$ is compact since W is a pointed cone. Also, the map

$$(A, x) \in \mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}^n \longmapsto x + Ax \in \mathbb{R}^n$$

is continuous. Hence, given $x \in \mathbf{B}$ and a neighborhood U of nf_1 in W, there are neighborhoods O_x of H and C_x of x such that for $A \in O_x$ and $y \in C_x$, it holds $y + Ay \in U \subset W$. By compactness of $\mathbf{B}$ there exists a neighborhood O of H such that $x + Ax \in U$ for all $x \in \mathbf{B}$ and $A \in O$. It follows that the open set O is contained in $\mathcal{L}(S_W)$, implying that $H \in \text{int}(\mathcal{L}(S_W))$. Clearly, this implies that $tH \in \text{int}(\mathcal{L}(S_W))$ for all $t > 0$. Hence, using the fact that the exponential mapping is a diffeomorphism around the identity we conclude that $\exp(tH) \in \text{int}S_W$ for small values of $t > 0$. Therefore the formula $\exp(tH) = \exp((t/n)H)^n$ implies that $\exp(tH) \in \text{int}S_W$ for all $t > 0$. $\qquad\square$

Taking H as in this proposition with $f_1 \in \text{int}W$ we have that $\exp(tH) \in \text{int}S_W$ if $t > 0$, hence for all $g \in S_W$, $\exp(tH)g$ and $g\exp(tH)$ belong to $\text{int}S_W$ if $t > 0$. Therefore, any $g \in S_W$ can be linked to $\text{int}S_W$ by a continuous path inside S_W. Since this fact is used in the proof that S_W is connected we emphasize it.

Corollary 5 *Let H be as in the previous proposition with $f_1 \in \text{int}W$. Take $g \in S_W$. Then $\exp(tH)g$ and $g\exp(tH)$ belong to $\text{int}S_W$ if $t > 0$.*

Before proceeding we note the following simple, but useful, fact about matrices in $\text{int}S_W$:

Lemma 6 *If $g \in \text{int}S_W$ then $gW \subset \text{int}W \cup \{0\}$.*

Proof: If $x \neq 0$, the assignment $h \in \text{Sl}(n, \mathbb{R}) \mapsto hx \in \mathbb{R}^n$ is an open mapping because $\text{Sl}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$. Hence $(\text{int}S_W)x = \{hx : h \in \text{int}S_W\}$ is open if $x \neq 0$. Since $(\text{int}S_W)x \subset W$, it follows that $gx \in \text{int}W$ for all $x \in W$, $x \neq 0$. $\qquad\square$

The following statement is central in the proof that S_W is connected, it concerns the Jordan decomposition of the matrices in $\text{int}S_W$.

Lemma 7 *Let $g \in \text{int}S_W$ be given. Then there exists a basis $\beta = \{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ with $f_1 \in \text{int}W$ and*
$$\text{span}\{f_2, \ldots, f_n\} \cap W = 0,$$
such that the matrix of g with respect to β is written in blocks as
$$g = \begin{pmatrix} \lambda & 0 \\ 0 & h \end{pmatrix}$$
where $\lambda > 0$ and h is an $(n-1) \times (n-1)$-matrix with $\det h > 0$. Furthermore λ is a principal eigenvalue, i.e., $|\mu| < \lambda$ if μ is an eigenvalue of h.

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This lemma is well known in the theory of matrices (see Berman and Plemmons [1]). Below we offer another proof of it, having a Lie theoretic flavor..

2.1 Proof of Theorem 1

In view of Corollary 5, in order to prove that S_W is path connected it is enough to show that $\mathrm{int} S_W$ is path connected. We prove this by exhibiting, for any $g \in \mathrm{int} S_W$, a path in S_W joining it to S_{inf}. Since S_{inf} is path connected, this implies that $\mathrm{int} S_W$ is path connected as well.

Fix $g \in \mathrm{int} S_W$, and let $\beta = \{f_1, \ldots, f_n\}$ be a basis given by Lemma 7, providing a block decomposition of g.

Let $P \subset \mathrm{Sl}(n, \mathbb{R})$ be the subgroup of those linear maps whose matrices with respect to β have the same block structure as g:

$$P = \{ \begin{pmatrix} \mu & 0 \\ 0 & Q \end{pmatrix} : \mu > 0, \, Q \in \mathrm{Gl}^+(n-1, \mathbb{R}), \, \mu \det Q = 1 \}.$$

Clearly, P is a closed and connected subgroup of $\mathrm{Sl}(n, \mathbb{R})$. By construction, $g \in (\mathrm{int} S_W) \cap P$. Let $H \in \mathfrak{sl}(n, \mathbb{R})$ be such that its matrix with respect to . β is

$$H = \mathrm{diag}\{n-1, -1, \ldots, -1\}. \tag{1}$$

The properties of β and Proposition 4 imply that $H \in \mathrm{int} \mathcal{L}(S_W)$ and $\exp(tH) \in (\mathrm{int} S_{\mathrm{inf}}) \cap P$ for all $t > 0$. Put

$$\Gamma = (\mathrm{int} S_{\mathrm{inf}}) \cap P.$$

Then Γ is a semigroup with nonempty interior in P (with respect to the topology of P).

Define the map $\phi : P \to \mathrm{Sl}(n-1, \mathbb{R})$ by

$$\phi \begin{pmatrix} \mu & 0 \\ 0 & Q \end{pmatrix} = (\det Q)^{-1/n-1} Q = \mu^{1/n-1} Q.$$

It is checked immediately that ϕ is a surjective homomorphism. Hence it is an open mapping, so that $\phi(\Gamma)$ is a semigroup with nonempty interior in $\mathrm{Sl}(n-1, \mathbb{R})$.

Now, $\exp(tH) \in \Gamma$, for all $t > 0$. Since

$$\exp(tH) = \mathrm{diag}\{e^{t(n-1)}, e^{-t}, \ldots, e^{-t}\},$$

it follows that $\phi(\exp(tH)) = 1$. Therefore, $1 \in \phi(\Gamma)$ implying that $\phi(\Gamma) = \mathrm{Sl}(n-1, \mathbb{R})$ because $\mathrm{Sl}(n-1, \mathbb{R})$ is connected. Combining this fact together with the definition of ϕ we get the

Lemma 8 *For all $h' \in \mathrm{Sl}\,(n-1, \mathbb{R})$ there exists $a > 0$ such that*

$$g' = \begin{pmatrix} a & 0 \\ 0 & a^{-1/n-1}h' \end{pmatrix} \in \Gamma = (\mathrm{int}\,S_{\mathrm{inf}}) \cap P \subset \mathrm{int}\,S_{\mathrm{inf}} \tag{2}$$

Let us show now that there is a path linking the given $g \in \mathrm{int}\,S_W$ to S_{inf}. We can write

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & (\det h)^{1/n-1}\,h' \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1/n-1}h' \end{pmatrix}$$

where $h' = (\det h)^{-1/n-1}\,h \in \mathrm{Sl}\,(n-1, \mathbb{R})$. For this h', the above lemma ensures the existence of $a > 0$ such that the corresponding g' as in (2) belongs to Γ. There are the following possibilities:

1. $\lambda \leq a$. Then $e^{(n-1)T}\lambda = a$ for some $T \geq 0$. Hence if H is given by (1) then

$$\exp\,(TH)\,g = \begin{pmatrix} e^{(n-1)T}\lambda & 0 \\ 0 & \left(e^{(n-1)T}\lambda\right)^{-1/n-1}h' \end{pmatrix}.$$

   Substituting in this equality $e^{(n-1)T}\lambda = a$ we get from (2) that

$$\exp\,(TH)\,g = g' \in S_{\mathrm{inf}}.$$

   Since $\exp\,(tH)\,g \in \mathrm{int}\,S_W$ for all $t \geq 0$, the path $t \mapsto \exp\,(tH)\,g$, $t \in [0, T]$, joins g to $g' \in S_{\mathrm{inf}}$, without leaving $\mathrm{int}\,S_W$.

2. $\lambda > a$. In this case we reverse the roles of g and g' to get $T > 0$ such that $\exp\,(TH)\,g' = g$, providing the path $t \mapsto \exp\,(tH)\,g'$, $t \in [0, T]$, linking g' to g inside $\mathrm{int}\,S_W$.

Therefore for arbitrary $g \in \mathrm{int}\,S_W$ there exists a path inside $\mathrm{int}\,S_W$ joining g to S_{inf} concluding the proof of Theorem 1.

2.2 Proof of Lemma 7

We start with the following lemma which holds for an arbitrary semigroup S contained in $\mathrm{Sl}\,(n, \mathbb{R})$ and having nonempty interior.

Lemma 9 *Given $h \in \text{int} S$ let $V \subset \mathbb{R}^n$ be an h-invariant subspace with $\dim V \geq 2$ and such that $|\mu|$ is constant as μ runs through the eigenvalues of the restriction $\bar{h}$ of h to V. Then S is transitive on the rays of V. Precisely, let P_V be the subgroup*

$$P_V = \{h \in \text{Sl}(n, \mathbb{R}) : hV = V\}.$$

Then $\Gamma = S \cap P_V$ is a semigroup with nonempty interior in P_V and for two rays r_1 and r_2 in V, starting at the origin, there exists $h' \in \Gamma$ such that $h'r_1 = r_2$.

Proof: The first step in the proof consists in projecting Γ to the group $\text{Sl}(V)$, of unimodular linear maps of V. This need to be done only if V is a proper subspace. In this case the restriction of P_V to V is the whole linear group $\text{Gl}(V)$, which has two connected components, say $\text{Gl}^{\pm}(V)$, with $1 \in \text{Gl}^+(V)$. Clearly $h \in \Gamma$ so that Γ is a semigroup with nonempty interior in P_V. Denote also by Γ its restriction to V. It follows that $\Gamma^+ = \Gamma \cap \text{Gl}^+(V)$ also has nonempty interior, because $Q^2 \in \text{Gl}^+(V)$ if $Q \in \text{Gl}(V)$.

Consider the onto homomorphism $\psi : \text{Gl}^+(V) \to \text{Sl}(V)$ given by

$$Q \in \text{Gl}^+(V) \longmapsto (\det Q)^{1/k} Q, \qquad k = \dim V.$$

The image $\Gamma_1 = \psi(\Gamma^+)$ is a semigroup with nonempty interior in $\text{Sl}(V)$.

Now, the restriction $\bar{h}$ of h to V belongs to $\text{int}\Gamma$. By assumption the eigenvalues of $\bar{h}$ are of the form

$$e^a(\cos\theta_1 + i\sin\theta_1), \ldots, e^a(\cos\theta_s + i\sin\theta_s),$$

with fixed a. So that $\bar{h}$ decomposes in Jordan blocks of the types

$$e^a \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad e^a \begin{pmatrix} \cos\theta_j & -\sin\theta_j & & & * \\ \sin\theta_j & \cos\theta_j & & & \\ & & \ddots & & \\ & & & \cos\theta_j & -\sin\theta_j \\ 0 & & & \sin\theta_j & \cos\theta_j \end{pmatrix}.$$

In case $\theta_j = 2\pi q_j$, $j = 1, \ldots, s$, with q_j rational, a quick glance at these blocks show that some power of $\bar{h}$ has real eigenvalues so that there exists $h_1 \in \text{int}\Gamma$ whose restriction $\bar{h}_1$ to V has the form

$$\bar{h}_1 = \lambda \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \tag{3}$$

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with $\lambda > 0$. The existence of such $\bar{h}_1$, coming from the semigroup, can be ensured without the restrictive assumption that the eigenvalues of $\bar{h}$ are rational multiples of π. In fact, since $h \in \mathrm{int}\Gamma$, there exists $h_2 \in \mathrm{int}\Gamma$ having the same block structure as h and such that the arguments of the eigenvalues of the restriction of h_2 to V are rational multiples of 2π. Thus we can argue with h_2 in place of h to get the desired element $\bar{h}_1$ like in (3).

Now,

$$\psi\left(\bar{h}_1\right) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

This implies that $\Gamma_1 = \mathrm{Sl}\,(V)$. In fact, $\psi\left(\bar{h}_1\right) \in \mathrm{int}\Gamma_1$ and $\psi\left(\bar{h}_1\right)$ can be approximated by a matrix of the form $\exp\left(X\right)$ with X having purely imaginary eigenvalues. This permits to show that $1 \in \mathrm{int}\Gamma_1$ concluding that $\Gamma_1 = \mathrm{Sl}\,(V)$ (see [6], Lemma 4.1, for details).

From $\Gamma_1 = \mathrm{Sl}\,(V)$ and $\dim V \geq 2$ it follows at once that Γ_1 is transitive on the rays of V. The lemma is then a direct consequence of the definition of ψ. $\qquad\square$

An application of the above lemma to g yields the

Corollary 10 *Fix $g \in \mathrm{int}S_W$. Let $V \subset \mathbb{R}^n$ be a g-invariant subspace such that $|\mu|$ is constant as μ runs through the eigenvalues of the restriction of g to V. Then $\dim V = 1$ if $V \cap W \neq 0$.*

Proof: If $V \cap W \neq 0$ there exists a ray of V contained in W. On the other hand the lemma implies that S_W is transitive on the rays of V if $\dim V \geq 2$. Hence $V \subset W$ if $\dim V \geq 2$ contradicting the assumption that W is a pointed cone. $\qquad\square$

In order to continue we put

$$\rho = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } g\}$$

for a fixed $g \in \mathrm{int}S_W$. Let V^+ be the direct sum of the generalized eigenspaces $V_\lambda = \ker\left(g - \lambda\right)^n$, with $|\lambda| = \rho$. Also, let V^- be the sum of the remaining generalized eigenspaces of g. We claim that $V^+ \cap W \neq \{0\}$. To see this write for $u \in \mathbb{R}^n$, $u = u^+ + u^-$ with $u^\pm \in V^\pm$. Then as $k \to +\infty$, $(1/\rho)^k g^k u^-$ converges to zero. Furthermore, the fact that the eigenvalues of g in V^+ have constant modulus ρ, implies that there exists a subsequence k_l such that $\lim(1/\rho)^{k_l} g^{k_l} u^+ = v$, as $l \to$

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$+\infty$. This limit is not zero if $u^+ \neq 0$. Thus when $u^+ \neq 0$, $(1/\rho)^{k_l} g^{k_l} u$ converges to $v \in V^+$. In particular take $u \in W$ such that $u^+ \neq 0$. The existence of such u follows from the assumption that W is generating. Then $0 \neq v \in V^+ \cap W$ because $(1/\rho)^{k_l} g^{k_l} u \in W$ and W is closed, showing the claim.

By Corollary 10 we conclude that $\dim V^+ = 1$. Hence there exists just one eigenvalue, say $\lambda_{\max}$, with $|\lambda_{\max}| = \rho$, which is by force real. Furthermore the eigenspace V^+ is contained in $W \cup (-W)$ and since $gW \subset W$, it follows that $\lambda_{\max} > 0$.

Take an eigenvector $f_1 \in V^+ \cap W$. Then $\lambda_{\max} f_1 = gf_1 \in \text{int}W$ by Lemma 6. Hence $f_1 \in \text{int}W$. Therefore, the proof of Lemma 7 follows as soon as we show that $V^- \cap W = \{0\}$.

To check that $V^- \cap W = \{0\}$ we note first that $V^- \cap \text{int}W = \{0\}$, since otherwise W would meet both half-spaces determined by the codimension one subspace V^-. But this would contradict the fact that W is a pointed cone. In fact, for v_1 and v_2 in different sides of V^- the ray defined by $g^k v_1$ approaches, say, the ray spanned by f_1, as $k \to +\infty$, whereas the ray $g^k v_2$ approaches the ray spanned by $-f_1$. Since $g^k v$, $v \in W$, does not leave W, we would have $\pm f_1 \in W$. Finally,

$$g\left(V^- \cap W\right) = gV^- \cap gW \subset V^- \cap (\text{int}W \cup \{0\})$$

because $gW \subset \text{int}W \cup \{0\}$ by Lemma 6. Hence $g\left(V^- \cap W\right) = \{0\}$ so that $V^- \cap W = \{0\}$ concluding the proof of Lemma 7.

3 Complements

This section is devoted to the proof of some facts related to the main result. We start with the

Proof of Corollary 2: Let T be a connected semigroup with nonempty interior containing S_W properly. Note first that T is not contained in $S[W]$. To see this suppose to the contrary that $T \subset S[W]$. Then $Tx \subset W \cup (-W)$ for all $x \in W$. However, T is connected so that if $0 \neq x \in W$ then Tx is contained in a connected component of $(W \cup (-W)) \setminus \{0\}$, which is by force W because Tx is connected and contains x, as $1 \in T$. Therefore, $T \subset S_W$ contradicting the assumption on T. Now, the proof that $T = \text{Sl}\,(n, \mathbb{R})$ follows the same steps as the proof that $S[W]$ is maximal (see [10], Theorem 6.12). We sketch it: By Proposition 4, any line outside $[W]$ is spanned by an eigenvector of some $h \in \text{int}S_W$. This implies that $[W]$ and

$\mathbb{P}^{n-1} \setminus [W]$ are the two control sets of S_W in $\mathbb{P}^{n-1}$. Therefore S_W is transitive in int$[W]$ as well as in $\mathbb{P}^{n-1} \setminus [W]$. Since T is not contained in $S[W]$, there exists $g \in T$ such that $gx \in \mathbb{P}^{n-1} \setminus [W]$ for some $x \in \text{int}[W]$. Also for any $y \in \mathbb{P}^{n-1}$ there exists $g_1 \in S_W$ with $g_1 y \in \text{int}[W]$ (because $[W]$ is the invariant control set of S_W in $\mathbb{P}^{n-1}$). It follows that T acts transitively in $\mathbb{P}^{n-1}$. Thus $T = \text{Sl}(n, \mathbb{R})$, by [10], Theorem 6.2. $\qquad\square$

Now, we discuss the relation between S_W and $S[W]$. By definition $S[W] = S_W \cup S_W^{\|}$, where

$$S_W^{\|} = \{g \in \text{Sl}(n, \mathbb{R}) : gW \subset -W\}.$$

The following lemma shows that $S_W^{\|}$ is not empty.

Lemma 11 S_W *is properly contained in* $S[W]$.

Proof: We must show that there exists $g \in \text{Sl}(n, \mathbb{R})$ such that $gW \subset -W$. For this purpose take $H = \text{diag}\{n-1, -1, \ldots, -1\}$ with respect to a basis $\beta = \{f_1, \ldots, f_n\}$ satisfying the requirements of Proposition 4, namely $f_1 \in \text{int}W$ and span$\{f_2, \ldots, f_n\} \cap W = \{0\}$. Since $\text{Sl}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n$, there exists $g_1 \in \text{Sl}(n, \mathbb{R})$ such that $g_1 f_1 = -f_1$. By continuity $U = g_1^{-1}(\text{int}(-W))$ is a neighborhood of f_1. Now, by construction of H there exists a large enough $t > 0$ such that if $h = \exp tH$ then $hW \subset U$. Hence $g_1 hW \subset -W$ so that $g = g_1 h$ belongs to $S[W]$ but not to S_W. $\qquad\square$

Clearly, there are the inclusions $S_W S_W^{\|} \subset S_W^{\|}$ and $\left(S_W^{\|}\right)^2 \subset S_W$. The former shows in particular that $S_W^{\|}$ has nonempty interior. In case n is even, $-1 \in \text{Sl}(n, \mathbb{R})$, hence $-1 \in S_W^{\|}$ for any W. Actually, -1 maps W exactly onto $-W$ hence the following statement implies that in even dimensions, $S_W^{\|} = -S_W$.

Proposition 12 *Suppose that there exists* $k \in \text{Sl}(n, \mathbb{R})$ *satisfying* $kW = -W$. *Then* $S_W^{\|} = kS_W = S_W k$.

Proof: Clearly, kS_W and $S_W k$ are contained in $S_W^{\|}$. For the reverse inclusions note that $k^{-1}W = -W$. Pick $g \in S_W^{\|}$. Then $gW \subset -W$, so that $gk^{-1}W \subset W$ and $k^{-1}gW \subset W$, that is, gk^{-1} and $k^{-1}g$ are in S_W. $\qquad\square$

Under the assumption of this proposition it follows at once that $S_W^{\|}$ is connected. Since the existence of k mapping W onto $-W$ depends on the geometry of the specific W, we prove next that in general

Proposition 13 $S_W^{\|}$ *is connected. Hence S_W and $S_W^{\|}$ are the connected components of $S[W]$.*

Proof: Take $g, h \in S_W^{\|}$. Both gW and hW are pointed generating cones contained in $-W$. Take H and β like in Proposition 4 with the first element f_1 of β contained in $\text{int}\,(hW)$. Like in that proposition $H \in \mathcal{L}\,(S_W)$ and for large enough t_0, $\exp\,(t_0 H)\,(-W) \subset hW$. In particular,

$$\exp\,(t_0 H)\,(gW) \subset hW.$$

Hence $h^{-1}\exp\,(t_0 H)\,g \in S_W$, that is, $\exp\,(t_0 H)\,g \in hS_W$. Since S_W is path connected, this implies the existence of a path in $S[W]$ linking $\exp\,(t_0 H)\,g$ to h. However, $H \in \mathcal{L}\,(S_W)$, so that $\exp\,(t_0 H)\,g$ and g are in the same path component of $S_W^{\|}$, concluding the proof of that $S_W^{\|}$ is connected. $\qquad\square$

In general $S_{\text{inf}} = \langle \exp\,(\mathcal{L}\,(S_W))\rangle$ is a proper subsemigroup of S_W. As observed by K.-H. Neeb (personal communication) the inclusion $S_{\text{inf}} \subset S_W$ is proper for the semigroup $\text{Sl}^+\,(n, \mathbb{R}) = S_W$ of positive matrices, where W is the orthant

$$W = \mathcal{O}^+\,(n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}.$$

To see this note that in case $n \geq 3$ the unit group of S_W, $H\,(S_W) = S_W \cap S_W^{-1}$ is not connected. In fact, it easy to check that g must permutes the basic vectors if $g \in H\,(S_W)$ so that $H\,(S_W) = \Pi \times A$ where A is the group of diagonal matrices with positive entries and Π is the group of permutation matrices with $\det = 1$. In case $n \geq 3$, Π – and hence $H\,(S_W)$ – is not connected. On the other hand, it is a general fact that the unit group of an infinitesimally generated semigroup like S_{inf} must be connected (see [2], Theorem V.2.8).

Finally, we observe that Corollary 3 completely determines the maximal connected semigroups of $\text{Sl}\,(n, \mathbb{R})$ for $n = 2, 3$. In fact, for $n = 2$, any semigroup is of the projective type so that any maximal connected semigroup is S_W for some pointed and generating cone $W \subset \mathbb{R}^2$. It should be remarked here that for any such cone W there exists $g \in \text{Sl}\,(2, \mathbb{R})$ such that $W = g\mathcal{O}^+\,(2)$. Since $S_{gW} = gS_W g^{-1}$, it follows that up to conjugation $\text{Sl}^+\,(2, \mathbb{R})$ is the only maximal connected semigroup of $\text{Sl}\,(2, \mathbb{R})$. For $n = 3$, there are two types of maximal semigroups, namely a semigroup is of type $\mathbb{P}^2$ or $\text{Gr}_2\,(3)$, the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$. However, if a semigroup is of type $\text{Gr}_2\,(3)$ then its inverse S^{-1} is of projective type (see [9], Proposition 6.3). Therefore there is the following characterization of the maximal connected semigroups in $\text{Sl}\,(3, \mathbb{R})$:

Proposition 14 *A semigroup $S \subset \mathrm{Sl}(3, \mathbb{R})$, with $\mathrm{int}S \neq \emptyset$, is maximal connected if and only if there exists a pointed and generating cone $W \subset \mathbb{R}^3$ such that either $S = S_W$ or $S = S_W^{-1}$.*

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