On the controllability of stationary magneto-micropolar fluids

†Jaime H. Ortega ‡Marko A. Rojas-Medar

Abstract

In this work we some results on the boundary controllability of the steady magneto-micropolar fluids. In particular we consider the boundary controllability of the fluid velocity on a subset of the boundary. In the same way, we can obtain some controllability results when we consider the case of two or more boundary controls for the microrotational velocity or the magnetic fields. In the case of the homogeneous boundary conditions, we can obtain some results for the internal controllability.

KEYWORDS: boundary control, Navier-Stokes equation, internal controllability, optimization, Lagrange multipliers.

1 Introduction

In this work we study the boundary controllability for the equations that describes the motion of a stationary viscous incompressible magneto-micropolar fluid in a bounded domain $\Omega \subseteq I\!R^n, n = 2$ or 3. Let $\Gamma^i, i = 0, 1, 2$, be subsets of the boundary Γ. Thus the equations are given by (see [1], for instance):

$$
\begin{aligned}
\mathbf{u} \cdot \nabla \mathbf{u} - (\mu + \chi)\Delta \mathbf{u} + \nabla(p + \frac{1}{2}\mathbf{h} \cdot \mathbf{h}) &= \chi \text{ rot } \mathbf{w} + r\mathbf{h} \cdot \nabla \mathbf{h} + \mathbf{f}, \text{ in } \Omega \\
j\mathbf{u} \cdot \nabla \mathbf{w} - \gamma \Delta \mathbf{w} + 2\chi \mathbf{w} - (\alpha + \beta)\nabla \text{ div } \mathbf{w} &= \chi \text{ rot } \mathbf{u} + \mathbf{g}, \text{ in } \Omega \\
-\nu \Delta \mathbf{h} + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u} &= 0, \text{ in } \Omega \\
\text{div } \mathbf{u} &= \text{div } \mathbf{h} = 0, \text{ in } \Omega \\
\mathbf{u} &= \mathbf{u}_0, \text{ on } \Gamma^0, \quad \mathbf{u} = 0, \text{ on } \Gamma \setminus \Gamma^0, \\
\mathbf{h} &= \mathbf{h}_0, \text{ on } \Gamma^1, \quad \mathbf{h} = 0, \text{ on } \Gamma \setminus \Gamma^1, \\
\mathbf{w} &= \mathbf{w}_0, \text{ on } \Gamma^2, \quad \mathbf{w} = 0, \text{ on } \Gamma \setminus \Gamma^2,
\end{aligned} \tag{1}
$$

being $\mathbf{u}(x) \in I\!R^n$ denotes the velocity of the fluid at a point $x \in \Omega$, $\mathbf{w}(x), \mathbf{h}(x) \in I\!R^n$ and $p(x) \in I\!R$ denote, respectively, the microrotational velocity, the magnetic field and the hydrostatic pressure; the constants

$\mu,\ \chi,\ r,\ \alpha,\ \beta,\ \gamma,\ j$ and ν are constants associated to properties of the material. From physical reasons, these constants satisfy $\min\{\mu,\ \chi,\ r,\ j,\ \gamma,\ \nu,\ \alpha+\beta+\gamma\}>0$; $\mathbf{f}(x)$ and $\mathbf{g}(x)\ \in I\!R^n$ are given external fields and $\mathbf{u}_0, \mathbf{h}_0, \mathbf{w}_0 \in I\!R^n$ are the boundary conditions.

Equation $(1.1)_i$ has the familiar form of the Navier-Stokes equations but it is coupled with equation $(1.1)_{ii}$, which essentially describes the motion inside the macrovolumes as they undergo microrotational effects represented by the microrotational velocity vector $\mathbf{w}$. For fluids with no microstructure this parameter vanishes. For Newtonian fluids, equations $(1.1)_i$ and $(1.1)_{ii}$ decouple since $\chi = 0$.

In this work we give a result of existence of the weak solutions of the problem $(1.1)_i$-$(1.1)_{vii}$ for $n = 2, 3$.

The problems of controllability in which we are interested are the following: Let $\mathbf{u}_0$ be the control function and $K \subset H^{\frac{1}{2}}(\Gamma^0)$ be a non empty set.

We want to study the constrained minimization problem:

To find $\tilde{\mathbf{u}}_0 \in K$ such that

$$\min_{\mathbf{u}_0 \in K} J(\mathbf{u}, \mathbf{h}, \mathbf{w}, u_0) = J(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_0) \tag{2}$$

$(\mathbf{u}, \mathbf{h}, \mathbf{w})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}})$ being a weak solution of $(1.1)_i$-$(1.1)_{vii}$ verifying $\mathbf{u}|_{\Gamma^0} = \mathbf{u}_0$ and $\tilde{\mathbf{u}}|_{\Gamma^0} = \tilde{\mathbf{u}}_0$ respectively.

Note that, in this case we consider a single control on the velocity of the fluid. We can consider also a single control on the magnetic field $\mathbf{h}$ or on the microrotational velocity $\mathbf{w}$. Another related problems are the case when we consider more than one control, for instante we can consider the controls $\mathbf{u}_0$ and $\mathbf{h}_0$. All these problems are called *boundary controllability,* because we are acting with the control on the boundary. The proof of the boundary controllability can be adapted to obtain results of *internal controllability.* In this case the control are the external forces $\mathbf{f}$ and $\mathbf{g}$. They are acting in a non empty open subset of ω of Ω such that $\overline{\omega} \subset \Omega$ and the functions satisfy that supp $\mathbf{f} \subset \omega$ and supp $\mathbf{g} \subset \omega$.

Some interesting functional are for instance

$$\begin{aligned}
J_0(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{u}|^2 + \alpha \left\|\mathbf{u}_0\right\|_{\frac{1}{2},\Gamma^1} \\
J_1(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) &= \frac{1}{2}\int_\Omega \left|(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T\right|^2 + \alpha \left\|\mathbf{u}_0\right\|_{\frac{1}{2},\Gamma^1}, \\
J_2(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{h}|^2 + \alpha \left\|\mathbf{u}_0\right\|_{\frac{1}{2},\Gamma^1}, \\
J_3(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{u} - \mathbf{u}_d|^2 + \alpha \left\|\mathbf{u}_0\right\|_{\frac{1}{2},\Gamma^1},
\end{aligned}$$

$(\mathbf{u}, \mathbf{h}, \mathbf{w})$ being a weak solution of the problem $(1.1)_i$-$(1.1)_{vii}$ with boundary control $\mathbf{u}_0$, $\alpha \geq 0$ and $\mathbf{u}_d \in L^2(\Omega)^n$ is a given function.

This work is organized as follows. In section 2 we give the main results of the paper without proof. In section 3 we give some preliminary results. In section 4 we proof the then existence of weak solutions of the system with non-homogeneous boundary conditions. In section 5 we give the proof of the control results.

2 Main Results

Firstly, we give a existence result of weak solutions of the problem (1.1).

Theorem 1

Let $\Omega \subset I\!\!R^n$ be a bounded set with Lipstchitz continuous boundary Γ, and let $\Gamma^0, \Gamma^1, \Gamma^2$ be open subsets of the boundary Γ. Assume that $\mathbf{u}_0, \mathbf{h}_0, \mathbf{w}_0 \in H^{\frac{1}{2}}(\Gamma)^n$, such that $\mathbf{u}_0 \cdot \mathbf{n} = \mathbf{h}_0 \cdot \mathbf{n} = 0$ on Γ.

Then the problem (1.1) has at least one weak solution.

Moreover, there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|_1 + \|\mathbf{h}\|_1 + \|\mathbf{w}\|_1 \leq C \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{g}\|_{H^{-1}(\Omega)} + \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^0} + \|\mathbf{h}_0\|_{\frac{1}{2},\Gamma_1} + \|\mathbf{w}_0\|_{\frac{1}{2},\Gamma^2} \right). \tag{3}$$

Analogously to G. V. Alekseev [2], we are interested in the study of some control problems of the following type:

Assume that $\Omega \subset I\!\!R^n$ is a bounded domain with Lipschitz continuous boundary and $\Gamma^0, \Gamma^1, \Gamma^2$ are bounded open subsets of the boundary Γ such that $\text{meas}(\Gamma \subset \Gamma^i) > 0$.

We prove some boundary control results for the problem (1.1) with a single control for the velocity of the fluid $\mathbf{u}_0$.

Let K be a non empty bounded convex subset of $H^{\frac{1}{2}}(\Gamma^0)$ and define the set of admissible functions

$$Z_K = \{(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) : (\mathbf{u}, \mathbf{h}, \mathbf{w}) \text{ is a weak solution of (\textbf{??})-(\textbf{??}) such that } \mathbf{u}|_{\Gamma^0} = \mathbf{u}_0, \mathbf{u}_0 \in K\}. \tag{4}$$

Our control problem is:

To find $(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_0) \in Z_K$ such that

$$\inf_{(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0) \in Z_K} J(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) = J(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}_0) \tag{5}$$

for some suitable functional J.

In particular we consider the functionals

$$
\begin{aligned}
J_0(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{u}|^2 + \alpha \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^1} \\
J_1(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0) &= \frac{1}{2}\int_\Omega \left|(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T\right|^2 + \alpha \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^1}, \\
J_2(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{h}|^2 + \alpha \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^1}, \\
J_3(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0) &= \frac{1}{2}\int_\Omega |\nabla \mathbf{u} - \mathbf{u}_d|^2 + \alpha \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^1},
\end{aligned}
$$

$(\mathbf{u},\mathbf{h},\mathbf{w})$ being a weak solution of the problem (1.1) with boundary control $\mathbf{u}_0$, for some suitable function $\mathbf{u}_d$ and $\alpha \geq 0$.

Thus we have the following result.

Theorem 2 *Let $\alpha \geq 0$ and K be a convex subset of $H^{\frac{1}{2}}(\Gamma^0)$. Then the problem (5), with $J = J_0, J_1$, has at least one solution.*

Theorem 3 *Asumme that $\alpha > 0$ and K be a convex subset of $H^{\frac{1}{2}}(\Gamma^0)$ or $\alpha \geq 0$ and K is a bounded convex subset of $H^{\frac{1}{2}}(\Gamma^0)$. Then the problem (5), with $J = J_2, J_3$, has at least one solution.*

3 Preliminaries

Let $\Omega \subseteq I\!R^n, n = 2$ or 3, be a bounded domain with Lipstchitz continuous boundary Γ. We denote by $L^p(\Omega)$ the usual Lebesgue spaces and by $\|\cdot\|_{L^p}$ the L^p-norm on Ω; in the case $p = 2$, we simply denote the L^2-norm by $|\cdot|$ and the corresponding inner product by $(\cdot,\cdot)$.The Sobolev spaces $H^s(\Omega), H_0^s(\Omega)$ (with $s \in I\!R$) are defined as usual; we denote by $\|\cdot\|_s$ and $(\cdot,\cdot)_{H^s}$, respectively the norm and the inner product in $H^s(\Omega)$ (or $H_0^s(\Omega)$) when is appropriate). We also consider the following spaces

$$
\begin{aligned}
&H(\text{div}\,,\Omega) = \left\{\varphi \in L^2(\Omega)^n \,:\, \text{div}\,\varphi \in L^2(\Omega)^n\right\} \\
&H_0(\text{div}\,,\Omega) = \{\varphi \in H(\text{div}\,,\Omega) \,:\, \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\} \\
&H_{\Gamma^i} = \left\{\varphi \in H_0(\text{div}\,,\Omega) \,:\, \mathbf{u} = 0 \text{ on } \Gamma \setminus \Gamma^i\right\} \\
&H = \{\varphi \in H_0(\text{div}\,,\Omega) \,:\, \text{div}\,\varphi = 0\} \\
&V = \left\{\varphi \in H_0^1(\Omega)^n \,:\, \text{div}\,\varphi = 0\right\} \\
&L_0^2(\Omega) = \left\{q \in L^2(\Omega) \,:\, \int_\Omega p(x)dx = 0\right\}.
\end{aligned}
$$

Notice that H is a closed subspace of $L^2(\Omega)^n$ and we have the following decomposition:

$$L^2(\Omega)^n = H \oplus H^\perp, \tag{6}$$

being $H^{\perp}$ the orthogonal of H in $L^2(\Omega)^n$. Moreover, if Ω is connected we have

$$H^{\perp} = \left\{ \nabla q \, : \, q \in H^1(\Omega) \right\}. \tag{7}$$

In addition if Ω is simply-connected we have that

$$H^{\perp} = \left\{ \mathbf{v} \in L^2(\Omega)^n \, : \, \mathrm{rot}\ \mathbf{v} = 0 \right\}. \tag{8}$$

On the other hand, if $\Omega \subset I\!R^n$ is a bounded set with Lipstchitz continuous boundary Γ we have that

$$H^1_0(\Omega)^n = V \oplus V^{\perp}, \tag{9}$$

being $V^{\perp}$ the orthogonal of H in $H^1_0(\Omega)^n$, characterized by

$$V^{\perp} = \left\{ (-\triangle)^{-1} q \, : \, q \in L^2(\Omega) \right\}, \tag{10}$$

being $(-\triangle)^{-1} \in (H^{-1}(\Omega)^n; H^1_0(\Omega)^n)$ the Green's operator related to Dirichlet's homogeneous problem for $-\triangle$ in $I\!R^n$.

Thus we have the following result.

Theorem 4 *([4, Theo. 3.1 and Cor. 3.4])*
Let $\Omega \subset I\!R^n$, $n = 2, 3$, be a bounded connected set with Lipschitz continuous boundary Γ. Then, every function $\mathbf{v} \in L^2(\Omega)^n$ verifies that

$$\mathbf{v} = \nabla q + \mathrm{rot}\ \phi, \tag{11}$$

where $q \in H^1(\Omega)/I\!R$ is the only solution of

$$(\nabla q, \nabla \mu) = (v, \nabla v), \quad \forall \mu \in H^1(\Omega)$$

and $\phi \in \Phi$ where $\Phi = \{\chi \in H^1(\Omega) \, : \, \chi|_{\Gamma_0} = 0, \chi|_{\Gamma_i} = \mathrm{constant}\}$ being Γ_0 the exterior boundary of Ω and Γ_i the other components of Γ, is a solution of

$$(\mathrm{rot}\ \phi, \mathrm{rot}\ \chi) = (v - \nabla q, \mathrm{rot}\ \chi), \quad \forall \phi \in \Phi.$$

Now we give a useful result on the trace operator.

Theorem 5 *([4, Theo. 1.5, p. 8]) Let $\Omega \subset I\!R^n$ be a bounded set with boundary Γ of class $C^{k,1}$, for some integer $k \geq 0$. Let $p \geq 1$ and $s \geq 0$ be two real numbers such that $s \leq k+1$, $s - \frac{1}{p} = l + \sigma$, where $l \geq 0$ is an integer and $0 < \sigma < 1$. Then the mapping*

$$\begin{aligned} \gamma : W^{s,p}(\Omega) \;&\rightarrow\; W^{s-\frac{1}{p},p}(\Gamma) \times W^{s-1-\frac{1}{p},p}(\Gamma) \\ \mathbf{u} \;&\rightarrow\; \gamma \mathbf{u} = (\gamma_0 \mathbf{u}, \gamma_1 \mathbf{u}) = (\mathbf{u}|_{\Gamma}, \frac{\partial \mathbf{u}}{\partial n}|_{\Gamma}), \end{aligned} \tag{12}$$

is an onto linear continuous operator.

On the other hand, let us denote

$$\begin{aligned} a(\mathbf{v},\mathbf{w}) &= \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i}\, dx, \\ b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial v_i}{\partial x_j}\, w_i\, dx, \\ c(\mathbf{w},\mathbf{v}) &= \gamma a(\mathbf{w},\mathbf{v}) + (\alpha+\beta)\int_\Omega (\text{div } \mathbf{w})(\text{div } \mathbf{v}) + 2\chi(\mathbf{w},\mathbf{v}), \end{aligned}$$

which we define for all vector-valued functions $\mathbf{u}, \mathbf{v}, \mathbf{w}$, for which the integrals are well defined.

Now, we need the following important technical result, which is a slight variation of the Hopf's result, the proof is analogous to [4, Lemma 2.3, p.287].

Theorem 6 *Let $\Omega \subset I\!R^n$, with $n = 2, 3$, be a bounded domain with Lipschitz continuous boundary. Then, given a function $\mathbf{g} \in H^{\frac{1}{2}}(\Gamma)^n$, there exists for any $\varepsilon > 0$ a function $\mathbf{u}_\varepsilon \in H^1(\Omega)^n$ such that*

$$\mathbf{u}_\varepsilon|_\Gamma = \mathbf{g}, \tag{13}$$

and for all $\phi, \psi \in V$:

$$|b(\mathbf{u}_\varepsilon, \phi, \psi)| \le \varepsilon \|\phi\|_1 \|\psi\|_1 , \tag{14}$$

$$|b(\phi, \mathbf{u}_\varepsilon, \psi)| \le \varepsilon \|\phi\|_1 \|\psi\|_1 . \tag{15}$$

Moreover, if $\mathbf{g}$ verifies

$$\int_{\Gamma_i} \mathbf{g}\cdot\mathbf{n} = 0, \quad \forall i, \tag{16}$$

we can choose $\mathbf{u}_\varepsilon$ verifying div $\mathbf{u}_\varepsilon = 0$ in Ω.

The following lemma stablishes a first coarse version of De Rham's Theorem.

Lemma 1 *([4, Lemma 2.1, p. 22])*
If $\mathbf{f} \in H^{-1}(\Omega)^n$ satisfies

$$\langle \mathbf{f}, \mathbf{v}\rangle_{H^{-1}(\Omega)^n \times H^1_0(\Omega)^n} = 0, \quad \forall \mathbf{v} \in V, \tag{17}$$

then there exists $p \in L^2_0(\Omega)$ such that

$$\mathbf{f} = \nabla p. \tag{18}$$

Moreover, when Ω is connected, p is unique up to an additive constant.

We can now define a notion of weak solution for (1.1).

Definition 1 *Let* $\mathbf{u}_0, \mathbf{h}_0, \mathbf{w}_0 \in H^{1/2}(\Gamma)^n$ *such that*

$$\mathbf{u}_0 \cdot \mathbf{n} = \mathbf{h}_0 \cdot \mathbf{n} = 0, \qquad on\ \Gamma, \tag{19}$$

being $\mathbf{n}$ *the unitary outward normal vector. We say that a triple of functions* $(\mathbf{u}, \mathbf{w}, \mathbf{h}) \in H_{\Gamma^0} \times H_{\Gamma^1} \times H_{\Gamma^2}$ *is a weak solution of (1.1) if only if they satisfy for all* $(\phi, \varphi, \psi) \in H_{\Gamma^0} \times H_{\Gamma^1} \times H_{\Gamma^2}$

$$(\mu+\chi)a(\mathbf{u},\phi)+b(\mathbf{u},\mathbf{u},\varphi)-(\mu+\chi)(\frac{\partial \mathbf{u}}{\partial \mathbf{n}},\phi)_{\Gamma^0} = rb(\mathbf{h},\mathbf{h},\phi)+\chi(\text{rot } \mathbf{w},\phi)+(\mathbf{f},\varphi), \tag{20}$$

$$c(\mathbf{w},\psi)+jb(\mathbf{u},\mathbf{w},\psi)-(\alpha+\beta)(\text{div } \mathbf{w},\psi\cdot\mathbf{n})_{\Gamma^2}-\gamma(\frac{\partial \mathbf{w}}{\partial \mathbf{n}},\psi)_{\Gamma^2}= \chi(\text{rot } \mathbf{u},\phi)+(\mathbf{g},\psi), \tag{21}$$

$$\nu a(\mathbf{h},\varphi) - \nu(\frac{\partial \mathbf{h}}{\partial \mathbf{n}},\varphi)_{\Gamma^1} + b(\mathbf{u},\mathbf{h},\varphi) - b(\mathbf{h},\mathbf{u},\varphi) = 0. \tag{22}$$

4 Existence of weak solutions

In this section we give a proof of the existence of weak solutions of the problem (1.1).

Theorem 7 *([4, theo. 1.2, p. 280]) Let* X *be a Hilbert space and we consider the operator*

$$b : X \times X \times X \to I\!R,$$

where for each $\mathbf{w} \in X$*, the mapping* $(\mathbf{u}, \mathbf{v}) \to b(\mathbf{w}, \mathbf{u}, \mathbf{v})$ *is a bilinear form on* $X \times X$*. Let* M *be a normed space and* $B \in (X, M')$ *and* $V =$ Ker (B).

Assume that there exists $\alpha > 0$ *such that*

$$b(\mathbf{v}, \mathbf{v}, \mathbf{v}) \geq \alpha \left\|\mathbf{v}\right\|_X^2, \qquad \forall \mathbf{v} \in V.$$

Let V *be a separable space and, for all* $\mathbf{v} \in V$*, the mapping*

$$\mathbf{u} \to b(\mathbf{u}, \mathbf{u}, \mathbf{v})$$

is sequentially weakly continuous.

Then, the problem: Find $\mathbf{u} \in V$ *such that*

$$b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{l}, \mathbf{v})$$

with $\mathbf{l} \in X'$*, has at least one solution.*

Now, we prove Theorem 1.

Let $\mathbf{u}_\Gamma, \mathbf{h}_\Gamma, \mathbf{w}_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ be functions verifying

$$\begin{array}{llll} \mathbf{u}_\Gamma = \mathbf{u}_0, & \text{on } \Gamma^0, & \mathbf{u}_\Gamma = 0, & \text{on } \Gamma \setminus \Gamma^0 \\ \mathbf{h}_\Gamma = \mathbf{h}_0, & \text{on } \Gamma^1, & \mathbf{h}_\Gamma = 0, & \text{on } \Gamma \setminus \Gamma^1, \\ \mathbf{w}_\Gamma = \mathbf{w}_0, & \text{on } \Gamma^2, & \mathbf{w}_\Gamma = 0, & \text{on } \Gamma \setminus \Gamma^2. \end{array} \tag{23}$$

Therefore, from Theorem 6, there exists functions $\mathbf{u}_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{w}_\varepsilon \in H^1(\Omega)^n$ such that

$$\mathbf{u}_\varepsilon = \mathbf{u}_\Gamma, \quad \mathbf{h}_\varepsilon = h_\Gamma, \quad \mathbf{w}_\varepsilon = \mathbf{w}_\Gamma, \quad \text{on } \Gamma, \tag{24}$$

$$\text{div } \mathbf{u}_\varepsilon = \text{ div } \mathbf{h}_\varepsilon = 0, \quad \text{in } \Omega, \tag{25}$$

and for all $\varphi, \phi \in V$ we have that

$$\begin{array}{ll} |b(\phi, \mathbf{u}_\varepsilon, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1, & |b(\mathbf{u}_\varepsilon, \phi, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1 \\ |b(\phi, \mathbf{h}_\varepsilon, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1, & |b(\mathbf{h}_\varepsilon, \phi, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1 \\ |b(\phi, \mathbf{w}_\varepsilon, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1, & |b(\mathbf{w}_\varepsilon, \phi, \varphi)| \leq \varepsilon \|\phi\|_1 \|\varphi\|_1. \end{array} \tag{26}$$

Thus, we can write

$$\mathbf{u} = \mathbf{u}_\varepsilon + \hat{\mathbf{u}}, \quad \mathbf{h} = \mathbf{h}_\varepsilon + \widehat{\mathbf{h}}, \quad \mathbf{w} = \mathbf{w}_\varepsilon + \widehat{\mathbf{w}}, \tag{27}$$

being $\hat{\mathbf{u}}, \widehat{\mathbf{h}} \in V$ and $\widehat{\mathbf{w}} \in H_0^1(\Omega)^n$.

From (27) we can write the problem (**??**)-(**??**) as follows.

To find $(\hat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}}) \in V \times V \times H_0^1(\Omega)$ *such that for all* $(\varphi, \phi, \psi) \in V \times V \times H_0^1(\Omega)$ *we have that:*

$$\begin{array}{l} (\mu + \chi) a(\hat{\mathbf{u}}, \varphi) + b(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \varphi) + b(\mathbf{u}_\varepsilon, \hat{\mathbf{u}}, \varphi) + b(\hat{\mathbf{u}}, \mathbf{u}_\varepsilon, \varphi) \\ = \chi(\textit{ rot } \widehat{\mathbf{w}}, \varphi) + r b(\widehat{\mathbf{h}}, \widehat{\mathbf{h}}, \varphi) + r b(\mathbf{h}_\varepsilon, \widehat{\mathbf{h}}, \varphi) + r b(\widehat{\mathbf{h}}, \mathbf{h}_\varepsilon, \varphi) + (\mathbf{f}_\varepsilon, \varphi), \end{array} \tag{28}$$

$$\begin{array}{l} j b(\hat{\mathbf{u}}, \widehat{\mathbf{w}}, \psi) + c(\widehat{\mathbf{w}}, \psi) + j b(\mathbf{u}_\varepsilon, \widehat{\mathbf{w}}, \psi) + j b(\hat{\mathbf{u}}, \mathbf{w}_\varepsilon, \psi) \\ = (\mathbf{g}_\varepsilon, \psi) + \chi(\textit{ rot } \hat{\mathbf{u}}, \psi) \end{array} \tag{29}$$

$$\begin{array}{l} \nu a(\widehat{\mathbf{h}}, \phi) + b(\hat{\mathbf{u}}, \widehat{\mathbf{h}}, \phi) - b(\widehat{\mathbf{h}}, \hat{\mathbf{u}}, \phi) \\ = b(\mathbf{h}_\varepsilon, \hat{\mathbf{u}}, \phi) + b(\hat{\mathbf{u}}, \mathbf{u}_\varepsilon, \phi) - b(\widehat{\mathbf{h}}, \mathbf{u}_\varepsilon, \phi) - b(\mathbf{h}_\varepsilon, \hat{\mathbf{u}}, \phi) + (\kappa_\varepsilon, \phi), \end{array} \tag{30}$$

where the functions $\mathbf{f}_\varepsilon, \mathbf{g}_\varepsilon \in V'$ *and* $\kappa_\varepsilon \in H^{-1}(\Omega)^n$ *are defined by*

$$\begin{array}{ll} \mathbf{f}_\varepsilon & = \mathbf{f} + \chi \textit{ rot } \mathbf{w}_\varepsilon + r(\mathbf{h}_\varepsilon \cdot \nabla \mathbf{h}_\varepsilon) - \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + (\chi + \mu) \triangle \mathbf{u}_\varepsilon, \\ \mathbf{g}_\varepsilon & = \mathbf{g} + \chi \textit{ rot } \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + \gamma \triangle \mathbf{w}_\varepsilon - 2\chi \cdot \mathbf{w}_\varepsilon + (\alpha + \beta) \nabla \textit{ div } \mathbf{w}_\varepsilon, \\ \kappa_\varepsilon & = \nu \triangle \mathbf{h}_\varepsilon + \mathbf{h}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon) - \mathbf{u}_\varepsilon \cdot \nabla \mathbf{h}_\varepsilon. \end{array} \tag{31}$$

We define the operators

$$
\begin{aligned}
a_1 : (V \times V \times H_0^1(\Omega)^n)^3 &\longrightarrow I\!R \\
a_1((\mathbf{u},\mathbf{h},\mathbf{w}),(\mathbf{u}',\mathbf{h}',\mathbf{w}'),(\varphi,\phi,\psi)) &= (\mu+\chi)a(\mathbf{u},\varphi)+b(\mathbf{u},\mathbf{u}',\varphi)+b(\mathbf{u}_\varepsilon,\mathbf{u},\varphi)+b(\mathbf{u},\mathbf{u}_\varepsilon,\varphi) \\
&\quad -\chi(\text{ rot } \mathbf{w},\varphi)-rb(\mathbf{h},\mathbf{h}',\varphi)-rb(\mathbf{h}_\varepsilon,\mathbf{h},\varphi)-rb(\mathbf{h},\mathbf{h}_\varepsilon,\varphi),
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
a_2 : (V \times V \times H_0^1(\Omega)^n)^2 &\longrightarrow I\!R \\
a_2((\mathbf{u},\mathbf{h},\mathbf{w}),(\varphi,\phi,\psi)) &= jb(u,w,\psi)+c(w,\psi)+ \\
&\quad jb(u_\varepsilon,w,\psi)+jb(u,w_\varepsilon,\psi)-\chi(\text{rot}u,\psi),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
a_3 : (V \times V \times H_0^1(\Omega)^n)^2 &\longrightarrow I\!R \\
a_3((\mathbf{u},\mathbf{h},\mathbf{w}),(\varphi,\phi,\psi)) &= \nu a(\mathbf{h},\phi)+b(\mathbf{u},\mathbf{h},\phi)-b(\mathbf{h},\mathbf{u},\phi)-b(\mathbf{h}_\varepsilon,\mathbf{u},\phi)+ \\
&\quad b(\mathbf{u},\mathbf{h}_\varepsilon,\phi)+b(\mathbf{h},\mathbf{u}_\varepsilon,\phi)+b(\mathbf{h}_\varepsilon,\mathbf{u},\phi).
\end{aligned} \tag{34}
$$

Thus we define the operator

$$
\begin{aligned}
\tilde{a} : (V \times V \times H_0^1(\Omega)^n)^3 &\longrightarrow I\!R \\
\tilde{a}((\mathbf{u},\mathbf{h},\mathbf{w}),(\mathbf{u}',\mathbf{h}',\mathbf{w}'),(\varphi,\phi,\psi)) &= a_1((\mathbf{u},\mathbf{h},\mathbf{w}),(\mathbf{u}',\mathbf{h}',\mathbf{w}'),(\varphi,\phi,\psi)) \\
&\quad +a_2((\mathbf{u},\mathbf{h},\mathbf{w}),(\varphi,\phi,\psi))+ra_3((\mathbf{u},\mathbf{h},\mathbf{w}),(\varphi,\phi,\psi)).
\end{aligned} \tag{35}
$$

We must prove that the map $\tilde{a}$ verifies the hypotheses of Theorem 7.

It is clear, from the compactness of the embedding of $H_0^1(\Omega)^n$ into $L^2(\Omega)^n$, we have that the map $\tilde{a}$ is sequentially weakly continuous on $V \times V \times H_0^1(\Omega)$ (for details, we refer to [4, p. 280]).

On the other hand, notice that

$$
b(\psi,\varphi,\psi)+b(\psi,\phi,\varphi)=0; \qquad b(\mathbf{w},\mathbf{v},\mathbf{v})=0,
$$

for all $\mathbf{u},\mathbf{v} \in H^1(\Omega)^n$ and $\mathbf{w} \in H^1(\Omega)^n$ such that div $\mathbf{w}=0$ and $\mathbf{w}\cdot\mathbf{n}|_\Gamma=0$. Taking $(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\mathbf{w}})=(\mathbf{u}',\mathbf{h}',\mathbf{w}')=(\phi,\varphi,\psi)$, we have that

$$
\begin{aligned}
\tilde{a}((\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\mathbf{w}}),(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\mathbf{w}}),(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\mathbf{w}})) &= (\mu+\chi)a(\hat{\mathbf{u}},\hat{\mathbf{u}})+b(\hat{\mathbf{u}},\hat{\mathbf{u}},\hat{\mathbf{u}})+b(\mathbf{u}_\varepsilon,(\hat{\mathbf{u}},\hat{\mathbf{u}})+b(\hat{\mathbf{u}},\mathbf{u}_\varepsilon,\hat{\mathbf{u}}) \\
&\quad -\chi(\text{ rot } \hat{\mathbf{w}},\hat{\mathbf{u}})-rb(\hat{\mathbf{h}},\hat{\mathbf{h}},\hat{\mathbf{u}})-rb(\mathbf{h}_\varepsilon,\hat{\mathbf{h}},\hat{\mathbf{u}})-rb(\hat{\mathbf{h}},\mathbf{h}_\varepsilon,\hat{\mathbf{u}})+ \\
&\quad jb(\hat{\mathbf{u}},\hat{\mathbf{w}},\hat{\mathbf{w}})+c(\hat{\mathbf{w}},\hat{\mathbf{w}})+jb(\mathbf{u}_\varepsilon,\hat{\mathbf{w}},\hat{\mathbf{w}})+ \\
&\quad jb(\hat{\mathbf{u}},\mathbf{w}_\varepsilon,\hat{\mathbf{w}})-\chi(\text{ rot } \hat{\mathbf{u}},\hat{\mathbf{w}})+r\nu a(\hat{\mathbf{h}},\hat{\mathbf{h}})+rb(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\mathbf{h}}) \\
&\quad +rb(\hat{\mathbf{u}},\mathbf{h}_\varepsilon,\hat{\mathbf{h}})+rb(\mathbf{u}_\varepsilon,\hat{\mathbf{h}},\hat{\mathbf{h}})-rb(\hat{\mathbf{h}},\hat{\mathbf{u}},\hat{\mathbf{h}})-rb(\mathbf{h}_\varepsilon,\hat{\mathbf{u}},\hat{\mathbf{h}}) \\
&\quad -rb(\hat{\mathbf{h}},\mathbf{u}_\varepsilon,\hat{\mathbf{h}}) \\
&= (\mu+\chi)a(\hat{\mathbf{u}},\hat{\mathbf{u}})+c(\hat{\mathbf{w}},\hat{\mathbf{w}})+r\nu a(\hat{\mathbf{h}},\hat{\mathbf{h}})+b(\hat{\mathbf{u}},\mathbf{u}_\varepsilon,\hat{\mathbf{u}}) \\
&\quad -rb(\mathbf{h}_\varepsilon,\hat{\mathbf{h}},\hat{\mathbf{u}})-rb(\hat{\mathbf{h}},\mathbf{h}_\varepsilon,\hat{\mathbf{u}})+jb(\mathbf{u}_\varepsilon,\hat{\mathbf{w}},\hat{\mathbf{w}})+jb(\hat{\mathbf{u}},\mathbf{w}_\varepsilon,\hat{\mathbf{w}}) \\
&\quad +rb(\hat{\mathbf{u}},\mathbf{h}_\varepsilon,\hat{\mathbf{h}})-rb(\mathbf{h}_\varepsilon,\hat{\mathbf{u}},\hat{\mathbf{h}})-rb(\hat{\mathbf{h}},\mathbf{u}_\varepsilon,\hat{\mathbf{h}})
\end{aligned}
$$

Since for all $\varphi \in H_0^1(\Omega)^n$ we have that there exists a constant $\eta > 0$ such that

$$
a(\varphi,\varphi) \geq \eta \, \|\varphi\|_1
$$

and from (26) we conclude that there exists a constant $\lambda > 0$ such that

$$\widetilde{a}((\widehat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}}), (\widehat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}}), (\widehat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}})) \geq \lambda \left\| (\widehat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}}) \right\|_{V \times V \times H_0^1(\Omega)^n}.$$

Choosing $\mathbf{l} = \mathbf{f}_\varepsilon + \mathbf{g}_\varepsilon + r\kappa_\varepsilon$, we have from Theorem 7 that the problem (28)-(31) has at least one weak solution $(\widehat{\mathbf{u}}, \widehat{\mathbf{h}}, \widehat{\mathbf{w}}) \in V \times V \times H_0^1(\Omega)^n$. Thus, the problem (**??**)-(**??**) has at least one weak solution.

Let $(\mathbf{u}, \mathbf{h}, \mathbf{w})$ be a weak solution of the system (**??**)-(**??**). Multipliying (**??**) by $\mathbf{u}$, (**??**) by $\mathbf{w}$ and (**??**) by $\mathbf{h}$ and integrating on Ω we have that

$$b(\mathbf{u},\mathbf{u},\mathbf{u}) + (\mu+\chi)a(\mathbf{u},\mathbf{u}) - (\mu+\chi)(\frac{\partial \mathbf{u}}{\partial \mathbf{n}}, \mathbf{u})_\Gamma = \chi(\text{ rot } \mathbf{w}, \mathbf{u}) + rb(\mathbf{h},\mathbf{h},\mathbf{u}) + (\mathbf{f},\mathbf{u}), \tag{36}$$

$$\begin{aligned} jb(\mathbf{u},\mathbf{w},\mathbf{w}) + \gamma c(\mathbf{w},\mathbf{w}) - \gamma(\frac{\partial \mathbf{w}}{\partial \mathbf{n}}, \mathbf{w})_\Gamma - (\alpha+\beta)(\text{ div } \mathbf{w}, \mathbf{w}\cdot\mathbf{n}) \\ = \chi(\text{ rot } \mathbf{u}, \mathbf{w}) + (\mathbf{g}, \mathbf{w}), \end{aligned} \tag{37}$$

$$\nu a(\mathbf{h},\mathbf{h}) - \nu(\frac{\partial \mathbf{h}}{\partial \mathbf{n}}, \mathbf{h})_\Gamma + b(\mathbf{u},\mathbf{h},\mathbf{h}) - b(\mathbf{h},\mathbf{u},\mathbf{h}) = 0. \tag{38}$$

Since $\text{div } \mathbf{u} = 0$, $\text{div } \mathbf{h} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$, $\mathbf{h}\cdot\mathbf{n} = 0$ on Γ, we have that

$$b(\mathbf{u},\mathbf{h},\mathbf{h}) = b(\mathbf{u},\mathbf{w},\mathbf{w}) = 0, \quad b(\mathbf{h},\mathbf{u},\mathbf{h}) + b(\mathbf{h},\mathbf{h},\mathbf{u}) = 0,$$

thus multipliying (38) by r and adding to (36) and (37) we have that

$$\begin{aligned} (\mu+\chi)a(\mathbf{u},\mathbf{u}) + c(\mathbf{w},\mathbf{w}) + r\nu a(\mathbf{h},\mathbf{h}) = (\mathbf{f},\mathbf{u}) + (\mathbf{g},\mathbf{w}) \\ + (\frac{\partial \mathbf{u}}{\partial \mathbf{n}}, \mathbf{u}_0)_{\Gamma^0} + \gamma(\frac{\partial \mathbf{w}}{\partial \mathbf{n}}, \mathbf{w}_0)_{\Gamma^2} + r\nu(\frac{\partial \mathbf{h}}{\partial \mathbf{n}}, \mathbf{h}_0)_{\Gamma^1} + (\alpha+\beta)(\text{ div } \mathbf{w}, \mathbf{w}_0\cdot\mathbf{n}). \end{aligned}$$

Notice that since Γ^0, Γ^1 are non empty bounded subset of the boundary Γ we have that there exists constants $c_1, c_2 > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq c_1 \|\mathbf{u}\|_1^2\,; \quad a(\mathbf{h},\mathbf{h}) \geq c_2 \|\mathbf{h}\|_1^2$$

and there exists $c_3 > 0$ such that

$$c(\mathbf{w},\mathbf{w}) \geq c_3 \|\mathbf{w}\|_1^2.$$

On the other hand, from the continuity of the trace operator (see Theorem 5) we have that there exists a constant $c_4 > 0$ such that

$$\|\mathbf{u}\|_{0,\Gamma} + \left\|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right\|_{0,\Gamma} \leq c_4 \|\mathbf{u}\|_1,$$

therefore we conclude that there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|_1 + \|\mathbf{h}\|_1 + \|\mathbf{w}\|_1 \leq C\left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{g}\|_{H^{-1}(\Omega)} + \|\mathbf{u}_0\|_{\frac{1}{2},\Gamma^0} + \|\mathbf{h}_0\|_{\frac{1}{2},\Gamma^1} + \|\mathbf{w}_0\|_{\frac{1}{2},\Gamma^2}\right),$$

and the proof is complete.

5 Proof of control results

Now, we will prove Theorem 2. Notice that it is enough to prove the theorem for the functional J_0, the proof is analogous for the other functionals.

Firstly, we can see that $J_0(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) \geq 0$, thus there exists $J^* \in I\!R$ such that

$$\inf_{(\mathbf{u},\mathbf{h},\mathbf{w},\mathbf{u}_0)\in Z_K} J_0(\mathbf{u}, \mathbf{h}, \mathbf{w}, \mathbf{u}_0) = J^*. \tag{39}$$

Let $\{(\mathbf{u}_m, \mathbf{h}_m, \mathbf{w}_m, \mathbf{u}_{0,m})\}_m \subset Z_K$ be a minimizing sequence. Thus we have that there exists a constant $c > 0$ such that for all $m \in I\!N$

$$\|\nabla \mathbf{u}_m\|_{L^2(\Omega)^n} \leq c.$$

Since $\mathbf{u}_m = 0$ on $\Gamma \setminus \Gamma_0$, we have that there exists $M > 0$ such that

$$\|\mathbf{u}_m\|_1 \leq M,$$

therefore we have that the sequence $\{\mathbf{u}_{0,m}\}_{m\in I\!N}$ is bounded in $H^{\frac{1}{2}}(\Gamma^0)^n$.

Hence, from (3) we have that the sequence $\{(\mathbf{u}_m, \mathbf{h}_m, \mathbf{w}_m, \mathbf{u}_{0,m})\}_{m\in I\!N} \subset Z_K$ is bounded in $H^1(\Omega)^n \times H^1(\Omega)^n \times H^1(\Omega)^n \times H^{\frac{1}{2}}(\Gamma^0)^n$. Then there exist $\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}}, \tilde{\mathbf{u}}^0$ such that

$$\begin{array}{rll} \mathbf{u}_m & \to \tilde{\mathbf{u}} & \text{weakly in } H^1(\Omega)^n \text{ and strngly in } L^4(\Omega)^n \\ \mathbf{h}_m & \to \tilde{\mathbf{h}} & \text{weakly in } H^1(\Omega)^n \text{ and strongly in } L^4(\Omega)^n \\ \mathbf{w}_m & \to \tilde{\mathbf{w}} & \text{weakly in } H^1(\Omega)^n \text{ and strongly in } L^4(\Omega)^n \\ \mathbf{u}_{0,m} & \to \tilde{\mathbf{u}}^0 & \text{weakly in } H^{\frac{1}{2}}1(\Gamma^0)^n \text{ and strongly in } L^2(\Gamma^0)^n. \end{array} \tag{40}$$

Since the operator b is weakly sequentally continuous, that is:

$$\lim_{m\to\infty} b(\phi_m, \varphi_m, \psi) = b(\tilde{\phi}, \tilde{\varphi}, \tilde{\psi}),$$

if $\phi_m \to \tilde{\phi}$ weakly in $H^1(\Omega)^n$ and $\varphi_m \to \tilde{\varphi}$ weakly in $H^1(\Omega)^n$, therefore we have that

$$\lim_{m\to\infty} b(\mathbf{u}_m, \mathbf{h}_m, \varphi) = b(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\varphi}).$$

Therefore, from the continuity of the mappings a and c on $H^1(\Omega)^n$ and from the definition of weak solution, we have that $(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}, \tilde{\mathbf{w}})$ is a weak solution of (**??**)-(**??**) satisfying that $\tilde{\mathbf{u}}|_{\Gamma^0} = \tilde{\mathbf{u}}_0$ and $\tilde{\mathbf{u}}|_{\Gamma\setminus\Gamma^0} = 0$.

This complete the proof of Theorem 2.

The proof of Theorem 3 is analogous to the above proof. In this case, if $\alpha > 0$ or $\alpha \geq 0$ and K is bounded, we have that the sequence $\{\mathbf{u}_m^0\}$ is bounded in $H^{\frac{1}{2}}(\Gamma^0)$ and therefore from (3) we obtain that the sequence $\{(\mathbf{u}_m, \mathbf{h}_m, \mathbf{w}_m, \mathbf{u}_{0,m})\}_{m\in I\!N}$ is bounded in $H^1(\Omega)^n \times H^1(\Omega)^n \times H^1(\Omega)^n \times H^{\frac{1}{2}}(\Gamma^0)^n$. Thus we can concluded analogously to the proof of Theorem 2.

References

[1] G. Ahmadi, M. Shahinpoor UNIVERSAL STABILITY OF MAGENTO-MICROPOLAR FLUID MOTIONS, Int. J. Enging Sci., **12**, 657-663 (1974).

[2] G. V. Alekséev. SOLVABILITY OF STATIONARY BOUNDARY CONTROL FOR HEAT CONVECTION EQUATIONS Siberian Mathematical Journal **39** (5), 844-858 (1998).

[3] V. Alexséev, V. Tikhomirov, S. Fomine. COMMANDE OPTIMALE, Ed. MIR (French translation), 1982.

[4] V. Girault, P.-A. Raviart. FINITE ELEMENT METHODS FOR NAVIER-STOKES EQUATIONS, Springer series in computational mathematics 5, Springer-Verlag, Berlin, 1986.

[5] I. V. Girsanov. LECTURES ON MATHEMATICAL THEORY OF EXTREMUM PROBLEMS, Lectures Notes in Economics and Mathematical Systems 67, Springer-Verlag, Berlin, 1972.

[6] A. D. Ioffe, V. M. Tihomirov THEORY OF EXTREMAL PROBLEMS, North-Holland Publishing Company, Amsterdam 1979.

[7] M. A. Rojas-Medar, J. L. Boldrini. MAGNETO-MICROPOLAR FLUID MOTION: EXISTENCE OF WEAK SOLUTIONS, Rev. Mat. Univ. Complutense de Madrid, V. 11 (2),(1998), 443-460.