The Hopf invariant conjecture and the homology of manifolds

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We show that a problem proposed by A. Bassi, about the Betti numbers of a manifold has no solution. J. Adem's work provided a partial solution. This is accomplished by adapting Atiyah's proof of the Hopf invariant conjecture. We also provide conditions for a graded group to be the homology of a closed, connected and orientable manifold, generalizing a theorem due to A. Bassi.

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1 Introduction

We work with CAT=DIFF, PL or TOP. If CAT is omitted, we mean TOP.

Let M^m be a closed, connected and orientable CAT manifold. The Betti numbers of M, $\beta_i(M) = rank H_i(M), i = 0, \dots, m$, satisfy by Poincaré duality, the identities below

$$\beta_i(M) = \beta_{m-i}(M), i = 0, \dots, m.$$

In addition, $\beta_{m/2}(M)$ is even for $m = 2 \bmod 4$.

In [Ba], A. Bassi have shown that if $\beta_o, \dots, \beta_m$ is a sequence of non-negative integers satisfying: $\beta_0 = 1, \beta_i = \beta_{m-i}, i = 0, \dots, m$ and $\beta_{m/2}$ is even for $m = 2 \bmod 4$, then for $m \neq 0 \bmod 4$ or $m = 4, 8$: There exists a closed, connected and orientable PL manifold such that $\beta_i(M) = \beta_i, i = 0, \dots, m$. An elementary proof can be provided for CAT=DIFF by using the connected sum #. For that purpose, we need a lemma.

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1.1. Lemma. Let $M^m, N^m, m > 1$, be closed, connected and orientable CAT manifolds. Then, $M\#N$ is a closed, connected, orientable CAT manifold and

$$H_i(M\#N) \cong H_i(M) \oplus H_i(N), 0 < i < m.$$

Proof. It is an exercise in homology. □

1.2. Theorem (A. Bassi). If $\beta_0, \ldots, \beta_m$ is a sequence of non-negative integers satisfying: $\beta_0 = 1, \beta_i = \beta_{m-i}, i = 0, \ldots, m$ and $\beta_{m/2}$ is even for $m = 2 \bmod 4$, then $\beta_i(M^m) = \beta_i, i = 0, \ldots, m$ where M^m is a closed, connected and orientable DIFF manifold. We are assuming $m \neq 0 \bmod 4$ or $m = 4, 8$.

Proof. For $m \neq 0 \bmod 4$, set $n = [\frac{m}{2}]$ and

$$M = \#_{i=0}^{n} \#_{j=1}^{\beta_i} S^i \times S^{m-i} \times \{j\}.$$

By the above lemma, $\beta_i(M) = \beta_i, 0 \leq i \leq n$. It follows that $\beta_i(M) = \beta_i, 0 \leq i \leq m$. The cases $m = 4, 8$ are similar. However, one must use the manifolds $\mathbb{C}P^2, \mathbb{H}P^2$ (the complex and quaternionic projective planes) in addition to generalized tori $S^i \times S^j, i + j = m$.

1.3. Observation. By using Hopf's octonionic projective plane $\mathbb{O}P^2$ ([Wh] p. 700), one can have in theorem 1.2, $m = 16$.

In [Ba], Bassi observed that the problem of extending theorem 1.2 to all m with CAT=PL, reduces to the problem of constructing for any positive integer $m, m = 0 \bmod 4$, a closed, connected and orientable PL manifold M^m such that $\beta_i(M) = 1$, for $i = 0, \frac{m}{2}, m$ and 0, otherwise. These manifolds are referred in [Ba] as "elementary manifolds of the third kind". In [Ba], Bassi proposed the problem of constructing such manifolds for all dimensions $m = 0 \mod 4, m > 8$. Of course we may take CAT=DIFF, TOP.

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3 Bassi's Problem.

3.1. Remark. Let $M^{4k}, k > 1$, be a closed, connected and orientable manifold. Then, by ([KS] p. 104) M^{4k} possesses a handle decomposition. As M^{4k} is connected, by ([RS] p. 85), we can, after a modification if necessary, assume that this handle decomposition contains just one $0, 4k$-handles. Associated with this handle decomposition, there is a CW-structure ([KS] p. 107) containing just one $0, 4k$-cells.

3.2. Remark. If G is a finitely generated abelian group, we define the *torsion part* $T(G)$ and the *free part* $F(G)$ by $T(G) = \{g \in G | \exists p \in \mathbb{Z}(pg = 0)\}, F(G) = G/T(G)$. The exact sequence $0 \to T(G) \to G \to F(G) \to 0$ splits and produces an identification $G \cong F(G) \oplus T(G)$. Moreover, this identification is natural with respect to homomorphisms and preserves grading and ring structure. Thus, we can write $G = F(G) \oplus T(G)$.

3.3. Remark. Let $M^{4k}, k > 0$, be a closed, connected and orientable manifold. Choose an orientation [M] for M. By Poincaré duality, the bilinear form $B : F(H^{2k}(M)) \times F(H^{2k}(M)) \to \mathbb{Z}$, given by $B(u, v) = \langle u \smile v, [M] \rangle, u, v \in F(H^{2k}(M))$ is a symmetric inner product ([MH] p. 1). By definition, the signature of $M, \sigma(M)$ is the signature of $B, \sigma(B)$. Notice that $\sigma(-M) = -\sigma(M)$. Therefore, the parity of the signature is independent from the choice of orientation. In addition, notice that $\sigma(M) = \beta_{2k}(M)$ mod 2. Symmetric inner products over $\mathbb{Z}$ are classified by their rank, type and signature ([MH] chapter 2). If $\sigma(M)$ is odd, then B must be of type I([MH] p. 24). I follows that there is a $u \in F(H^{2k}(M))$ such that $B(u,u)$ is odd.

The theorem below completely solves Bassi's problem.

3.4. Theorem. Let M^{4k} be a closed, connected and orientable manifold such that $\beta_{2k}(M)$ is odd and $\beta_i(M) = 0, 0 < i < 2k$. Then, $k = 1, 2, 4$.

Proof. We can work with $k > 2$. Assume $k \neq 4$. We will generate a contradiction.

By remark 3.1, M^{4k} is endowed with a CW-structure. By remark 3.3, there is an element $u \in H^{2k}(M)$ such that $u \smile u = h[M]^*$, where h is an odd integer and $[M]^* \in H^{4k}(M)$ is a fundamental cohomology class, algebraically dual to an (orientation) fundamental class of M. Next, represent u by a cellular chain and choose a $2k$-cell e^{2k} in it. Let X be the CW-complex obtained by collapsing all the cells e of M such that $\dim e \leq 2k, e \neq e^{2k}$. It turns out that the cohomology ring of $X, H(X)$ possesses just two torsion free generators c, d such that $c^2 = hd, \dim c = 2k$. The generator c is represented by the only one $2k$-dimensional cell of X, e^{2k}. Similarly for d. Notice that $e^0 \cup e^{2k}$ is a sphere S^{2k}, where e^0 is the only one 0-dimensional cell of X.

Consider the exact sequence of maps $S^{2k} \xrightarrow{i} X \xrightarrow{j} X/S^{2k}$, where i is the inclusion and j is the colapse. We have a commutative diagram

$$
\begin{array}{ccccc}
0 \leftarrow \widetilde{K}(S^{2k}) & \overset{i*}{\longleftarrow} & \widetilde{K}(X) & \overset{j*}{\longleftarrow} & \widetilde{K}(X/S^{2k}) \leftarrow 0 \\
ch \downarrow & & ch \downarrow & & ch \downarrow \\
0 \leftarrow \widetilde{H}^{ev}(S^{2k};Q) & \overset{i*}{\longleftarrow} & \widetilde{H}^{ev}(X;Q)) & \overset{j*}{\longleftarrow} & \widetilde{H}^{ev}(X/S^{2k};Q) \leftarrow 0
\end{array}
$$

$\widetilde{K}(\ ), \widetilde{H}^{ev}(\ ;Q)$ mean reduced K-theory and reduced even dimensional rational cohomology ring respectively [Hu]. The horizontal maps are induced and the vertical ones are Chern characters [Hu]. It is straightforward to verify that the rows are exact sequences except for the surjectivity of the top i^* and the injectivity of the top j^*. Let us prove that.

As $\widetilde{K}^1(S^{2k}) = \widetilde{K}(\Sigma S^{2k}) \cong \widetilde{K}(S^{2k+1}) \cong 0$ (Σ means suspension) the top $j*$ is injective. The top $j*$ is surjective. To see that, attach cells to X of dimension at least $2k+2$ ad infinitum to kill the homology of X in dimensions bigger than $2k$. The process yields a CW-complex Y containing X such that if $\bar{i} : X \to Y$ is the inclusion, then $\bar{i} \circ i$ is a homology equivalence. As both X, Y are simply connected, by the Hurewicz's and Whitehead's theorems [Wh] it follows that $\bar{i} \circ i$ is a homotopy equivalence. Therefore $(\bar{i} \circ i)^* : \widetilde{K}(Y) \to \widetilde{K}(S^{2k})$ is an isomorphism. Conclusion: $i^* : \widetilde{K}(X) \to \widetilde{K}(S^{2k})$ is surjective.

Now, by Atiyah and Hirzebruch's version of Bott periodicity based on integrality ([Hu] p. 280) it follows that $ch : K(S^{2k}) \to H^{ev}(S^{2k};Q)$ is a monomorphism onto $F(\widetilde{H}^{ev}(S^{2k}))$. In addition, $i^* : H^{2k}(S^{2k}) \leftarrow H^{2k}(X)$ is an isomorphism. It follows that there is an element $a \in \widetilde{K}(X)$ such that $cha = c$.

Next, as M has just one top cell, the same holds for X. So, let e^{4k} be the top cell of X. Let D be a disk in e^{4k}. Let $\pi : X \to S^{4k}$ be the colapse map $X \to X/(X - D) = D/\partial D$ followed by a homomorphism $D/\partial D \to S^{4k}$.

The exact sequence of maps $D \to X/S^{2k} \xrightarrow{\pi} S^{4k}$ induces a commutative diagram ($\widetilde{K}(D) \cong 0$) where the rows are isomorphisms

$$\begin{array}{ccc} \widetilde{K}(X/S^{2k}) & \stackrel{\pi^*}{\longleftarrow} & \widetilde{K}(S^{4k}) \\ ch \downarrow & & ch \downarrow \\ \widetilde{H}^{ev}(X/S^{2k};Q) & \stackrel{\pi^*}{\longleftarrow} & \widetilde{H}^{ev}(S^{4k};Q) \end{array}.$$

In addition $H^{4k}(X/S^{2k}) \stackrel{\pi^*}{\leftarrow} H^{4k}(S^{4k})$ is an isomorphism. Therefore by Atiyah and Hirzebruch's integrality result again, we obtain an element $b \in \widetilde{K}(X)$ such that $chb = c$.

By tensoring with Q the exact sequence

$$0 \leftarrow \widetilde{K}(S^{2k}) \stackrel{i^*}{\leftarrow} \widetilde{K}(X) \stackrel{j^*}{\leftarrow} \widetilde{K}(X/S^{2k}) \leftarrow 0,$$

it follows that $a, b \in \widetilde{K}(X)$ are the only torsion-free elements of $\widetilde{K}(X)$. As $cha^2 = hchb$, we conclude that $ch(a^2 - hb) = 0$. Therefore $a^2 - hb$ is a torsion element of $\widetilde{K}(X)$.

Now, assume that $f : G \to H$ is a homomorphism of finitely generated abelian groups. Then, f maps torsion elements to torsion elements and therefore, there is an induced homomorphism

$$F(f) : F(G) \to F(H).$$

We can apply this procedure to Adam's K-theoretic operations [Hu]. Its properties will remain because although $F(\ )$ does not preserve exact sequences, it does preserve the tensor sum and the direct product. Therefore, Atiyah's proof of the Hopf invariant conjecture as modified by Husemoller (in lemma 4.2 p. 137 [Hu], the Hopf invariant may be odd) will work on $F(K(X))$ and will generate an absurd because $k \neq 1, 2, 4$. Conclusion: X can not exist. Therefore $\beta_i(M) > 0, i \neq 0, 2k, 4k$ even. By Poincaré duality $\beta_i(M) > 0, 0 < i < 2k$.

4 Generalization of Bassi's Theorem

Denote by $T_*(M), F_*(M) = H_*(M)/T_*(M)$, the torsion part and the free part of the homology $H_*(M)$ of a closed, connected and orientable m-dimensional CAT manifold M^m, respectively. By Poincaré duality $F_i(M) \cong F_{m-i}(M), T_i(M) \cong T_{m-i-1}(M)(T_{-1}(M) = 0), i = 0, \ldots, m$. Poincaré duality implies that the intersection form is non-singular and so $F_{m/2}(M) \cong F \oplus F, F$ abelian for $m = 2$ mod 4. Furthermore, Poincaré duality implies that for $m = 1$ mod 4 $T_{(m-1)/2}(M) \cong T \oplus T$ or $T \oplus T \oplus \mathbb{Z}_2, T$ abelian [Bro].

Of course $F_0(M) \cong \mathbb{Z}, T_0(M) = 0$. Section 2 implies that these restrictions on $H_*(M)$ are not the only ones.

5.1. Lemma. Given $m, k, s \in \mathbb{Z} \cap (0, \infty)$ with $s \geq 2, m < 2$, there is a DIFF closed manifold $M^m = M^m(s, k)$ such that

$$H_i(M^m) \cong \begin{cases} \mathbb{Z}, i = 0, m \\ \mathbb{Z}_s^{n(i)}, i = k, m-k-1, \quad \text{where} \\ 0, i \neq 0, k, m-k-1, m \end{cases}$$

$$n(k) = n(m-k-1) = 1 \quad \text{if} \quad k \neq m-k-1, 2 \quad \text{if} \quad k = m-k-1.$$

Proof. Let $M^m = \chi(S^k \times S^{m-k}, f)(\chi(\ ,\ )$ means the result of performing surgery [Mi] where f is such that the composite map

$$S^k \times 0 \subset S^k \times D^{m-k} \xrightarrow{f} S^k \times S^{m-k} \xrightarrow{\text{projection}} S^k,$$

has degree s.

To construct f, first choose a link L on S^m, such that L is composed of $k, m-k-1$ dimensional unknotted subspheres K^k, K^{m-k-1} with linking number s (apply the suggestion in [RS] p. 72 for CAT=DIFF). We have,

$$S^m = S^k \times D^{m-k} \cup D^{k+1} \times S^{m-k-1}.$$

After an ambient isotopy, if necessary, we can assume that $K^{m-k-1} = 0 \times S^{m-k-1}, K^k \subset int(S^k \times D^{m-k})$. Thus, $K^k \subset S^k \times D^{m-k} \subset S^k \times D^{m-k} \cup_{id} S^k \times D^{m-k} = S^k \times S^{m-k}$. Choose a closed tubular neighborhood T of K^k such that $T \subset int(S^k \times D^{m-k})$. Choose a diffeomorphism of pairs $\varphi : (S^k \times D^{m-k}, S^k \times 0) \to (T, K^k)$. By definition, f is φ with its codomain enlarged to $S^k \times S^{m-k}$. The definition of linking numbers in terms of degree gives that f has the required homological property. Now, it is an exercise in homology to verify that $H_*(M^m)$ is given up to isomorphism by the above formulas. □

5.2. Lemma. Let $s, k \in \mathbb{Z} \cap (0, \infty), s \geq 2, k$ odd. There is a DIFF closed manifold of dimension $2k+1, L = L^{2k+1}(s)$ such that

$$H_i(L) \cong \begin{cases} \mathbb{Z}, i = 0, 2k+1 \\ \mathbb{Z}_s, i = k \\ 0, i \neq 0, k, 2k+1 \end{cases}$$

Proof. Set $L = L^{2k+1}(s) = S^k \times D^{k+1} \cup_h S^k \times D^{k+1}$, where $h : S^k \times S^k \to S^k \times S^k$ is a diffeomorphism to be chosen later. It is an exercise in homology to verify that $H_i(L), i \neq k$, up isomorphism is given by the above formulas.

Choose a fundamental class for $S^k, [S^k] \in H_k(S^k)$. Let $i_1, i_2 : S^k \to S^k \times S^k, j : S^k \to S^k \times D^k$ be the maps specified by the equations below

$$i_1(x) = (x, 1), i_2(x) = (1, x), i(x) = (x, 1), x \in S^k.$$

Set

$$p = i_{1*}([S^k]) \in H_k(S^k \times S^k), q = i_{2*}([S^k]) \in H_k(S^k \times S^k).$$

Notice that we can identify p with $j_*([S^k]) \in H_k(S^k \times D^{k+1})$. In the classical $k = 1$ case, p, q are referred to as a longitude and a meridian respectively of $S^k \times D^{k+1}$ ([Ro] p.29). By ([Bre1] p. 51) there is a diffeomorphism h such that

$$h_*(p) = (1+s)p + sq, h_*(q) = -sp + (1-s)q.$$

The Mayer-Vietoris exact sequence gives

$$H_k(S^k \times S^k) \to H_k(S^k \times D^{k+1}) \oplus H_k(S^k \times D^{k+1}) \to H_k(L) \to 0.$$

We have: The map on the left maps p to $(p, (1+s)p)$ and q to $(p, (1+s)p)$ and q to $(0, -sp)$. Now, set $x = (p, 0), y = (0, p).H_k(L)$ is generated by both x, y. We have: $x - (1+s)y = 0, sy = 0$. This means that $x = (1+s)y$, that is, y generates $H_k(L)$. In addition, y has order s. Thus, $H_k(L) \cong \mathbb{Z}_s$. □

5.3. Remark. A diffeomorphism h as above and such that $h_*(p) = ap + cq, h_*(q) = bp + dq$ must satisfy the condition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

unimodular. In addition ([Bre2] p. 404) for $k \neq 1, 3, 7$ the Hopf invariant imposes the condition ab, cd even. There are no further conditions to look for ([Wa] p. 426). For the case k even, the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can take only the forms ([Bre2 p. 333)

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

5.4. Remark. By performing surgery on a closed, connected and orientable DIFF 5-dimensional manifold discovered by Wu ([Sm], [Wu]) one obtains a closed, connected and orientable DIFF manifold W^5 such that

$$H_i(W^5) \cong \begin{cases} \mathbb{Z}, i = 0, 5 \\ \mathbb{Z}_2, i = 2 \\ 0, i \neq 1, 3, 4 \end{cases}$$

We call W^5 the Wu manifold. Let $k \in \mathbb{Z} \cap (0, \infty)$. Let M^{4k+1} be a closed, connected and orientable CAT manifold such that

$$H_i(M) \cong \begin{cases} \mathbb{Z}, i = 0, 4k+1 \\ \mathbb{Z}_2, i = 2k \\ 0, i \neq 0, 2k, 2k+1 \end{cases}$$

By the theorem of [LMP], M must satisfy: $w_2(M) \neq 0, w_{4k-1}(M) \neq 0$. In particular, $H_2(M; \mathbb{Z}_2) \neq 0, H_{4k-1}(M; \mathbb{Z}_2) \neq 0$. Conclusion: $k = 2$.

By forming appropriate connected sums of the manifolds of lemmas 3.1, 3.2, generalized tori, $\mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2$ and W^5 it is straightforward to prove the theorem below by using lemma 1.1.

5.5. Theorem (Generalized Bassi's Theorem). Let $G_* = \{G_i\}_{0 \leq i \leq m}$ be a graded group where G_i is finitely generated and abelian, $0 \leq i \leq m$. Let F_i, T_i be the free and torsion parts of G_i, respectively, $0 \leq i \leq m$. Assume $G_0 \cong \mathbb{Z}, F_i \cong F_{m-i}, T_i \cong T_{m-i-1} (T_{-1} = 0), 0 \leq i \leq m$. For even $m, m \leq 4, 8, 16$, assume $F_{m/2} \cong F \oplus F, F$ abelian. For $m = 1 \bmod 4, m \neq 5$, assume $T_{(m-1)/2} \cong T \oplus T, T$ abelian. For $m = 5$, assume $T_2 \cong T \oplus T$ or $T \oplus T \oplus \mathbb{Z}_2, T$ abelian. Then, there is a closed DIFF manifold M^m such that $H_*(M) \cong G_*$. Here, $m \geq 3$.

For $m < 3$ a theorem such as this one is unnecessary because it is well known that $0, 1, 2$-dimensional closed CAT manifolds have been classified. The above theorem is best possible only for $m = 2, 3 \bmod 4$ or $m = 4, 5, 8, 16$. The problem of improving this theorem to a best possible one is more difficult than Bassi's problem. Here's an easier problem.

Problem. Let M^{4k} be a closed, connected and orientable manifold. Prove or disprove. For k odd, $\beta_*(M^{4k}) \geq \beta_*(CP^{2k})$. For k even, $\beta_*(M^{4k}) \geq \beta_*(\mathbb{H}P^k)$.

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