Stationary Micropolar Fluid Flows with Boundary Data in L^2

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Abstract

We consider the Dirichlet boundary value problem for the equations of a stationary micropolar fluid in a bounded three dimensional domain. We show the existence and uniqueness of a distributional solution with boundary values in L^2.

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1. Introduction

The micropolar fluid model is an essential generalization of the well-established Navier–Stokes model in the sense that it takes into account the microstructure of the fluid. It may represent fluids consisting of randomly oriented (or spherical) particles suspended in a viscous medium, when the deformation of fluid particles is ignored. Micropolar fluids were introduced in [5]. They are non–Newtonian fluids with nonsymmetric stress tensor.

The governing system of equations of micropolar fluids expresses the balance of momentum, mass, and moment of momentum [5], [16], which is the following:

(1.1) $\mathbf{v}_t - (\nu + \nu_r)\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 2\nu_r \ \mathrm{rot}\mathbf{w} + \mathbf{f}$

(1.2) $\mathrm{div}\, \mathbf{v} = 0$

(1.3) $\mathbf{w}_t - (c_a + c_d)\Delta \mathbf{w} - (c_0 + c_d - c_a)\nabla \mathrm{div}\mathbf{w} + (\mathbf{v}\cdot\nabla)\ \mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \ \mathrm{rot}\mathbf{v} + \mathbf{g}$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity field, p is the pressure, and $\mathbf{w} = (w_1, w_2, w_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles. The fields $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ are external forces and moments, respectively. Positive constants $\nu, \nu_r, c_0, c_a, c_d$ represent viscosity coefficients, ν is the usual Newtonian viscosity and ν_r is called the microrotation viscosity. It is assumed that the density of the fluid is equal to one.

Observe that if the microrotation viscosity ν_r equals zero then system (1.1), (1.2) reduces to the Navier–Stokes system and the velocity field is independent of the microrotation field. Thus, the size of the microrotation viscosity allows us to measure, in a certain sense, the deviation of flows of micropolar fluids from that of the Navier–Stokes model.

Several experiments show that solutions of the micropolar fluid model better describe behavior of numerous real fluids (eg., blood, cf., e.g., [23], [24], [19]) then corresponding solutions of the Navier–Stokes model, especially, when the characteristic dimensions of the flow (eg. the diameter of the channel) become small. It well agrees with our expectations that the influence of the internal structure of the fluid is the greater, the smaller the characteristic dimensions of the flow.

For flows in narrow films the microstructure plays an important role as it usually increases the load capacity and stabilizes the flow; cf., e.g., [6], [26]. In general, as part of the momentum is lost in rotating of the particles, the flow of a micropolar fluid is less prone to instability then that of a classical fluid. Stability problems for micropolar fluids were studied, e.g., in [1], [11], [12] and [25], and control problems in [21].

In this paper we are interested in the system (1.1)-(1.3) in a stationary regime, i.e. $\mathbf{v}_t = \mathbf{w}_t = 0$, in a bounded domain Ω with irregular boundary data on $\partial\Omega$, i.e. with boundary data that belong to $L^2(\partial\Omega)$. The system (1.1)-(1.3) in a stationary regime was studied by Łukaszewicz [17] in a bounded domain Ω with null Dirichlet's boundary conditions (see also [16]), and in [4] in the case of exterior domain. The case where the boundary data are not null but sufficiently regular, such that they can be extended to the interior of the domain Ω accordingly with trace theorems, can be treated in a similar way as in [16]. (The case of stationary Navier-Stokes system with data in $H^{1/2}(\partial\Omega)$ goes back to the classical method of Leray, see e.g. [29], and with data in in $W^{1-1/q,q}(\Gamma)$, $3/2 < q < 2$, was solved in [28].) However, if they are not regular, for instance, if the boundary data are not the traces at the boundary of Ω of some functions in Sobolev spaces on Ω, then the problem is quite more difficult. This problem for the Stokes equations was treated by Conca [2], where the concept of *very weak solution* was introduced (see the Appendix A in [2] or [3]). Then, more recently, Marusič-Paloka proved the existence of a *very weak solution* for the stationary Navier-Stokes equations.

There are some physical motivations for considering fluid equations with irregular boundary data, e.g. in [2] it is considered the Stokes equations modeling a fluid in a domain containing a sieve and then it is shown that when the sieve becomes finer and finer the solution of the problem converges to a solution of a Stokes problem with boundary data only in L^2. Other examples, for the stationary Navier-Stokes equations with boundary data in some Sobolev space $W^{1-1/q,q}$, are pointed out in [28], namely, the problem of a stationary fluid in "domain with a cavity", i.e. the union of a semi-space with a bounded domain (the "cavity"), and the *Taylor* *problem*, i.e. the problem of equilibrium of a fluid between two co-centered cylinders with the external cylinder fixed and the internal one in a rotational motion about its axis.

The main idea used by Conca in [2] is the transposition method (see e.g. [15]), which is very useful for linear equations. Marusič-Paloka [18] was able to extend Conca's result, first for small data by using a linearization of the Navier-Stokes equations and an iterative argument (in fact, the Banach's fixed point theorem) based on penalisation method and an estimate on the Oseen's problem solution, and then for no small data assumption by splitting the data into a small irregular part and a large regular part.

We combine ideas from Conca [2], Marusič-Paloka [18], and Łukaszewicz [17], to obtain the existence of a *very weak solution* for the stationary micropolar fluid equations. That is, first we use the transposition method for obtaining a solution to the microrotational field equation, which depends on the velocity field $\mathbf{v}$ that lives in $L^3(\Omega)$. This microrotational field solution obeys a good estimate with respect to $\mathbf{v}$, as we prove below, provided $\mathbf{v}$ is split into a small irregular part in $L^3(\Omega)$ and a regular part $\mathbf{u}^\varepsilon$ in $H^1(\Omega)$ (see Lemma 3.1). To attain that, we needed to prove a regularity result for a second order linear strongly elliptic system with an irregular coeefficient (see the proof of Lemma 3.1). Then taking the small part of $\mathbf{v}$ as a solution for the Navier-Stokes equations, via Marusič-Paloka's theorem, we prove the existence of $\mathbf{u}^\varepsilon$ using an appropriate Leray-Hopf extension of a smooth approximation of the boundary value for $\mathbf{v}$, and the Leray-Schauder fixed point theorem, following [17].

Besides the existence of solutions, we obtain a result of continuous dependence on the boundary data for $\mathbf{w}$ and given external forces, which implies, in particular, uniqueness of solution.

The plan of the paper is as follows. In Section 2. we introduce our main problem, give the notations used throughout the paper and the definition of a *very weak solution*. Then we state our main theorem and make some brief comments on the relation between *very weak* and *weak solutions*. We end the Section showing that the *very weak solution* has a "trace"on the boundary of the domain that coincides with the boundary data and with the usual trace when the boundary are regular and the *very weak solution* is a *weak solution.* Section 3. deals with the system for the microrotational field $\mathbf{w}$ assuming that $\mathbf{v}$ is split into an appropriate sum, as explained above. In Section 4. we show a way of reducing the system for $\mathbf{v}$ to a new system for an unknown $\mathbf{u}$ in the space $\mathcal{V}$ of divergent free functions in $H_0^1(\Omega)$. That is, $\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, where $\mathbf{v}^\varepsilon$ is the small part of $\mathbf{v}$ in $L^3(\Omega)$. This small part $\mathbf{v}^\varepsilon$ is a *very weak solution* of the stationary Navier-Stokes system, which exists due to the Marusič-Paloka's theorem [18], with null external force and with a boundary data very small in the norm of $L^2(\partial\Omega)$, depending on a smooth approximation $\mathbf{v}_0^\varepsilon$ of $\mathbf{v}_0$. The part $\mathbf{u}^\varepsilon$ is the "large"regular part of $\mathbf{v}$ in $H^1(\Omega)$. It is equal to $\mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, where $\widetilde{\mathbf{v}_0^\varepsilon}$ is an appropriate Leray-Hopf extension of $\mathbf{v}_0^\varepsilon$ to Ω which is in $\mathcal{V}$, and $\mathbf{u}$ is the new unknown which satisfies its own system shown in Section 4.. This system for $\mathbf{u}$ is a nonlinear one, where the nonlinearities come from the term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and from $\mathbf{w}$ that depends on $\mathbf{v}$. In Section 5. we prove the existence of a solution $\mathbf{u}$ in $\mathcal{V}$ for this system using the Leray-Schauder fixed point theorem, with the help of a good choice of $\mathbf{v}_0^\varepsilon$ and $\widetilde{\mathbf{v}_0^\varepsilon}$. Finally, in Section 6 we prove the continuous dependence of the *very weak solution* on the data $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{w}_0$.

2. The equations of stationary micropolar fluids, notations, and definition of a very weak solution

We begin this Section wtih the presentation of the problem we study is this paper. Then we give the notation used throughout the paper, the definition of a *very weak solution*, and state our main theorem. We end it with a brief explanation on the definition of a *very weak solution*, its relation with a *weak solution* and show that a *very weak solution* attains the boundary data in a "trace"sense.

Let Ω be an open, bounded and connected set in $\mathbf{R}^3$ with a boundary of class C^2, which we denote by Γ. We are interested in solving the following boundary value problem.

Problem 2.1 Assume that $\mathbf{v}_0 \in L^2(\Gamma)$ with $\int_\Gamma \mathbf{v}_0 \cdot \mathbf{n} ds = 0$, where $\mathbf{n}$ is the unit outward normal on Γ, $\mathbf{w}_0 \in L^2(\Gamma)$, $\mathbf{f}, \mathbf{g} \in L^2(\Omega)$, and prove existence of functions $\mathbf{v} \in L^3(\Omega)$, $\mathbf{w} \in L^2(\Omega)$ such that, together with some distribution p, satisfy in a *very weak* sense

$$-\mu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = a\mathrm{rot}\mathbf{w} + \mathbf{f} \tag{2.1}$$

$$\mathrm{div}\,\mathbf{v} = 0 \tag{2.2}$$

$$-\alpha\Delta\mathbf{w} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \beta\nabla\mathrm{div}\mathbf{w} + \gamma\mathbf{w} = a\mathrm{rot}\mathbf{v} + \mathbf{g}, \tag{2.3}$$

in Ω, with boundary data

$$\mathbf{v}|_\Gamma = \mathbf{v}_0 \tag{2.4}$$

$$\mathbf{w}|_\Gamma = \mathbf{w}_0. \tag{2.5}$$

In (2.1)-(2.3) and hereafter, for short we write $\mu = \nu + \nu_r$, $a = 2\nu_r$, $\alpha = c_a + c_d$, $\beta = c_o + c_d - c_a$, and $\gamma = 4\nu_r$ (cf. Section 1.).

Notations: Throughout this paper we fix the following notations:

x : A generic point in Ω ($x = (x_1, x_2, x_3)$)

Γ : The boundary of Ω

$\mathbf{n}$: The unit outward normal on Γ

$W^{k,p}$: The Sobolev space of order k modelled in $L^p(\Omega; \mathbf{R}^3)$

$W_0^{k,p}$: The space of function in $W^{k,p}$ whose derivatives up to the order $k-1$ have null trace in Γ

H^k : $W^{k,2}$

H_0^k : $W_0^{k,2}$

$\mathcal{V}$: The closure in H_0^1 of the functions in C_0^∞ with null divergent

$((\ ,\ ))$: The inner product in $\mathcal{V}$, given by

$$((\mathbf{u}, \mathbf{v})) \stackrel{\text{def}}{=} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j},$$

$\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathcal{V}$, where here, and in what follows, we use the notation of repeated indices, with summation from 1 to 3.

$\dfrac{\partial \varphi}{\partial \mathbf{n}}$: For $\varphi \in W^{2,p}$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, this notation stands for the Jacobian matrix $\nabla\varphi$ times $\mathbf{n}$, i.e.

$$\frac{\partial \varphi}{\partial \mathbf{n}} \stackrel{\text{def}}{=} (\nabla\varphi)\mathbf{n} = ((\nabla\varphi_1) \cdot \mathbf{n},\ (\nabla\varphi_2) \cdot \mathbf{n},\ (\nabla\varphi_3) \cdot \mathbf{n}) = (\frac{\partial \varphi_1}{\partial \mathbf{n}}, \frac{\partial \varphi_2}{\partial \mathbf{n}}, \frac{\partial \varphi_3}{\partial \mathbf{n}})$$

$||\ ||$: The norm associated with $((\ ))$ in H_0^1 or $\mathcal{V}$

$||\ ||_{k,p}$: The norm of $W^{k,p}$

$||\ ||_k$: The norm of H^k

$(\ ,\ )$: The inner produtc of L^2

$|\ ,\ |$: The norm of L^2

$|\ ,\ |_p$: The norm of L^p

$\mathcal{B}(\ ,\ ,\ )$ The trilinear form given by $\mathcal{B}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \stackrel{\text{def}}{=} (\,(\mathbf{u} \cdot \nabla)\mathbf{v}\,,\ \mathbf{w}\,)$

c : A universal constant, i.e. some positive constant that does not depend on the unknowns.

Definition 2.1 (Very weak solution). *Let* $\mathbf{v}_0 \in L^2(\Gamma)$ *such that* $\int_\Gamma \mathbf{v}_0 \cdot \mathbf{n} ds = 0$, $\mathbf{w}_0 \in L^2(\Gamma)$, *and* $\mathbf{f}, \mathbf{g} \in L^2(\Omega)$. *A triple* $(\mathbf{v}, \mathbf{w}, p)$ *in* $L^3 \times L^2 \times W^{-1,3}$ *is a* very weak solution *of Problem 2.1 if*

$$(\mathbf{v}, \nabla\theta) = \int_\Gamma (\mathbf{v}_0 \cdot \mathbf{n})\theta ds \tag{2.6}$$

for all θ *in* $W^{1,3/2}$,

$$\begin{aligned} -\mu(\mathbf{v}, \Delta\varphi) \; &- \; \mathcal{B}(\mathbf{v}, \varphi, \mathbf{v}) - (p, \operatorname{div}\varphi) \\ &= a(\mathbf{w}, rot\,\varphi) + (\mathbf{f}, \varphi) - \mu\int_\Gamma \mathbf{v}_0 \cdot \frac{\partial\varphi}{\partial\mathbf{n}} ds \end{aligned} \tag{2.7}$$

for all φ *in* $W^{2,3/2} \cap W_0^{1,3/2}$, *and*

$$\begin{aligned} -\alpha(\mathbf{w}, \Delta\psi) \; &- \; \mathcal{B}(\mathbf{v}, \psi, \mathbf{w}) - \beta(\mathbf{w}, \nabla\, div\,\psi) + \gamma(\mathbf{w}, \psi) \\ &= \; a(\mathbf{v}, rot\,\psi) + (\mathbf{g}, \psi) \\ &\quad -\alpha\int_\Gamma \mathbf{w}_0 \cdot \frac{\partial\psi}{\partial\mathbf{n}} ds - \beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) div\psi ds \end{aligned} \tag{2.8}$$

for all ψ *in* $H^2 \cap H_0^1$.

The main goal of this paper is to prove the following theorems.

Theorem 2.1 (Existence) *There exists a very weak solution of Problem 2.1 in the sense of the above definition, provided the viscosity* μ *is larger than some constant depending only on* Ω *and on the parameters* a, α, β, *and* γ.

Theorem 2.2 (Continuous dependence on f, g and $\mathbf{w}_0$, and uniqueness) *Let* $(\mathbf{v}_i, \mathbf{w}_i)$, $i = 1, 2$ *be very weak solutions of Problem 2.1 corresponding to the external fields* $\mathbf{f} = \mathbf{f}_i$, $\mathbf{g} = \mathbf{g}_i$, *and boundary data* $\mathbf{w}_{0i}$, $i = 1, 2$, *respectively. Then there exists a constant* $\mu^* > 0$ *such that for all* $\mu \geq \mu^*$,

$$|\mathbf{v}_1 - \mathbf{v}_2|_3 + |\mathbf{w}_1 - \mathbf{w}_2| \leq c(\,|\mathbf{f}_1 - \mathbf{f}_2| + |\mathbf{g}_1 - \mathbf{g}_2| \, + |\mathbf{w}_{01} - \mathbf{w}_{02}|\,), \tag{2.9}$$

where the constant c *depends only on the data of the problem and on* Ω. *In particular, for* $\mu \geq \mu^*$ *the problem is uniquely solvable.*

Remark 2..1 *In view of our construction of the solution we are not able to prove the continuous dependence of the solution on the boundary data* $\mathbf{v}_0$.

A brief explanation of the definition of a very weak solution: The definition of a *very weak solution* comes out naturally when we multiply the equations in (2.1)-(2.3) by test functions θ, φ, ψ, respectively, vanishing on the boundary Γ, formally integrate by parts, and impose the boundary data (2.4), (2.5). Then it is easy to check that the spaces $W^{2,3/2} \cap W_0^{1,3/2}$, $H^2 \cap H_0^1$ and $W^{1,3/2}$ are the appropriate Sobolev spaces to the test functions φ, ψ and θ, respectively, in view of the conditions $\mathbf{v}_0 \in L^2(\Gamma)$ and $\mathbf{w}_0 \in L^2(\Gamma)$. When $\mathbf{v}_0$ and $\mathbf{w}_0$ are regular, that is, $\mathbf{v}_0, \mathbf{w}_0 \in H^{1/2}(\Gamma)$, the concepts of *very weak solutions* and *weak solutions* are equivalent.

Next, we show that a *very weak solution* $(\mathbf{v}, \mathbf{w})$ satisfies the boundary conditions (2.4), (2.5) in the sense that $\mathbf{v}$ and $\mathbf{w}$ have whole "traces" on Γ which are equal to $\mathbf{v}_0$ and $\mathbf{w}_0$, respectively. From (2.1) it is classical that $\mathbf{v}$ has a normal trace on Γ, which we denote by $\gamma_{\mathbf{n}}(\mathbf{v})$, and

$$\gamma_{\mathbf{n}}(\mathbf{v}) = \mathbf{v}_0 \cdot \mathbf{n} \tag{2.10}$$

in $W^{-1/3,3}(\Gamma)$. Now, to obtain the tangent trace we consider the space (cf. [2] and [18])

$$X \stackrel{\text{def}}{=} \{\zeta \in W^{1/3,3/2}(\Gamma) \,:\, \zeta \cdot \mathbf{n} = 0\}$$

and define the functional $\gamma_{\mathbf{t}}(\mathbf{v}) : X \to \mathbf{R}$ by

$$< \gamma_{\mathbf{t}}(\mathbf{v}), \zeta > \stackrel{\text{def}}{=} \frac{1}{\mu}\left\{\mu(\mathbf{v}, \Delta\varphi) + \mathcal{B}(\mathbf{v}, \varphi, \mathbf{v}) + (p,\ \operatorname{div}\varphi) + a(\mathbf{w}, \operatorname{rot}\varphi) + (\mathbf{f}, \varphi)\right\}, \tag{2.11}$$

where for a given $\zeta \in X$, φ is any function in $W^{2,3/2} \cap W_0^{1,3/2}$ such that $\frac{\partial\varphi}{\partial\mathbf{n}}|_\Gamma = \zeta$. This map is well defined since $\mathbf{v}$ satisfies (2.7), and so

$$< \gamma_{\mathbf{t}}(\mathbf{v}), \zeta >= \int_\Gamma \mathbf{v}_0 \cdot \frac{\partial\varphi}{\partial\mathbf{n}} ds = \int_\Gamma \mathbf{v}_0 \cdot \zeta ds = \int_\Gamma \mathbf{v}_0^{\mathbf{t}} \cdot \zeta ds,$$

for all $\zeta \in X$, where ${\mathbf{v}_0}^{\mathbf{t}} \stackrel{\text{def}}{=} \mathbf{v}_0 - (\mathbf{v}_0 \cdot \mathbf{n})\mathbf{n}$. From this equality we see that

$$\gamma_{\mathbf{t}}(\mathbf{v}) = {\mathbf{v}_0}^{\mathbf{t}} \tag{2.12}$$

regarding $\gamma_{\mathbf{t}}(\mathbf{v})$, ${\mathbf{v}_0}^{\mathbf{t}}$ as elements in X'–the dual space of X. From (2.10) and (2.12) we can conclude that $\mathbf{v}$ admits a "trace"on Γ which coincides with $\mathbf{v}_0$.

Using equation (2.8) we get a similar notion of trace for $\mathbf{w}$ on Γ. For this case it is convenient to write the boundary term $\alpha\int_\Gamma \mathbf{w}_0 \cdot \frac{\partial\psi}{\partial\mathbf{n}} ds + \beta\int_\Gamma (\mathbf{w}_0\cdot\mathbf{n})\mathrm{div}\psi ds$ in the following way:

$$\begin{aligned}
& \alpha\int_\Gamma \mathbf{w}_0 \cdot \frac{\partial\psi}{\partial\mathbf{n}} ds + \beta\int_\Gamma (\mathbf{w}_0\cdot\mathbf{n})\mathrm{div}\psi ds \\
= & \int_\Gamma [\alpha\mathbf{w}_0^{\mathbf{t}} \cdot \frac{\partial\psi}{\partial\mathbf{n}} + (\mathbf{w}_0\cdot\mathbf{n})(\alpha\mathbf{n}\cdot\frac{\partial\psi}{\partial\mathbf{n}} + \beta\ \mathrm{div}\,\psi)]ds \\
= & \int_\Gamma [\alpha\mathbf{w}_0^{\mathbf{t}} \cdot \frac{\partial\psi}{\partial\mathbf{n}} + (\alpha+\beta)(\mathbf{w}_0\cdot\mathbf{n})\ \mathrm{div}\,\psi)]ds,
\end{aligned} \tag{2.13}$$

where $\mathbf{w}_0^{\mathbf{t}} \stackrel{\mathrm{def}}{=} \mathbf{w}_0 - (\mathbf{w}_0\cdot\mathbf{n})\mathbf{n}$. We have used the fact that $\mathbf{n}\cdot\frac{\partial\psi}{\partial\mathbf{n}}|_\Gamma = \mathrm{div}\,\psi|_\Gamma$ for all ψ in $H^2\cap H_0^1$. Now, we consider the spaces

$$Y \stackrel{\mathrm{def}}{=} \{g\,\mathbf{n}\ :\ g\in H^{1/2}(\Gamma;\mathbf{R})\} \quad \text{and} \quad Z \stackrel{\mathrm{def}}{=} \{\xi\in H^{1/2}(\Gamma;\mathbf{R}^3)\ ;\ \xi\cdot\mathbf{n}=0\},$$

and define the functionals $\gamma_{\mathbf{n}}(\mathbf{w}) : Y\to\mathbf{R}$, $\gamma_{\mathbf{t}}(\mathbf{w}) : Z\to\mathbf{R}$, respectively, by

$$(\alpha+\beta) < \gamma_{\mathbf{n}}(\mathbf{w}), g\mathbf{n} > \stackrel{\mathrm{def}}{=} \alpha(\mathbf{w},\Delta\psi)+\mathcal{B}(\mathbf{v},\psi,\mathbf{w})+\beta(\mathbf{w},\nabla\ \mathrm{div}\,\psi)+\gamma(\mathbf{w},\psi)+a(\mathbf{v},\mathrm{rot}\psi)+(\mathbf{g},\psi),$$

$$\alpha < \gamma_{\mathbf{t}}(\mathbf{w}), \xi > \stackrel{\mathrm{def}}{=} \alpha(\mathbf{w},\Delta\psi)+\mathcal{B}(\mathbf{v},\psi,\mathbf{w})+\beta(\mathbf{w},\nabla\ \mathrm{div}\,\psi)+\gamma(\mathbf{w},\psi)+a(\mathbf{v},\mathrm{rot}\,\psi)+(\mathbf{g},\psi),$$

where in the first case, ψ is any function in $H^2\cap H_0^1$ such that $\frac{\partial\psi}{\partial\mathbf{n}}|_\Gamma = g\,\mathbf{n}$, and in the second case, ψ is any function in $H^2\cap H_0^1$ such that $\frac{\partial\psi}{\partial\mathbf{n}}|_\Gamma = \xi$. Since $\mathbf{w}$ satisfies (2.8) and we have (2.13), it follows that

$$\begin{aligned}
(\alpha+\beta) < \gamma_{\mathbf{n}}(\mathbf{w}), g\,\mathbf{n} > &= \int_\Gamma [\alpha\mathbf{w}_0^{\mathbf{t}}\cdot(g\,\mathbf{n}) + (\alpha+\beta)(\mathbf{w}_0\cdot\mathbf{n})\mathbf{n}\cdot\frac{\partial\psi}{\partial\mathbf{n}})]ds \\
&= \int_\Gamma (\alpha+\beta)(\mathbf{w}_0\cdot\mathbf{n})\mathbf{n}\cdot(g\,\mathbf{n})ds
\end{aligned}$$

for all $g\,\mathbf{n}\in Y$. Then

$$\gamma_{\mathbf{n}}(\mathbf{w}) = (\mathbf{w}_0\cdot\mathbf{n})\mathbf{n},$$

as elements in Y'. Similarly, we have

$$\gamma_{\mathbf{t}}(\mathbf{w}) = \mathbf{w}_0^{\mathbf{t}}$$

in Z'. Therefore, $\mathbf{w}$ also has a "trace"on Γ which coincides with $\mathbf{w}_0$.

3. Problem in w

In this section we study the following problem in $\mathbf{w}$:

Problem 3.1 Given $\mathbf{w}_0 \in L^2(\Gamma)$ and $\mathbf{v} \in L^3$ with $\operatorname{div} \mathbf{v} = 0$ (see Remark 3..1 below) and such that $\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$ where $\mathbf{u}^\varepsilon \in H^1$ and $\mathbf{v}^\varepsilon \in L^3$ with $|\mathbf{v}^\varepsilon|_3$ sufficiently small, find $\mathbf{w} \in L^2$ such that (2.8) is satisfied, i.e.

$$
\begin{aligned}
-\alpha(\mathbf{w}, \Delta\psi) \quad &- \quad \mathcal{B}(\mathbf{v}, \psi, \mathbf{w}) - \beta(\mathbf{w}, \nabla \operatorname{div} \psi) + \gamma(\mathbf{w}, \psi) \\
&= \quad a(\mathbf{v}, \operatorname{rot} \psi) + (\mathbf{g}, \psi) \\
&\qquad -\alpha \int_\Gamma \mathbf{w}_0 \cdot \frac{\partial \psi}{\partial \mathbf{n}} ds - \alpha \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div} \psi ds
\end{aligned} \tag{3.1}
$$

for all ψ in $H_0^1 \cap H^2$.

Remark 3..1 *Above, the condition* $\operatorname{div} \mathbf{v} = 0$ *is understood in the weak sense, i.e.* $(\mathbf{v}, \nabla\theta) = 0$ *for all* $\theta \in W_0^{1,3/2}$. *As a consequence of this condition we have that the bilinear form*

$$
B(\phi, \psi) \stackrel{\text{def}}{=} \alpha(\nabla\phi, \psi) - \mathcal{B}(\mathbf{v}, \psi, \phi) + \beta(\operatorname{div} \phi, \operatorname{div} \psi) + \gamma(\phi, \psi), \tag{3.2}
$$

which is associated with the left hand side of (3.1), is strongly elliptic, i.e. it is bilinear continuous and coercive. Indeed,

$$
\mathcal{B}(\mathbf{v}, \phi, \phi) = -\frac{1}{2}(\mathbf{v}, \nabla(|\phi|^2)) = 0, \tag{3.3}
$$

for all $\phi \in H_0^1$, *since* $\operatorname{div} \mathbf{v} = 0$ *and* $H_0^1 \subset W_0^{1,3/2}$.

Lemma 3.1 *There exists a unique solution of Problem 3.1. Moreover, the following estimate holds:*

$$
|\mathbf{w}| \le c(1 + ||\mathbf{u}^\varepsilon||_1), \tag{3.4}
$$

where c is independent of $\mathbf{v}$.

Proof: We use the transposition method [15]. Let

$$
L(\psi) \stackrel{\text{def}}{=} -\alpha\Delta\psi - (\mathbf{v} \cdot \nabla)\psi - \beta\nabla \operatorname{div}\psi + \gamma\psi, \tag{3.5}
$$

the adjoint operator associated with the left hand side of (3.1). Given $h \in L^2$, let ψ be the unique weak solution in H_0^1 of the adjoint equation $L(\psi) = h$, i.e.

$$B(\phi, \psi) = (h, \phi) \tag{3.6}$$

for all $\phi \in H_0^1$. Existence and uniqueness of such solution ψ in H_0^1 easily follows from Lax-Milgram's lemma, since $\operatorname{div} \mathbf{v} = 0$ (cf. Remark 3..1 above). Besides, we can easily get the estimates

$$||\psi|| \leq \alpha^{-1}|h|, \qquad |\psi| \leq \gamma^{-1}|h| \tag{3.7}$$

by taking $\phi = \psi$ in (3.6).

Next we prove higher regularity of the solution of (3.6), i.e. we show that $\psi \in H^2$. Moreover, we obtain the following estimate:

$$||\psi||_2 \leq c(1 + ||\mathbf{u}^\varepsilon||_1^2)|h|, \tag{3.8}$$

where c is independent of $\mathbf{v}$. Although the operator L (defined in (3.5)) is strongly elliptic, this not follows straightforward from known results for elliptic systems because the operator L contains an irregular coefficient for the derivatives of the first order, namely, $\mathbf{v}$ is in L^3, and we do not assume it is bounded, i.e. in $L^\infty(\Omega)$. We do not know if $\psi \in H^2$ or if (3.8) holds true for a general $\mathbf{v}$ in L^3. In our case we gain that result due to the special form of $\mathbf{v}$ that is decomposed as a sum of a "regular part"$\mathbf{u}^\varepsilon$ in H^1 and a small part $\mathbf{v}^\varepsilon$ in L^3. In Section 4. we obtain the *very weak solution* of Problem 2.1 with $\mathbf{v}$ in L^3 by writing $\mathbf{v}$ as a such decomposition.

To attain our purpose of showing that ψ is in H^2 and to show that we have the estimate (3.8) we first regularize $\mathbf{v}$ by making the convolution of it with a smooth family of mollifiers $\{\rho_\eta\}, \eta > 0$. Then writing $\mathbf{v}_\eta \overset{\text{def}}{=} \mathbf{v} * \rho_\eta = \mathbf{u}^\varepsilon * \rho_\eta + \mathbf{v}^\varepsilon * \rho_\eta \equiv \mathbf{u}_\eta^\varepsilon + \mathbf{v}_\eta^\varepsilon$ we let ψ_η be the solution in H_0^1 of the following regularization of system $L(\psi) = h$:

$$-\alpha\Delta\psi_\eta - \beta\nabla\text{div}\psi_\eta + \gamma\psi_\eta = F_\eta, \tag{3.9}$$

where

$$F_\eta \overset{\text{def}}{=} h + (\mathbf{v}_\eta \cdot \nabla)\psi_\eta = h + (\mathbf{u}_\eta^\varepsilon \cdot \nabla)\psi_\eta + (\mathbf{v}_\eta^\varepsilon \cdot \nabla)\psi_\eta.$$

Since $\mathbf{u}^\varepsilon_\eta, \mathbf{v}^\varepsilon{}_\eta \in C^\infty(\overline{\Omega})$ and $\nabla\psi_\eta \in L^2$, we have that $F_\eta \in L^2$, thus by Nečas result on strongly elliptic systems (Theorem 5 in [20]) we obtain

$$||\psi_\eta||_2 \le c|F_\eta|, \tag{3.10}$$

where c is independent of $\mathbf{v}_\eta$. But

$$\begin{aligned} |(\mathbf{u}^\varepsilon_\eta \cdot \nabla)\psi_\eta| &\le |\mathbf{u}^\varepsilon_\eta|_6\, |\nabla\psi|_3 \le c||\mathbf{u}^\varepsilon_\eta||_1\, |\nabla\psi_\eta|_3 \\ &\le c||\mathbf{u}^\varepsilon_\eta||_1\, ||\psi_\eta||^{1/2}||\psi_\eta||_2^{1/2} \\ &\le \tfrac{c^2}{4\sigma}||\mathbf{u}^\varepsilon_\eta||_1^2||\psi_\eta|| + \sigma||\psi_\eta||_2 \\ &\le \tfrac{c^2}{4\sigma}||\mathbf{u}^\varepsilon||_1^2||\psi_\eta|| + \sigma||\psi_\eta||_2, \end{aligned} \tag{3.11}$$

for any $\sigma > 0$. (On the second inequality above we used the Gagliardo-Nirenberg (see e.g. [8]) inequality

$$||u||_{W^{k,p}} \le c||u||^\theta_{W^{m,q}}\, |u|_r^{1-\theta}$$

with $k = 1$, $p = n = 3$, $m = q = 2, \theta = 1/2$ and $r = 6$.) Besides,

$$\begin{aligned} |(\mathbf{v}^\varepsilon_\eta \cdot \nabla)\psi_\eta| &\le |\mathbf{v}^\varepsilon_\eta|_3\, |\nabla\psi_\eta|_6 \le \sigma c||\psi_\eta||_2 \\ &\le |\mathbf{v}^\varepsilon|_3\, |\nabla\psi_\eta|_6 \le \sigma c||\psi_\eta||_2, \end{aligned} \tag{3.12}$$

if $|\mathbf{v}^\varepsilon|_3 \le \sigma$. Then, using (3.11) and (3.12) in (3.10) with an appropriate σ, we obtain

$$||\psi_\eta||_2 \le c(|h| + ||\mathbf{u}^\varepsilon||_1^2||\psi_\eta||).$$

As

$$||\psi_\eta|| \le c|h|,$$

we arrive at (3.8) with ψ_η in place of ψ. Then we pass to the limit for a subsequence of $\{\eta\}$ and get (3.8). Here we used Banach-Alaoglu's theorem in H^2 and the uniqueness of solution of (3.6) in H^1_0.

Now we consider the map that takes h in L^2 into the unique solution ψ of (3.6) which is in H^2. Since we have (3.8) and the equation (3.6) is linear, this is a continuous linear map from L^2 into H^2. Then the expression

$$l(h) \stackrel{\text{def}}{=} a(\mathbf{v}, \text{rot}\psi) + (\mathbf{g}, \psi) - \alpha\int_\Gamma \mathbf{w}_0 \cdot \frac{\partial\psi}{\partial\mathbf{n}} ds - \beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n})\text{div}\psi ds$$

(given by the right hand side of (3.1)) defines a continuous linear functional in h acting on L^2. Writing (3.1) in the form

$$(\mathbf{w}, h) = l(h) \tag{3.13}$$

for all $h \in L^2$, we conclude directly from the Riesz representation theorem that there exists a unique $\mathbf{w}$ in L^2 such that (3.13) holds. This prove the existence and uniqueness part of the Lemma.

Next we proceed to get the estimate (3.4). Setting $h = \mathbf{w}$ in (3.13) we get

$$\begin{aligned} |\mathbf{w}|^2 &= l(\mathbf{w}) \\ &= a(\mathbf{v}, \mathrm{rot}\psi) + (\mathbf{g}, \psi) - \alpha \int_\Gamma \mathbf{w}_0 \tfrac{\partial \psi}{\partial \mathbf{n}} ds - \beta \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \mathrm{div}\psi ds, \end{aligned} \tag{3.14}$$

where $L(\psi) = \mathbf{w}$, that is, $\psi \in H_0^1 \cap H^2$ and

$$-\alpha \Delta \psi - (\mathbf{v} \cdot \nabla)\psi - \beta \nabla \mathrm{div}\psi + \gamma \psi = \mathbf{w}. \tag{3.15}$$

We shall show that the right hand side of (3.14) can be estimated by $c(1 + ||\mathbf{u}^\varepsilon||_1)|\mathbf{w}|$, where c is independent of $\mathbf{v}$.

Multiplying (3.15) by ψ and integrating in Ω we obtain, in particular

$$\alpha ||\psi||^2 + \gamma |\psi|^2 \leq (\mathbf{w}, \psi) \leq \frac{1}{2\gamma}|\mathbf{w}|^2 + \frac{\gamma}{2}|\psi|^2,$$

whence

$$||\psi|| \leq \frac{1}{\sqrt{2\alpha\gamma}}|\mathbf{w}| \quad \text{and} \quad |\psi| \leq \frac{1}{\gamma}|\mathbf{w}|. \tag{3.16}$$

The difficult term in (3.14) is $\int_\Gamma \mathbf{w}_0 \frac{\partial \psi}{\partial \mathbf{n}} ds$. To estimate it we need to use the fact that

$$|z|_{L^2(\Gamma)} \leq c(|\nabla z|^{1/2}|z|^{1/2} + |z|) \tag{3.17}$$

for any z in $H^1(\Omega)$. This estimate can be inferred from

$$|z|_{L^2(\Gamma)} \leq c|\nabla z|^{1/2}|z|^{1/2}$$

for all $z \in H^1(\Omega)$ with null average in Ω (see e.g. [10], p.50) by applying it to z minus its average in Ω.

Now we can estimate the terms on the right hand side of (3.14) in terms of $\mathbf{w}$. Using (3.16), (3.17) with $z = \nabla\psi$, and (3.8), we have:

$$a(\mathbf{v}, \mathrm{rot}\psi) \leq a|\mathbf{v}|||\psi|| \leq \frac{a}{\sqrt{2\alpha\gamma}}|\mathbf{v}|\,|\mathbf{w}| \leq c(1 + ||\mathbf{u}^\varepsilon||_1)|\mathbf{w}|,$$

$$(\mathbf{g}, \psi) \leq |\mathbf{g}||\psi| \leq \frac{1}{\gamma}|\mathbf{g}||\mathbf{w}| \leq c(1 + ||\mathbf{u}^\varepsilon||_1)|\mathbf{w}|,$$

$$\begin{aligned}\alpha\int_\Gamma \mathbf{w}_0 \tfrac{\partial\psi}{\partial\mathbf{n}} ds &\leq \alpha|\mathbf{w}_0|_{L^2(\Gamma)}|\nabla\psi|_{L^2(\Gamma)} \\ &\leq \alpha|\mathbf{w}_0|_{L^2(\Gamma)} c(||\psi||_2^{1/2}||\psi||^{1/2} + ||\psi||) \\ &\leq \alpha c|\mathbf{w}_0|_{L^2(\Gamma)}\Big((1 + ||\mathbf{u}^\varepsilon||_1^2)^{1/2}|\mathbf{w}|^{1/2}|\mathbf{w}|^{1/2} + |\mathbf{w}|\Big) \\ &\leq c(1 + ||\mathbf{u}^\varepsilon||_1)|\mathbf{w}|\end{aligned}$$

and

$$\beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n})\mathrm{div}\psi ds \leq c\beta|\mathbf{w}_0|_{L^2(\Gamma)}|\mathbf{w}| \leq c(1 + ||\mathbf{u}^\varepsilon||_1)|\mathbf{w}|.$$

In conclusion, (3.14) together with the above estimates gives (3.4). ■

We finish this Section with the following Lemma which will be used in the end of the proof of Lemma 5.2.

Lemma 3.2 *Let $(\mathbf{u}_n^\varepsilon)$ be a bounded sequence in H^1, $\mathbf{v}_n \stackrel{\mathrm{def}}{=} \mathbf{u}_n^\varepsilon + \mathbf{v}^\varepsilon$, and $\mathbf{w}_n$ the unique solution of Problem 3.1 with $\mathbf{v} = \mathbf{v}_n$. Then there exists a subsequence $(\mathbf{w}_{n_k})$ that is strongly convergent in L^2.*

Proof: From inequality (3.4) we conclude that the sequence $(\mathbf{w}_n)$ is bounded in L^2. Thus, there exists a subsequence $(\mathbf{w}_{n_k})$ that is weakly convergent in L^2. From (3.14) written for $\mathbf{w}_{n_k}$ and $\mathbf{w}_{n_l}$, we get

$$\begin{aligned}|\mathbf{w}_{n_k}|^2 - |\mathbf{w}_{n_l}|^2 = \; & a(\mathbf{v}_{n_k} - \mathbf{v}_{n_l},\ \mathrm{rot}\,\psi_{n_k}) + a(\mathbf{v}_{n_k},\ \mathrm{rot}\,(\psi_{n_k} - \psi_{n_l})) \\ & +(\mathbf{g}, \psi_{n_k} - \psi_{k_l}) \\ & -\alpha\int_\Gamma \mathbf{w}_0 \frac{\partial(\psi_{n_l} - \psi_{n_k})}{\partial\mathbf{n}} ds - \beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n})\ \mathrm{div}\,(\psi_{n_l} - \psi_{n_k}) ds,\end{aligned} \tag{3.18}$$

where $L(\psi_{n_k}) = \mathbf{w}_{n_k}$ and $L(\psi_{n_l}) = \mathbf{w}_{n_l}$.

From the boundedness of $(\mathbf{w}_n)$ in L^2 and inequality (3.8) it follows that the sequence (ψ_{n_k}) is bounded in H^2. From the compact embedding $H^1 \hookrightarrow\hookrightarrow L^{3/2}$ we conclude the existence of a subsequence $(\psi_{n_{k_m}})$, $m = 1, 2, \ldots$, such that $(\nabla\psi_{n_{k_m}})$ converges strongly in $L^{3/2}$. Since $H^2 \hookrightarrow H^{3/2}(\Gamma) \hookrightarrow\hookrightarrow H^1(\Gamma)$, we can assume also that $|\nabla(\psi_{n_{k_m}} - \psi_{nk_i})|_{L^2(\Gamma)}$ converges to zero as m, i go to infinity. Taking that into account, we can see easily from (3.18) that

$$|\mathbf{w}_{n_{k_m}}|^2 - |\mathbf{w}_{n_{k_i}}|^2 \to 0,$$

as m, i go to infinity. This, together with the weak convergence of $(\mathbf{w}_{n_{k_m}})$ in L^2 gives the strong convergence of $(\mathbf{w}_{n_{k_m}})$ in L^2. ■

4. Problem in v and a related problem

Assume that $\mathbf{w} \in L^2$ is given and consider the problem (2.6), (2.7) in $\mathbf{v}$. We want to get rid of the pressure (it can be recovered when needed from De Rham's Lemma) and to this end we take test functions that are divergent free. Then the problem (2.6), (2.7) reduces to the following one.

Problem 4.1 Given $\mathbf{w} \in L^2$, $\mathbf{v}_0 \in L^2(\Gamma)$ and $\mathbf{f} \in L^2$, find $\mathbf{v} \in L^3$ such that

$$(\mathbf{v}, \nabla\theta) = \int_\Gamma (\mathbf{v}_0 \cdot \mathbf{n})\theta ds \tag{4.1}$$

for all θ in $W^{1,3/2}$, and

$$-\mu(\mathbf{v}, \Delta\varphi) - \mathcal{B}(\mathbf{v}, \varphi, \mathbf{v}) = a(\mathbf{w}, \operatorname{rot}\varphi) + (\mathbf{f}, \varphi) - \mu\int_\Gamma \mathbf{v}_0 \cdot \frac{\partial\varphi}{\partial\mathbf{n}} ds \tag{4.2}$$

for all φ in $W^{2,3/2} \cap W_0^{1,3/2}$ with $\operatorname{div}\varphi = 0$.

Now, we introduce a problem that is related to Problem 4.1. Assume that $\mathbf{v}$ is a solution of Problem 4.1 and that we can write $\mathbf{v}$ in the form

$$\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon \qquad (\varepsilon > 0) \tag{4.3}$$

where $\mathbf{u}^\varepsilon$ is a "large regular part": $\mathbf{u}^\varepsilon \in H^1$, $\operatorname{div}\mathbf{u}^\varepsilon = 0$, $\mathbf{u}^\varepsilon|_\Gamma = \mathbf{v}_0^\varepsilon$ ($\mathbf{v}_0^\varepsilon$ is a smooth approximation of $\mathbf{v}_0$ in $L^2(\Gamma)$ such that $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)} << 1$) and $\mathbf{v}^\varepsilon$ is a "small regular part": $\mathbf{v}^\varepsilon \in L^3$ and is very weak solution of the problem (cf. Lemma 4.2 below):

$$\left\{\begin{array}{ll} -\mu\Delta\mathbf{v}^\varepsilon + (\mathbf{v}^\varepsilon\cdot\nabla)\mathbf{v}^\varepsilon + \nabla p^\varepsilon = 0 & \text{in } \Omega \\ \mathrm{div}\mathbf{v}^\varepsilon = 0 & \text{in } \Omega \\ \mathbf{v}^\varepsilon|_\Gamma = \mathbf{v}_0 - \mathbf{v}_0^\varepsilon. & \end{array}\right. \tag{4.4}$$

According to the definition of a very weak solution, we have, in particular,

$$-\mu(\mathbf{v}^\varepsilon, \Delta\varphi) - \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{v}^\varepsilon) = -\mu\int_\Gamma (\mathbf{v}_0 - \mathbf{v}_0^\varepsilon)\frac{\partial\varphi}{\partial\mathbf{n}}ds \tag{4.5}$$

for all $\varphi \in W^{2,3/2}\cap W_0^{1,3/2}$ with $\mathrm{div}\,\varphi = 0$. From (4.2), (4.3) and (4.5) it follows that

$$-\mu(\mathbf{u}^\varepsilon, \Delta\varphi) = \mathcal{B}(\mathbf{u}^\varepsilon, \varphi, \mathbf{v}) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{u}^\varepsilon) + a(\mathbf{w},\ \mathrm{rot}\,\varphi) + (\mathbf{f}, \varphi) - \mu\int_\Gamma \mathbf{v}_0^\varepsilon\frac{\partial\varphi}{\partial\mathbf{n}}ds.$$

Observe that $\mathbf{v}_0^\varepsilon$ is smooth and that $\mathbf{u}^\varepsilon$ belongs to H^1. We can integrate by parts on the left hand side of this equation to get

$$\mu((\mathbf{u}^\varepsilon, \varphi)) = \mathcal{B}(\mathbf{u}^\varepsilon, \varphi, \mathbf{v}) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{u}^\varepsilon) + a(\mathbf{w},\ \mathrm{rot}\,\varphi) + (\mathbf{f}, \varphi). \tag{4.6}$$

Now we write $\mathbf{u}^\varepsilon$ in the form

$$\mathbf{u}^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{u}, \tag{4.7}$$

where $\widetilde{\mathbf{v}_0^\varepsilon}$ is a suitable Leray-Hopf extension of $\mathbf{v}_0^\varepsilon$ to Ω (cf. Lemma 4.1 below), and $\mathbf{u} \in \mathcal{V}$. From (4.6) and (4.7) we can derive the equation for $\mathbf{u}$. We also write

$$\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{u} + \mathbf{v}^\varepsilon = \mathbf{u} + V^\varepsilon, \tag{4.8}$$

where $V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$. We observe that V^ε belongs to L^3 and $V^\varepsilon|_\Gamma = \mathbf{v}_0$. Applying (4.7) and (4.8) to (4.6) we obtain

$$\begin{aligned} \mu\left((u, \varphi)\right) &= \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{B}(V^\varepsilon, \varphi, \mathbf{u}) + \mathcal{B}(\mathbf{u}, \varphi, V^\varepsilon) + a(\mathbf{w},\ \mathrm{rot}\,\varphi) \\ &\quad + (\mathbf{f}, \varphi) - \mu((\widetilde{\mathbf{v}_0^\varepsilon}, \varphi)) + \mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi, V^\varepsilon) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \widetilde{\mathbf{v}_0^\varepsilon}). \end{aligned}$$

Denote

$$\mathcal{L}(\mathbf{u}, \varphi) \stackrel{\text{def}}{=} \mathcal{B}(V^\varepsilon, \varphi, \mathbf{u}) + \mathcal{B}(\mathbf{u}, \varphi, V^\varepsilon), \qquad V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon \tag{4.9}$$

and

(4.10) $< \mathcal{F}, \varphi > \stackrel{\text{def}}{=} (\mathbf{f}, \varphi) - \mu((\widetilde{\mathbf{v}_0^\varepsilon}, \varphi)) + \mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi)) + \mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi, V^\varepsilon) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \widetilde{\mathbf{v}_0^\varepsilon}).$

Then

(4.11) $$\mu((\mathbf{u}, \varphi)) = \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{L}(\mathbf{u}, \varphi) + a(\mathbf{w}, \text{ rot}\,\varphi) + < \mathcal{F}, \varphi >$$

for all $\varphi \in W^{2,3/2} \cap W_0^{1,3/2}$ with $\text{div}\,\varphi = 0$. If $\mathbf{u}$ is a solution of problem (4.11) then it is also a variational solution, that is,

(4.12) $$\mu((\mathbf{u}, \varphi)) = \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{L}(\mathbf{u}, \varphi) + a(\mathbf{w}, \text{ rot}\,\varphi) + < \mathcal{F}, \varphi >$$

for all $\varphi \in \mathcal{V}$, as from (4.9), (4.10) we can see that $\mathcal{L}(\mathbf{u}, \varphi)$, $< \mathcal{F}, \varphi >$, and $\mathcal{B}(\mathbf{u}, \varphi, \mathbf{u})$ are continuous in φ with respect to the H^1 topology.

Let us assume now that $\mathbf{u} \in \mathcal{V}$ is a solution of (4.12). From the above considerations it follows then that $\mathbf{v} = \mathbf{u} + V^\varepsilon$, $V^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, is a very weak solution of Problem 4.1.

In the next section we prove existence of a very weak solution of Problem 2.1, where the velocity field is of the form $\mathbf{v} = \mathbf{u} + V^\varepsilon = \mathbf{u}^\varepsilon + \mathbf{u}$, with $\mathbf{u} \in \mathcal{V}$, and with $V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, $\mathbf{u}^\varepsilon \stackrel{\text{def}}{=} \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, suitably constructed on the basis of the boundary data $\mathbf{v}_0 \in L^2$.

For the just mentioned construction we use the following Lemmas.

Lemma 4.1 (Leray-Hopf extension) *Let Ω be an open connected and bounded set in $\mathbf{R}^3$ of class C^2 and $\mathbf{z}_0 \in H^{1/2}(\Gamma)$ with $\int_\Gamma \mathbf{z}_0 \cdot \mathbf{n} ds = 0$. Then for every $\sigma > 0$ there exists a function $\widetilde{\mathbf{z}_0}$ such that*

$$\widetilde{\mathbf{z}_0} \in H^1(\Omega), \quad \text{div}\,\widetilde{\mathbf{z}_0} = 0 \ \ \textit{in } \Omega, \quad \widetilde{\mathbf{z}_0} = \mathbf{z}_0 \ \ \textit{on } \Gamma,$$

and

$$|\mathcal{B}(\mathbf{u}, \widetilde{\mathbf{z}_0}, \mathbf{u})| \le \sigma ||\mathbf{u}||^2$$

for all $\mathbf{u} \in \mathcal{V}$.

Proof: See [29],Chapter II, §1.4 and Appendix 1. ■

Lemma 4.2 (Marusič-Paloka) *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain in $\mathbf{R}^3$ with a boundary Γ of class C^2. Consider the following boundary value problem for the Navier-Stokes equations with data $\mathbf{g}$ in $L^2(\Gamma)$ satisfying $\int_\Gamma \mathbf{g}\cdot\mathbf{n}ds = 0$:*

$$\begin{cases} -\mu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div}\mathbf{z} = 0 & \text{in } \Omega \\ \mathbf{z} = \mathbf{g} & \text{on } \Gamma. \end{cases}$$

If $|\mathbf{g}|_{L^2(\Gamma)}$ is sufficiently small then there exists a unique very weak solution $\mathbf{z}$ in L^3 of the above problem. Furthermore, there is a constant c_1 depending only on μ such that

$$|\mathbf{z}|_3 < \frac{c_1\mu|\mathbf{g}|_{L^2(\Gamma)}}{\mu - c_1|\mathbf{g}|_{L^2(\Gamma)}}. \tag{4.13}$$

Proof: See Theorem 4 in [18]. ■

5. Existence theorem

At the beginning of this Section we shall show how to construct a map $\mathcal{A}:\mathcal{V}\to\mathcal{V}$ whose fixed point gives a very weak solution of Problem 2.1 in the sense of Definition 2.1. Then we prove two lemmas which yield the proof of Theorem 2.1.

We start with $\mathbf{v}_0 \in L^2(\Gamma)$–the irregular boundary condition. We take a smooth approximation $\mathbf{v}_0^\varepsilon$ of $\mathbf{v}_0$ in $L^2(\Gamma)$ such that $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)}$ is small enough with respect to μ, and let $\mathbf{v}^\varepsilon$ to be a very weak solution of (4.4) (cf. Lemma 4.2); we take $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)}$ so small that the Problem 3.1 has a solution for each $\mathbf{u}^\varepsilon$ in H^1 and that the last inequality in (5.4) below holds true. Then we construct the Leray-Hopf extension $\widetilde{\mathbf{v}_0^\varepsilon}$ of $\mathbf{v}_0^\varepsilon$ satisfying

$$\mathcal{B}(\mathbf{u}, \widetilde{\mathbf{v}_0^\varepsilon}, \mathbf{u}) \le \frac{\mu}{2}||\mathbf{u}||^2 \tag{5.1}$$

for all $\mathbf{u} \in \mathcal{V}$ (cf. Lemma 4.1).

Now, for $\mathbf{u} \in \mathcal{V}$, we define $\mathbf{v} = \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, $\mathbf{u}^\varepsilon \stackrel{\text{def}}{=} \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, and for this $\mathbf{v}$ we solve Problem 3.1 in $\mathbf{w}$. Having $\mathbf{w}$–the unique solution of Problem 3.1, we can define $\mathcal{A}(\mathbf{u}) \in \mathcal{V}$ by the relation

$$(5.2) \qquad E(\mathcal{A}(\mathbf{u}), \varphi) = a(\mathbf{w},\ \mathrm{rot}\,\varphi) + < \mathcal{F}, \varphi > + \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u})$$

for all $\varphi \in \mathcal{V}$, where $E(\mathbf{u}, \varphi) \stackrel{\text{def}}{=} \mu((\mathbf{u}, \varphi)) - \mathcal{L}(\mathbf{u}, \varphi)$ ($\mathcal{L}$ defined in (4.9)) is continuous and coercive under our assumptions. For each $\mathbf{w} \in L^2$ and $\mathbf{u} \in \mathcal{V}$ the right hand side of (5.2) defines a linear and bounded functional in φ on $\mathcal{V}$. Thus, by the Lax-Milgram lemma, the map $\mathcal{A}$ is well defined.

Observe that each fixed point $\mathbf{u}$ of the map $\mathcal{A}$ defines a pair $(\mathbf{v}, \mathbf{w}) = (\mathbf{u}^\varepsilon + V^\varepsilon, \mathbf{w})$, $V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, which is a very weak solution of Problem 2.1; $(\mathbf{v}, \mathbf{w})$ satisfies (4.1), (4.2) and (3.1). Using the De Rham lemma we show then that there exists a $p \in W^{-1,3}$ such that the triple $(\mathbf{v}, \mathbf{w}, p)$ satisfies all conditions in Definition 2.1.

For μ big enough, we can prove that the operator $\mathcal{A}$ is completely continuous and that all $\mathbf{u} \in \mathcal{V}$ such that for some $\lambda \in [0,1]$ it is $\mathbf{u} = \lambda \mathcal{A}\mathbf{u}$ are contained in a ball $||\mathbf{u}|| \le M$. The existence of a fixed point of $\mathcal{A}$ follows then from the Leray-Schauder fixed point theorem.

Lemma 5.1 *If μ is sufficiently large then there exists a constant $M > 0$ such that for all $\mathbf{u} \in \mathcal{V}$ satisfying the equation $\mathbf{u} = \lambda \mathcal{A}\mathbf{u}$ for some $\lambda \in [0,1]$ we have $||\mathbf{u}|| \le M$.*

Proof: If $\lambda = 0$ then $\mathbf{u} = 0$. Now, if $0 < \lambda \le 1$ then setting $\mathcal{A}\mathbf{u} = \frac{1}{\lambda}\mathbf{u}$ in (5.2) with $\varphi = \mathbf{u}$, we obtain

$$(5.3) \qquad \mu||\mathbf{u}||^2 - \mathcal{L}(\mathbf{u}, \mathbf{u}) = \lambda\{a(\mathbf{w},\ \mathrm{rot}\,\mathbf{u}) + < \mathcal{F}, \mathbf{u} >\}.$$

By the definition of $\mathcal{L}$ (see (4.9)) together with the fact that $\mathrm{div}\, V^\varepsilon = 0$ and $V^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, and by the estimates (5.1) and (4.13) in Lemma 4.2 with $\mathbf{g} = \mathbf{v}_0 - \mathbf{v}_0^\varepsilon$

(cf. problem (4.4)) we have

$$
\begin{aligned}
|\mathcal{L}(\mathbf{u},\mathbf{u})| = |\mathcal{B}(\mathbf{u},\mathbf{u},V^\varepsilon)| &= \mathcal{B}(\mathbf{u},\mathbf{u},\widetilde{\mathbf{v}_0^\varepsilon}) + \mathcal{B}(\mathbf{u},\mathbf{u},\mathbf{v}^\varepsilon)| \\
&= |-\mathcal{B}(\mathbf{u},\widetilde{\mathbf{v}_0^\varepsilon},\mathbf{u}) + \mathcal{B}(\mathbf{u},\mathbf{u},\mathbf{v}^\varepsilon)| \\
&\le \tfrac{\mu}{2}||\mathbf{u}||^2 + c|\mathbf{v}^\varepsilon|_3||\mathbf{u}||^2 \\
&\le (\tfrac{\mu}{2} + c\tfrac{c_1\mu|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}}{\mu - c_1|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}})||\mathbf{u}||^2 \le \tfrac{\mu}{4}||\mathbf{u}||^2,
\end{aligned} \tag{5.4}
$$

for $|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}$ sufficiently small with respect to μ.

Also, by (3.4),

$$
\begin{aligned}
a(\mathbf{w},\ \mathrm{rot}\,\mathbf{u}) &\le a|\mathbf{w}|||\mathbf{u}|| \le c(1 + ||\mathbf{u}^\varepsilon||_1)||\mathbf{u}|| \\
&\le c(1 + ||\mathbf{u}||_1 + ||\widetilde{\mathbf{v}_0^\varepsilon}||_1)||\mathbf{u}|| \le c(1 + ||\mathbf{u}|| + ||\widetilde{\mathbf{v}_0^\varepsilon}||_1)||\mathbf{u}|| \\
&\le c||\mathbf{u}||^2 + c'||\mathbf{u}|| \le \tfrac{\mu}{4}||\mathbf{u}||^2 + c'||\mathbf{u}||
\end{aligned} \tag{5.5}
$$

for μ large enough, and, by the definition of $\mathcal{F}$ (see (4.10)),

$$
< \mathcal{F},\mathbf{u} >= (\mathbf{f},\mathbf{u}) - \mu((\widetilde{\mathbf{v}_0^\varepsilon},\mathbf{u})) + \mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon},\mathbf{u},V^\varepsilon) + \mathcal{B}(\mathbf{v}^\varepsilon,\mathbf{u},\widetilde{\mathbf{v}_0^\varepsilon}) \le c||\mathbf{u}||. \tag{5.6}
$$

From (5.3), together with (5.4)–(5.6), we obtain the desired result. ■

Lemma 5.2 *The operator $\mathcal{A}$ is completely continuous.*

Proof: Let $(\mathbf{u}_n)$ be a bounded sequence in $\mathcal{V}$. We shall show that then $(\mathcal{A}\mathbf{u}_{n_k})$ is a Cauchy sequence in $\mathcal{V}$ (for a subsequence (n_k)). Let

$$
E(\mathcal{A}\mathbf{u}_m,\varphi) = a(\mathbf{w}_m,\ \mathrm{rot}\,\varphi) + < \mathcal{F},\varphi > + \mathcal{B}(\mathbf{u}_m,\varphi,\mathbf{u}_m) \tag{5.7}
$$

$$
E(\mathcal{A}\mathbf{u}_n,\varphi) = a(\mathbf{w}_n,\ \mathrm{rot}\,\varphi) + < \mathcal{F},\varphi > + \mathcal{B}(\mathbf{u}_n,\varphi,\mathbf{u}_n) \tag{5.8}
$$

for all $\varphi \in \mathcal{V}$, where

$$
\begin{aligned}
&(\mathbf{w}_m, -\alpha\Delta\psi + (\mathbf{v}_m\cdot\nabla)\psi - \beta\nabla\ \mathrm{div}\,\psi + \gamma\psi) \\
&= a(\mathbf{v}_m,\ \mathrm{rot}\,\psi) + (\mathbf{g},\psi) - \alpha\int_\Gamma \mathbf{w}_0\frac{\partial\psi}{\partial\mathbf{n}}ds - \beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\ \mathrm{div}\,\psi ds
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
(5.10)\qquad & (\mathbf{w}_n, -\alpha\Delta\psi + (\mathbf{v}_n\cdot\nabla)\psi - \beta\nabla\,\mathrm{div}\,\psi + \gamma\psi) \\
&= a(\mathbf{v}_n,\ \mathrm{rot}\,\psi) + (\mathbf{g},\psi) - \alpha\int_\Gamma \mathbf{w}_0\frac{\partial\psi}{\partial\mathbf{n}}ds - \beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\,\mathrm{div}\,\psi ds,
\end{aligned}
$$

$$
\mathbf{v}_m = \mathbf{u}_m + V^\varepsilon, \qquad \mathbf{v}_n = \mathbf{u}_n + V^\varepsilon, \qquad V^\varepsilon \stackrel{\mathrm{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon.
$$

Taking the difference of (5.7) and (5.8) we obtain

$$
(5.11)\qquad
\begin{aligned}
E(\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n, \varphi) &= a(\mathbf{w}_m - \mathbf{w}_n,\ \mathrm{rot}\,\varphi) \\
&+\mathcal{B}(\mathbf{u}_m - \mathbf{u}_n, \varphi, \mathbf{u}_n) + \mathcal{B}(\mathbf{u}_m, \varphi, \mathbf{u}_m - \mathbf{u}_n).
\end{aligned}
$$

Set $\varphi = \mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n$ and we have

$$
\begin{aligned}
\tfrac{3}{4}\mu||\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n||^2 &\le a|\mathbf{w}_m - \mathbf{w}_n|\,||\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n|| \\
&+c(||\mathbf{u}_m|| + ||\mathbf{u}_n||)||\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n||\,|\mathbf{u}_m - \mathbf{u}_n|_3,
\end{aligned}
$$

where for obtaining the left hand side we used $E(\mathbf{u},\varphi) \stackrel{\mathrm{def}}{=} \mu((\mathbf{u},\varphi)) - \mathcal{L}(\mathbf{u},\varphi)$ and the estimate for $\mathcal{L}(\mathbf{u},\mathbf{u})$ in (5.4). Thus

$$
(5.12)\qquad
\begin{aligned}
\tfrac{3}{4}\mu||\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n|| &\le a|\mathbf{w}_m - \mathbf{w}_n| \\
&+c(||\mathbf{u}_m|| + ||\mathbf{u}_n||)|\mathbf{u}_m - \mathbf{u}_n|_3.
\end{aligned}
$$

Now, as $(\mathbf{u}_m)$ is a bounded sequence in $\mathcal{V}$, there exists a subsequence (we denote it also by $(\mathbf{u}_n)$) that is convergent is L^3. Moreover, in view of Lemma 3.1, $(\mathbf{w}_m)$ converges in L^2. Thus, by (5.12), $(\mathcal{A}\mathbf{u}_n)$ is a Cauchy sequence in $\mathcal{V}$. In consequence, the operator $\mathcal{A}$ is compact.

Observe that from inequality (5.12) the continuity of $\mathcal{A}$ in $\mathcal{V}$ immediately follows.

■

6. Continuous dependence on the data f, g and $\mathbf{w}_0$

In this section we prove Theorem 2.2. Let

$$
(6.1)\qquad \mu((\mathbf{u}_i,\phi)) = \mathcal{B}(\mathbf{u}_i,\phi,\mathbf{u}) + \mathcal{L}(\mathbf{u}_i,\phi) + a(\mathbf{w}_i,\ \mathrm{rot}\,\phi) + <\mathcal{F}_i,\phi>
$$

where

$$\mathcal{L}(\mathbf{u}_i,\phi) \stackrel{\text{def}}{=} \mathcal{B}(V^\varepsilon,\phi,\mathbf{u}_i)+\mathcal{B}(\mathbf{u}_i,\phi,V^\varepsilon), \qquad V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon}+\mathbf{v}^\varepsilon \tag{6.2}$$

and

$$< \mathcal{F}_i,\phi > \stackrel{\text{def}}{=} (\mathbf{f}_i,\phi)-\mu((\widetilde{\mathbf{v}_0^\varepsilon},\phi))+\mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon},\phi))+\mathcal{B}(\widetilde{\mathbf{v}_0^\varepsilon},\phi,V^\varepsilon)+\mathcal{B}(\mathbf{v}^\varepsilon,\phi,\widetilde{\mathbf{v}_0^\varepsilon}). \tag{6.3}$$

for $i=1,2$ and $\phi\in H_0^1$. We recall that $\mathbf{v}^\varepsilon$ is the very weak solution of (4.4) with $\mathbf{v}^\varepsilon|_\Gamma=\mathbf{v}_0-\mathbf{v}_0^\varepsilon$, where $\mathbf{v}_0^\varepsilon$ is a smooth approximation of $\mathbf{v}_0$ such that $\operatorname{div}\mathbf{v}_0^\varepsilon=0$, and $|\mathbf{v}_0-\mathbf{v}_0^\varepsilon|_3$ is very small with respect to μ (cf. (5.4)), and $\widetilde{\mathbf{v}_0^\varepsilon}$ is a Leray-Hopf extension of $\mathbf{v}_0^\varepsilon$ satisfying (5.1).

From (5.4) we have

$$\mathcal{L}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)\le\frac{\mu}{4}||\mathbf{u}_1-\mathbf{u}_2||^2. \tag{6.4}$$

Then, writing (6.1) for $i=1,2$, taking the difference and setting $\phi=\mathbf{u}_1-\mathbf{u}_2$, we obtain

$$\begin{aligned}\tfrac{3}{4}\mu||\mathbf{u}_1-\mathbf{u}_2||^2 &\le \mathcal{B}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2)+a(\mathbf{w}_1-\mathbf{w}_2,\ \operatorname{rot}(\mathbf{u}_1-\mathbf{u}_2))+(\mathbf{f}_1-\mathbf{f}_2,\mathbf{u}_1-\mathbf{u}_2)\\ &\le c||\mathbf{u}_2||\,||\mathbf{u}_1-\mathbf{u}_2||^2+a|\mathbf{w}_1-\mathbf{w}_2|\,||\mathbf{u}_1-\mathbf{u}_2||+c|\mathbf{f}_1-\mathbf{f}_2|\,||\mathbf{u}_1-\mathbf{u}_2||,\end{aligned}$$

whence

$$\frac{3}{4}\mu||\mathbf{u}_1-\mathbf{u}_2||\le c||\mathbf{u}_2||\,||\mathbf{u}_1-\mathbf{u}_2||+a|\mathbf{w}_1-\mathbf{w}_2|+c|\mathbf{f}_1-\mathbf{f}_2|.$$

From Lemma 5.1 we have that $||\mathbf{u}_2||\le M$, where M is a constant that does not increase with μ, thus for μ large enough such that $c||\mathbf{u}_2||\le\mu/4$, we obtain

$$\frac{\mu}{2}||\mathbf{u}_1-\mathbf{u}_2||\le a|\mathbf{w}_1-\mathbf{w}_2|+c|\mathbf{f}_1-\mathbf{f}_2|. \tag{6.5}$$

Now, we use equation (3.13). Assume at first that $\mathbf{w}_{01}=\mathbf{w}_{02}$. Then from (3.13) written for $\mathbf{w}=\mathbf{w}_1$ and $\mathbf{w}=\mathbf{w}_2$ we have

$$(\mathbf{w}_1,h_1)=a(\mathbf{v}_1,\ \operatorname{rot}\psi_1)+(\mathbf{g}_1,\psi_1)-\alpha\int_\Gamma\mathbf{w}_0\frac{\partial\psi_1}{\partial\mathbf{n}}ds-\beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\operatorname{div}\psi_1 ds \tag{6.6}$$

and

$$(\mathbf{w}_2,h_2)=a(\mathbf{v}_2,\ \operatorname{rot}\psi_2)+(\mathbf{g}_2,\psi_2)-\alpha\int_\Gamma\mathbf{w}_0\frac{\partial\psi_2}{\partial\mathbf{n}}ds-\beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\operatorname{div}\psi_2 ds, \tag{6.7}$$

where

$$h_1 = L_{\mathbf{v}_1}(\psi_1) \qquad \text{and} \qquad h_2 = L_{\mathbf{v}_2}(\psi_2),$$

for

$$L_{\mathbf{v}} \stackrel{\text{def}}{=} -\alpha\Delta\psi - (\mathbf{v}\cdot\nabla)\psi - \beta\nabla \operatorname{div}\psi + \gamma\psi.$$

Taking the test functions ψ_1 and ψ_2 in (6.6) and (6.7), respectively, such that $L_{\mathbf{v}_1}(\psi_1) = L_{\mathbf{v}_2}(\psi_2) = \mathbf{w}_1 - \mathbf{w}_2$ and subtracting (6.7) from (6.6) we obtain

$$\begin{aligned} |\mathbf{w}_1 - \mathbf{w}_2|^2 = \ & a(\mathbf{v}_1 - \mathbf{v}_2,\ \operatorname{rot}\psi_2) + a(\mathbf{v}_1,\ \operatorname{rot}(\psi_1 - \psi_2)) \\ & +(\mathbf{g}_1 - \mathbf{g}_2, \psi_2) + (\mathbf{g}_1, \psi_1 - \psi_2) \\ & \alpha\int_\Gamma \mathbf{w}_0 \tfrac{\partial}{\partial \mathbf{n}}(\psi_1 - \psi_2)ds + \beta\int_\Gamma (\mathbf{w}_0\cdot\mathbf{n}) \operatorname{div}(\psi_1 - \psi_2)ds. \end{aligned} \tag{6.8}$$

Now, we estimate the terms on the right hand side of (6.8). The first term is easily estimated:

$$\begin{aligned} & a(\mathbf{v}_1 - \mathbf{v}_2,\ \operatorname{rot}\psi_2) = a(\mathbf{u}_1 - \mathbf{u}_2,\ \operatorname{rot}\psi_2) = a(\operatorname{rot}(\mathbf{u}_1 - \mathbf{u}_2), \psi_2) \\ & \leq a||\mathbf{u}_1 - \mathbf{u}_2||\,|\psi_2| \\ & \leq \tfrac{a}{\gamma}||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|, \end{aligned} \tag{6.9}$$

where we used (3.7).

The second term can be estimated as follows:

$$a(\mathbf{v}_1,\ \operatorname{rot}(\psi_1 - \psi_2)) \leq a|\mathbf{v}_1|\,||\psi_1 - \psi_2||. \tag{6.10}$$

We have

$$-\alpha\Delta\psi_1 + (\mathbf{v}_1\cdot\nabla)\psi_1 - \beta\nabla \operatorname{div}\psi_1 + \gamma\psi_1 = \mathbf{w}_1 - \mathbf{w}_2$$

and

$$-\alpha\Delta\psi_2 + (\mathbf{v}_2\cdot\nabla)\psi_2 - \beta\nabla \operatorname{div}\psi_2 + \gamma\psi_1 = \mathbf{w}_1 - \mathbf{w}_2,$$

then taking the difference, we get

$$\begin{aligned} -\alpha\Delta(\psi_1 - \psi_2) \quad & +(\mathbf{v}_1\cdot\nabla)(\psi_1 - \psi_2) - \beta\nabla \operatorname{div}(\psi_1 - \psi_2) + \gamma(\psi_1 - \psi_2) \\ & = -((\mathbf{v}_1 - \mathbf{v}_2))\cdot\nabla)\psi_2 = -((\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla)\psi_2. \end{aligned} \tag{6.11}$$

Multiplying by $\psi_1 - \psi_2$ and integrating in Ω we obtain, in particular,

$$||\psi_1 - \psi_2|| \leq c||\mathbf{u}_1 - \mathbf{u}_2||\,||\psi_2||,$$

so, using again (3.7), it follows that

$$||\psi_1 - \psi_2|| \leq c||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|.$$

Using this estimate in (6.10) we obtain

$$|a(\mathbf{v}_1,\ \mathrm{rot}\,(\psi_1 - \psi_2)| \leq c|\mathbf{v}_1|\,||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|. \tag{6.12}$$

Next, we have

$$(\mathbf{g}_1 - \mathbf{g}_2, \psi_2) \leq |\mathbf{g}_1 - \mathbf{g}_2|\gamma^{-1}|\mathbf{w}_1 - \mathbf{w}_2| \tag{6.13}$$

and

$$\begin{aligned}(\mathbf{g}_1, \psi_1 - \psi_2) &\leq |\mathbf{g}_1|\,|\psi_1 - \psi_2| \\ &\leq c|\mathbf{g}_1|\,||\psi_1 - \psi_2|| \\ &\leq c|\mathbf{g}_1|\,||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|.\end{aligned} \tag{6.14}$$

The boundary integrals give, by (6.11) and (3.8),

$$\begin{aligned}&\alpha\int_\Gamma \mathbf{w}_0 \tfrac{\partial}{\partial \mathbf{n}}\psi_1 - \psi_2) ds \leq \alpha|\mathbf{w}_0|_{L^2(\Gamma)}c||\psi_1 - \psi_2||_2 \\ &\leq c|\mathbf{w}_0|_{L^2(\Gamma)}|\,(\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla)\,\psi_2| \leq c|\mathbf{w}_0|_{L^2(\Gamma)}||\mathbf{u}_1 - \mathbf{u}_2||\,||\psi_2||_2 \\ &\leq c|\mathbf{w}_0|_{L^2(\Gamma)}||\mathbf{u}_1 - \mathbf{u}_2||(1 + ||\mathbf{u}_2||^2)|\mathbf{w}_1 - \mathbf{w}_2| \\ &\leq c|\mathbf{w}_0|_{L^2(\Gamma)}||\mathbf{u}_1 - \mathbf{u}_2||(1 + M^2)|\mathbf{w}_1 - \mathbf{w}_2| \\ &\equiv c|\mathbf{w}_0|_{L^2(\Gamma)}||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|,\end{aligned} \tag{6.15}$$

and

$$\beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n})\ \mathrm{div}\,(\psi_1 - \psi_2) ds \leq \beta|\mathbf{w}_0|_{L^2(\Gamma)}c||\mathbf{u}_1 - \mathbf{u}_2||\,|\mathbf{w}_1 - \mathbf{w}_2|. \tag{6.16}$$

From (6.8)-(6.16) we obtain

$$|\mathbf{w}_1 - \mathbf{w}_2| \leq c(||\mathbf{u}_1 - \mathbf{u}_2|| + |\mathbf{g}_1 - \mathbf{g}_2|). \tag{6.17}$$

Using this estimate in (6.5) we have

$$||\mathbf{u}_1 - \mathbf{u}_2|| \leq c(|\mathbf{g}_1 - \mathbf{g}_2| + |\mathbf{f}_1 - \mathbf{f}_2|)$$

for μ large enough. Then, from (6.17), it follows an estimate of the same type for $|\mathbf{w}_1 - \mathbf{w}_2|$. Therefore, we can write

$$|\mathbf{v}_1 - \mathbf{v}_2|_3 + |\mathbf{w}_1 - \mathbf{w}_2| \leq c(|\mathbf{f}_1 - \mathbf{f}_2| + |\mathbf{g}_1 - \mathbf{g}_2|), \tag{6.18}$$

as

$$|\mathbf{v}_1 - \mathbf{v}_2|_3 = |\mathbf{u}_1 - \mathbf{u}_2|_3 \leq c||\mathbf{u}_1 - \mathbf{u}_2||.$$

Estimate (6.18) gives the continuous dependence of solutions $(\mathbf{v}, \mathbf{w})$ on the data $\mathbf{f}$, $\mathbf{g}$, provided μ is large enough.

Now, to prove (2.9), we observe that if $\mathbf{w}_{01} \neq \mathbf{w}_{02}$ then in (6.6) and (6.7) we have $\mathbf{w}_{01}$ and $\mathbf{w}_{02}$ instead of $\mathbf{w}_0$, respectively, and subtracting these equations we obtain two new terms, namely,

$$\alpha \int_\Gamma (\mathbf{w}_{01} - \mathbf{w}_{02}) \frac{\partial}{\partial \mathbf{n}} \psi_2 ds \quad \text{and} \quad \beta \int_\Gamma \left((\mathbf{w}_{01} - \mathbf{w}_{02}) \cdot \mathbf{n} \right) \, \mathrm{div}\, \psi_2 ds,$$

which can be estimate from above by

$$c|\mathbf{w}_{01} - \mathbf{w}_{02}| \, |\mathbf{w}_1 - \mathbf{w}_2|,$$

whence we have (2.9) in view of the above considerations.

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