# On a Combinatorial Result Related to the Rogers-Ramanujan Identities.

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**Abstract**

We give a generating function for partitions with difference conditions and a combinatorial proof for a bijection between these partitions and another class of partitions. New combinatorial interpretations for the Rogers-Ramanujan identities are included as special cases.

In [1], page 59, Andrews present a bijective proof, given by Bressoud, for the following theorem:

**Theorem A:** The number of partitions of $n$ with minimal difference at least 2 between parts equals the number of partitions of $n$ into distinct parts wherein each even part is larger than twice the number of odd parts.

It is clear that this is related to the first Rogers-Ramanujan identity since the left side of (1) is the generating function for partitions as described in the first part of Theorem A.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{((1-q^{5n-1})(1-q^{5n-4})} \tag{1}$$

where we are using the standard notation

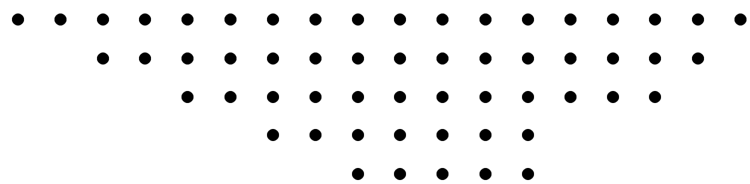
$$\begin{aligned} (a;q)_0 &= 1 \\ (a)_n = (a;q)_n &= (1-a)(1-aq)\dots(1-aq^{n-1}) \end{aligned}$$

The general result that we are going to prove has as special case, not only this Theorem A, but also one related to the second Rogers-Ramanujan identity which is the following:

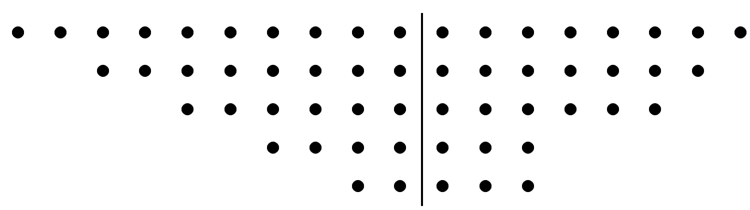
**Theorem 1.** The number of partitions of $n$ with minimal difference at least 2 between parts, with parts greater than 1 equals the number of partitions of $n$ into distinct parts wherein each odd part is larger than 2 plus twice the number of even parts.

The proof for this theorem is similar to the one given by Bressoud for Theorem A.

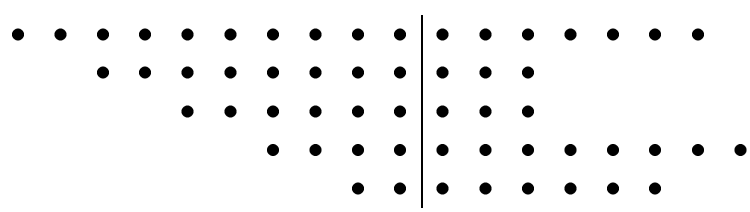
**Proof.** We consider a partition $\pi$ as described in the first part of the theorem. We represent $\pi$ with a modified Ferrers graph in which we indent each now by two nodes. Thus if $\pi : 18 + 15 + 12 + 7 + 5$, our representations is:



We now put a vertical bar in our graph so that to the left are rows of 2, 4, 6, 8, etc. nodes going from botton to top.



We reorder the rows to the right of the bar putting first the rows with an odd number of nodes (in descending order) and then the rows with an even number of nodes (in descending order). Thus our new graph is:



and reading the new rows as parts of a transformed partition we have in this instance $17 + 12 + 11 + 9 + 8$.

It is immediate from our construction that all parts are distinct and that the smallest odd part is larger than 2 plus twice the number of even parts. The process is clearly reversible thus giving us a bijection between the two classes of partitions presented in Theorem 1.

We state, next, our main theorem.

**Theorem 2.** Let $A(n,\ell)$ be the number of partitions of $n$ of the form $n = b_1 + b_2 + \cdots + b_s$ such that $b_j - b_{j+1} \geq 2$ and $b_s > \ell$, and $B(n,\ell)$ be the number of partition of $n$ in distinct parts such that the smallest part is greater than $\ell$ and each part$\equiv \ell (mod\ 2)$ is greater than $2t + \ell + 1$ where $t$ is the number of parts $\equiv \ell + 1 (mod\ 2)$. Then, for $\ell \geq 0$, $A(n,\ell) = B(n,\ell)$ for all $n$ and

$$\sum_{n=0}^{\infty} A(n,\ell) q^n = \sum_{s=0}^{\infty} \frac{q^{s^2+\ell s}}{(q)_s}$$

**Proof.** Let $n = b_1 + b_2 + \cdots + b_s$ be a partition enumerated by $A(n,\ell)$. If we substract $\ell + 1$ from $b_s$, $\ell + 3$ from $b_{s-1}, \ldots, \ell + (2s-1)$ from $b_1$ we are left with a partition of $n - (\ell + 1 + \ell + 3 + \cdots + \ell + (2s-1)) = n - \ell s - s^2$ in at most $s$ parts and this is generated by

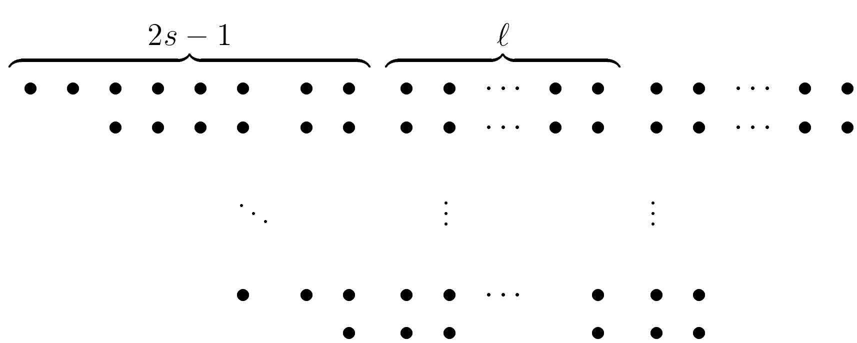
$$\frac{q^{s^2+\ell s}}{(q)_s}, s \geq 1.$$

Hence the generating function for the partitions enumerated by $A(n,\ell)$ is

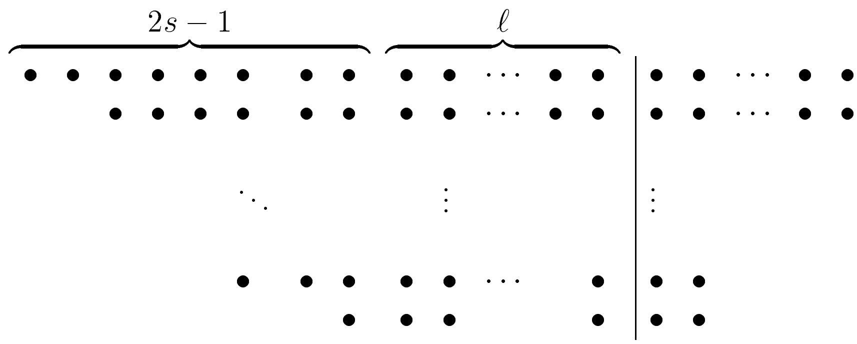
$$1 + \sum_{s=1}^{\infty} \frac{q^{s^2+\ell s}}{(q)_s}$$

Now in order to prove that $A(n,\ell) = B(n,\ell)$ we are going to construct a bijection between the elements enumerated by these two numbers.

We take a partition enumerated by $A(n,\ell)$. Considering that the difference between parts is at least 2 we may represent $\pi$ with a modified Ferrers graph in which we indent each row by two nodes and, in doing so, our representation is:



We now put a vertical bar in our graph so that to the left are rows of $\ell+1, \ell+3, \ldots, \ell+(2s-1)$ nodes going from botton to top.



Now we reorder the rows to the right of the bar putting first the rows with an odd number of nodes and after the rows with an even number of nodes, both in descending order. If we consider now the new rows as parts of a transformed partition it is easy to see that from our construction all parts are distinct, each one is greater than $\ell$ and the smallest part$\equiv \ell(mod\ 2)$ is greater than $2t+\ell+1$ where $t$ is the number of parts $\equiv \ell+1(mod\ 2)$. In fact if there are $r$ parts$\equiv \ell(mod\ 2)$ then the $r$-$th$ is $\geq 2(s-r)+\ell+2 > 2(s-r)+\ell+1$.

What we have described is clearly reversible thus giving us a bijection between the two classes of partitions enumerated by $A(n,\ell)$ and $B(n,\ell)$.

$\square$

We illustrate, below, the partitions enumerated by $A(n,\ell)$ and $B(n,\ell)$ and the correspondence between them given by the bijection described in the theorem for $n=19$ and $\ell=2$.

| $A(19,2)$ | | $B(19,2)$ |
|---|---|---|
| $19$ | $\longleftrightarrow$ | $19$ |
| $16+3$ | $\longleftrightarrow$ | $16+3$ |
| $15+4$ | $\longleftrightarrow$ | $13+6$ |
| $14+5$ | $\longleftrightarrow$ | $14+5$ |
| $13+6$ | $\longleftrightarrow$ | $11+8$ |
| $12+7$ | $\longleftrightarrow$ | $12+7$ |
| $11+8$ | $\longleftrightarrow$ | $10+9$ |
| $11+5+3$ | $\longleftrightarrow$ | $11+5+3$ |
| $10+6+3$ | $\longleftrightarrow$ | $10+6+3$ |
| $9+7+3$ | $\longleftrightarrow$ | $9+7+3$ |
| $9+6+4$ | $\longleftrightarrow$ | $8+6+5$ |

We observe that if in the proof of this theorem we reorder putting first the even ones we get the following result:

**Theorem 3.** Let $C(n, \ell)$ be the number of partitions of $n$ in distinct parts greater than $\ell$ such that each part $\equiv \ell + 1 (mod\ 2)$ is greater than $2r + \ell$ where $r$ is the number of parts $\equiv \ell (mod\ 2)$. Then $C(n, \ell) = A(n, \ell)$ for $\ell \geq 0$.

Finally we observe that the cases $\ell = 0$ and $\ell = 1$ are the special cases described in Theorem $A$ and Theorem 1, respectively, that are related to the Rogers-Ramanujan identities.

# References


[1] Andrews, G. E. (1986) q. Series: *Their Development and Application In Analysis, Number Theory, Combinatorics, Physics, And Computer Algebra*, AMS, Providence n$^{\underline{o}}$ 66.

[2] Bressoud, D.M. (1978). *A New Family of Partition Identities*, Pacific J. Math. 77, 71-74.

[3] __________ (1980). *Analytic and Combinatorial Generalizations of The Rogers-Ramanujan Identities*, Mem. Amer. Math. Soc. 24, n$^{\underline{o}}$ 227.

[4] Bressoud, D. M. and Zeilberger, D. (1982). *A Short Rogers-Ramanujan Bijection*, Discrete Math. 38, 313-315.

[5] MacMahon, P. A. (1916). *Combinatory Analysis*, Vol. 2, Cambridge University Press, London (Reprint: Chelsea, New York, 1960).



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