Covariant Derivatives on Minkowski Manifolds

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Abstract

We present a general theory of covariant derivative operators (linear connections) on a Minkowski manifold (represented as an affine space $(M, \mathcal{M}^*)$ using the powerful multiform calculus. When a gauge metric extensor G (generated by a gauge distortion extensor h) is introduced in the Minkowski manifold, we get a theory that permits the introduction of general Riemann-Cartan-Weyl geometries. The concept of gauge covariant derivatives is introduced as the key notion necessary to generate linear connections that are compatible with G, thus permitting the construction of Riemann-Cartan geometries. Many results of genuine mathematical interest are obtained. Moreover, such results are fundamental for building a consistent formulation of a theory of the gravitational field in flat spacetime. Some important examples of applications of our theory are worked in details.

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1 Introduction

In this paper we study the theory of covariant derivative operators on the Minkowski manifold using the multiform calculus (Hestenes and Sobczyk, 1984: Moya, Fernández, and Rodrigues, 1999, Moya, 1999). Before making use of this formalism, we recall the main ideas and problems discussed in this paper in the language of ordinary tensor calculus[1]. Let $(M,\boldsymbol{\eta},\tau_{\boldsymbol{\eta}})$ be the Minkowski manifold and $(M,\boldsymbol{\eta},\tau_{\boldsymbol{\eta}},D^{\boldsymbol{\eta}})$ Minkowski spacetime (Wu and Sachs, 1993, Rodrigues and Rosa, 1989)[2]. The quadruple $(M,\boldsymbol{\eta},\tau_{\boldsymbol{\eta}},\nabla)$, where ∇ is an arbitrary covariant derivative operator (not necessarily compatible with $\boldsymbol{\eta}$) and such that its torsion and curvature tensors are non zero will be called a *Riemann-Cartan-Weyl (RCW) spacetime*[3]. We can introduce into Minkowski manifold an infinite number of nondegenerate symmetric tensors $\boldsymbol{G}\in\sec T_2^0(M)$ (not necessarily of Lorentzian signature). A quadruple $(M,\boldsymbol{G},\tau_{\boldsymbol{G}},\nabla)$, where ∇ is an arbitrary covariant derivative operator (not necessarily compatible with $\boldsymbol{G}$) and such that its torsion and curvature are non zero will be called a *RCW space*. Such a quadruple will be called a spacetime only if $\boldsymbol{G}$ has Lorentzian signature, in which case it will be denoted by $\boldsymbol{g}$, and will be called a gauge metric field. Now, given an invertible mapping $\boldsymbol{h}:\sec TM\to\sec TM$, called a gauge distortion field, it induces on the Minkowski manifold a nondegenerate symmetric tensor $\boldsymbol{G}$, by

$$\boldsymbol{G}(\boldsymbol{u},\boldsymbol{v})=\boldsymbol{\eta}(\boldsymbol{h}(\boldsymbol{u}),\boldsymbol{h}(\boldsymbol{v})),\forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{sec}TM. \tag{1.1}$$

A *nontrivial* mathematical question is the following: given an arbitrary RCW space (or spacetime) $(M,\boldsymbol{G},\tau_{\boldsymbol{G}},\nabla)$, where $\boldsymbol{G}$ is generated by $\boldsymbol{h}$, construct a

[1] We hope that this will encourage physicists to read the paper.

[2] M is a 4-dimensional manifold oriented by τ_η (the volume element 4-form) and time oriented, which is diffeomorphic to R^4, $\eta\in\sec T_2^0(M)$ is a Lorentzian flat metric and D^η is the Levi-Civita connection of η.

[3] More details and precise definitions using the multiform calculus are given in section 5.

covariant derivative operator from ∇ and $\boldsymbol{h}$, denoted by $\hat{\nabla}$, such that it is compatible with $\boldsymbol{G}$, i.e., $\hat{\nabla}\boldsymbol{G} = 0$. In this paper a solution for this problem is found by introducing the concept of gauge covariant derivatives.

Now, from the physical point of view, our interest in this problem comes from the so called flat spacetime formulations of gravitational theory (Logunov and Mestvirishvili, 1989: Rodrigues and Souza, 1993; Pommaret, 1994), where the gravitational field is supposed to be a field in the Faraday sense and not a manifestation of a geometry, as in Einstein's general relativity. Recently, (Doran, Lasenby and Gull (DLG), 1998) developed a theory of this kind (using the multivector calculus (Hestenes and Sobczyk, 1984)) which they called the gauge theory of gravitation[4]. They believe to have produced a theory that is a generalization of the flat spacetime formulation of Einstein's general relativity. One of the main ingredients of their theory is the introduction of a certain covariant derivative operator that they called also a gauge covariant derivative. However, it happens that what DLG called a gauge covariant derivative is one like ∇ and not one as $\hat{\nabla}$ (this statement is proved below). It follows that in their theory[5] $\nabla g \neq 0$, contrary to their statements (see, in particular Appendix C of (DLG,1993)) and as a consequence their theory has the same deficiency as regards physical interpretation as Weyl theory (Weyl, 1918, 1922, Adler, Bazin and Schiffer, 1965). We are not going to discuss gravitational theories in this paper. We only observe that a gauge theory of gravitational field in flat spacetime, i.e., on Minkowski manifold, incorporating the compatibility of the gauge covariant derivative with g (the gauge metric field) and using the multiform calculus, is given in (Fernández, Moya and Rodrigues, 1999a).

Before proceeding, we would like to observe that one of the authors of the present paper already studied the geometry of Riemann-Cartan-Weyl spaces in (Rodrigues and Souza, 1993; Souza and Rodrigues, 1994; Rodrigues et al, 1995) using the Clifford bundle of differential forms. However, the formalism in the general case is somewhat artificial, the main reason being that a given covariant derivative operator defined in the tensor bundle, in general does not define a covariant derivative operator in the Clifford bundle of differential forms, unless it is compatible with the metric field used in the definition of the Clifford bundle. Thus, in this approach only Riemann-Cartan geometries can be properly studied, the RCW cases being treated in a correct, but artificial way [6]. To overcome this problem it is necessary to develop the multiform calculus in a special way that enhances its power (Moya, Fernández, and Rodrigues,1999, Moya 1999). This can be done once we represent the Minkowski manifold as an *affine space*, with vector space $\mathcal{M}^*$.

Due to limitation of space, in this paper we study only the action of covariant derivative operators on form fields and on $(1,1)$-extensor fields (which are

[4]Applications of his theory appears in several papers, as e.g., (Challinor et al, 1997; Dabrowski et al 1999; Doran et al,1999)

[5]g is a $(1,1)$- extensor field representing a metric $\boldsymbol{g} \in \sec T_2^0(M)$. The definition of (p,q)-extensor fields is given in definition(1.8)

[6]This does not means that a gauge theory of the gravitational field in the sense of (Fernández, Moya and Rodrigues, 1999b) cannot be done in the Clifford bundle formalism.

the objects that represent the tensor fields $\boldsymbol{h}$ and $\boldsymbol{g}$ in our formalism). A study of the action of covariant derivative operators on general multiform and extensor fields, including a generalization of Cartan differential operator acting on exform fields (i.e., completely antisymmetric extensor fields) and Lie derivatives of multiform and extensor fields, is given in (Fernández, Moya and Rodrigues, 1999b). Despite these limitations the results to be presented are really worth studying and we hope that our readers will enjoy them. Finally, it is necessary to recall that the developments that follow can be only understood by readers who are familiar with Clifford algebras as presented, e.g., in (Lounesto, 1997)[7] and the multivector calculus as presented in (Hestenes and Sobczyk, 1984: Moya, Fernández, and Rodrigues, 1999; Moya, 1999).

1.1 Some preliminaries and notations

Given a *global coordinate system* over M, say $M \ni x \leftrightarrow x^\mu(x) \in R$ $(\mu = 0,1,2,3)$ associated to a inertial reference frame (Rodrigues and Rosa, 1989) at $x \in M$. $\langle \frac{\partial}{\partial x^\mu}\big|_x \rangle$ and $\langle dx^\mu|_x \rangle$ are the *natural basis* for the tangent vector space T_xM and the tangent covector space T_x^*M.

We have,

$$\begin{aligned} \boldsymbol{\eta} &= \eta_{\mu\nu} dx^\mu \otimes dx^\nu, \\ \eta_{\mu\nu} &= \boldsymbol{\eta}(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) = diag(1,-1,-1,-1). \end{aligned} \tag{1.2}$$

Definition 1.1. *$T_xM \ni \boldsymbol{v}_x$ is said to be equipolent to $\boldsymbol{v}_{x'} \in T_{x'}M$ (written $\boldsymbol{v}_x = \boldsymbol{v}_{x'}$) if and only if*

$$\boldsymbol{\eta}_{(x)}(\frac{\partial}{\partial x^\mu}\bigg|_x, \boldsymbol{v}_x) = \boldsymbol{\eta}_{(x')}(\frac{\partial}{\partial x^\mu}\bigg|_{x'}, \boldsymbol{v}_{x'}), \ (\mu = 0,1,2,3). \tag{1.3}$$

Note that $\dfrac{\partial}{\partial x^\beta}\bigg|_x = \dfrac{\partial}{\partial x^\beta}\bigg|_{x'}$ $(\beta = 0,1,2,3)$.

Definition 1.2. *The set of equivalent classes of tangent vectors over the tangent bundle,*

$$\mathcal{M} = \{\mathcal{C}_{\boldsymbol{v}_x} \mid \text{for all } x \in M\}, \tag{1.4}$$

has a natural structure of vector space, it is called Minkowski vector space.

Note that $\left\langle \mathcal{C}_{\frac{\partial}{\partial x^\mu}\big|_x} \right\rangle$ is a natural basis for $\mathcal{M}$ $(\dim \mathcal{M} = 4)$. With the notations: $\vec{v} \equiv \mathcal{C}_{\boldsymbol{v}_x}$ and $\vec{e}_\mu \equiv \mathcal{C}_{\frac{\partial}{\partial x^\mu}\big|_x}$, we can write $\vec{v} = v^\mu \vec{e}_\mu$.

[7] It is particularly important to emphasize that in our approach (contrary to the approach in (Hestenes and Sobczyk, 1984), it is necessary to distinguish between contractions and internal products, as done in (Lounesto, 1997; Moya, Fernández and Rodrigues, 1999a; Moya,1999).

Definition 1.3. *The 2-tensor over $\mathcal{M}$,*

$$\eta : \mathcal{M} \times \mathcal{M} \to R, \tag{1.5}$$

such that for each $\vec{v} = \mathcal{C}_{\boldsymbol{v}_x}$ and $\vec{w} = \mathcal{C}_{\boldsymbol{w}_x} \in \mathcal{M} : \eta(\vec{v}, \vec{w}) = \boldsymbol{\eta}_{(x)}(\boldsymbol{v}_x, \boldsymbol{w}_x)$, for all $x \in M$, is called Minkowski metric tensor.

Note that, for each pair of basis vectors $\vec{e}_\mu \equiv \mathcal{C}_{\left.\frac{\partial}{\partial x^\mu}\right|_x}$ and $\vec{e}_\nu \equiv \mathcal{C}_{\left.\frac{\partial}{\partial x^\nu}\right|_x}$, it holds

$$\eta(\vec{e}_\mu, \vec{e}_\nu) = diag(1, -1, -1, -1). \tag{1.6}$$

Definition 1.4. *The dual basis of $\langle \vec{e}_\mu \rangle$ will be symbolized by $\langle \gamma^\mu \rangle$, i.e., $\gamma^\mu \in \mathcal{M}^* \equiv \Lambda^1(\mathcal{M})$ and $\gamma^\mu(\vec{e}_\nu) = \delta^\mu_\nu$.*

To continue, we observe the existence of a fundamental isomorphism between $\mathcal{M}$ and $\Lambda^1(\mathcal{M})$ given by,

$$\mathcal{M} \ni \vec{a} \leftrightarrow a \in \Lambda^1(\mathcal{M}), \tag{1.7}$$

such that if $\vec{a} = a^\mu \vec{e}_\mu$ then $a = \eta_{\mu\nu} a^\mu \gamma^\nu$ and if $a = a_\mu \gamma^\mu$ then $\vec{a} = \eta^{\mu\nu} a_\mu \vec{e}_\nu$, where $\eta_{\mu\nu} = \eta(\vec{e}_\mu, \vec{e}_\nu)$, $\eta^{\mu\nu} = \eta_{\mu\nu}$.

Remark 1. *To each basis vector $\vec{e}_\mu$ correspond a basis form $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$.*

Definition 1.5. *A scalar product of forms can be defined by*

$$\Lambda^1(\mathcal{M}) \times \Lambda^1(\mathcal{M}) \ni (a, b) \mapsto a \cdot b \in R, \tag{1.8}$$

such that if $\vec{a} \leftrightarrow a$ and $\vec{b} \leftrightarrow b$ then $a \cdot b = \eta(\vec{a}, \vec{b})$.

Remark 2. $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$, $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$ *($\langle \gamma_\mu \rangle$ is called the reciprocal basis of $\langle \gamma^\mu \rangle$) and $\gamma^\mu \cdot \gamma^\nu = \eta^{\mu\nu}$. Thus, η admits the expansions $\eta = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu = \eta^{\mu\nu}\gamma_\mu \otimes \gamma_\nu$.*

Remark 3. *The oriented affine space $(M, \mathcal{M}^*)$ (oriented by $\gamma^5 = \gamma^0 \wedge \gamma^1 \wedge \gamma^2 \wedge \gamma^3$) is a representation of the Minkowski manifold.*

Remark 4. *$(M, \mathcal{M}^*)$ equipped with the scalar product given by eq.(1.8) is a representation of Minkowski spacetime.*

Definition 1.6. *Let $\langle x^\mu \rangle$ be a global affine coordinate system for $(M, \mathcal{M}^*)$, relative to an arbitrary point $o \in M$. A position form associated to $x \in M$, is the form over $\mathcal{M}$ (designed by the same letter), given by the following correspondence*

$$M \ni x \leftrightarrows x = x^\mu \gamma_\mu \in \Lambda^1(\mathcal{M}). \tag{1.9}$$

Remark 5. *We denote by $\mathcal{C}\ell(\mathcal{M}) \approx \mathcal{C}\ell(1,3) \approx \boldsymbol{H}(2)$, the spacetime algebra, i.e., the Clifford algebra (Lounesto, 1997) of $\mathcal{M}^*$ equipped with the scalar product defined by eq.(1.8).*

Remark 6. *As a vector space over the reals, we have $\mathcal{C}\ell(\mathcal{M}) = \sum\limits_{p=0}^{4} \Lambda^p(\mathcal{M})$.*

Definition 1.7. *A smooth multiform field A on Minkowski spacetime is a multiform valued function of position form,*

$$\Lambda^1(\mathcal{M}) \ni x \mapsto A(x) \in \Lambda(\mathcal{M}) \tag{1.10}$$

Definition 1.8. *Let $0 \leq p, q \leq 4$. A (p,q)-extensor t is a linear mapping*

$$t : \Lambda^p(\mathcal{M}) \to \Lambda^q(\mathcal{M}). \tag{1.11}$$

Remark 7. *The set of all (p,q)-extensors is denoted by $ext(\Lambda^p(\mathcal{M}), \Lambda^q(\mathcal{M}))$.*

Definition 1.9. *A smooth (p,q)-extensor field t on Minkowski spacetime is a differentiable (p,q)-extensor valued function of position form,*

$$\Lambda^1(\mathcal{M}) \ni x \mapsto t_x \in ext(\Lambda^p(\mathcal{M}), \Lambda^q(\mathcal{M})). \tag{1.12}$$

Definition 1.10. *The a-directional derivative (a is an arbitrary form) of a smooth multiform field X, denoted as $a \cdot \partial X$, is defined by*

$$a \cdot \partial X = \lim \lambda \to 0 \frac{X(x + \lambda a) - X(x)}{\lambda} = \frac{d}{d\lambda} X(x + \lambda a)\bigg|_{\lambda=0}. \tag{1.13}$$

Remark 8. *The γ_μ-directional derivative $\gamma_\mu \cdot \partial X$ coincides with the coordinate derivative $\frac{\partial X}{\partial x^\mu}$. For short, we will use the notation $\partial_\mu \equiv \gamma_\mu \cdot \partial$.*

Definition 1.11. *The gradient, divergence and curl of a smooth multiform field X, respectively denoted by ∂X, $\partial \lrcorner X$ and $\partial \wedge X$, are defined by*

$$\partial X = \gamma^\mu (\partial_\mu X). \tag{1.14}$$
$$\partial \lrcorner X = \gamma^\mu \lrcorner (\partial_\mu X). \tag{1.15}$$
$$\partial \wedge X = \gamma^\mu \wedge (\partial_\mu X). \tag{1.16}$$

Remark 9. *For any X, it holds $\partial X = \partial \lrcorner X + \partial \wedge X$.*

2 Covariant derivative of form fields

Definition 2.1. *Let a be any form. A covariant derivative operator ∇ (or connection) acting on the set of smooth form fields on Minkowski manifold, modeled by $\mathcal{M}^*$, is the mapping*

$$\nabla_a : \hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})] \to \hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})], \tag{2.1}$$

satisfying the axioms: (i) for all scalars α, α' and forms a, a' it holds $\nabla_{\alpha a + \alpha' a'} b = \alpha \nabla_a b + \alpha' \nabla_{a'} b$ ($b \in \hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})]$), (ii) for all smooth scalar fields f, f' and form fields b, b' it holds $\nabla_a(fb + f'b') = (a \cdot \partial f)b + f \nabla_a b + (a \cdot \partial f')b' + f' \nabla_a b'$ ($a \in \Lambda^1(\mathcal{M})$). $\nabla_a b$ is called the directional covariant derivative of the form field b.

Remark 10. *The definition above is not empty. For example, the ordinary directional derivative $a \cdot \partial b$ is a well-defined directional covariant derivative. If h is a smooth $(1,1)$-extensor field which has inverse h^{-1}, then $h^{-1}(a \cdot \partial h(b))$, is also a well-defined directional covariant derivative.*

2.1 Connection extensor fields

We will show that, associated to ∇_a, there exist just two fundamental extensor fields, say $b \mapsto \boldsymbol{\gamma}_a(b)$ and $a \mapsto \Omega(a)$, the first one being of type $(1,1)$ and the second one of type $(1,2)$. They are smooth extensor fields over the Minkowski spacetime.

Proposition 2.2. *There exists a unique smooth* $(1,1)$*-extensor field* $b \mapsto \boldsymbol{\gamma}_a(b)$,

$$\boldsymbol{\gamma}_a(b) = b \cdot \partial_n \nabla_a n, \tag{2.2}$$

such that for any smooth form field b, *it holds*

$$\nabla_a b = a \cdot \partial b + \boldsymbol{\gamma}_a(b). \tag{2.3}$$

Proof. Let b be an arbitrary form field; by using ax. (ii) into def.(2.1), we have

$$\nabla_a b = \nabla_a (b \cdot \gamma^\nu \gamma_\nu) = (a \cdot \partial b \cdot \gamma^\nu)\gamma_\nu + b^\nu \nabla_a \gamma_\nu = a \cdot \partial b + b \cdot \gamma^\nu \nabla_a \gamma_\nu. \tag{2.4}$$

This shows that there exists an $(1,1)$-extensor field, defined as

$$n \mapsto \boldsymbol{\gamma}_a(n) = n \cdot \gamma^\nu \nabla_a \gamma_\nu, \tag{2.5}$$

(note the linearity with respect to $n \in \Lambda^1(\mathcal{M})$) such that for any form field b, it holds

$$\nabla_a b = a \cdot \partial b + \boldsymbol{\gamma}_a(b). \tag{2.6}$$

Now, by applying the *directional derivative operator* $b \cdot \partial_n$ (b is an arbitrary form) on $\nabla_a n = a \cdot \partial n + \boldsymbol{\gamma}_a(n)$, we get

$$b \cdot \partial_n \nabla_a n = b \cdot \partial_n a \cdot \partial n + b \cdot \partial_n \gamma_a(n) = 0 + \boldsymbol{\gamma}_a(b). \tag{2.7}$$

Thus, the $(1,1)$-extensor field $\boldsymbol{\gamma}_a$ is given by

$$b \mapsto \boldsymbol{\gamma}_a(b) = b \cdot \partial_n \nabla_a n.$$

□

Remark 11. $\boldsymbol{\gamma}_a$ *is indeed a smooth* $(1,1)$*-extensor field over Minkowski spacetime, associated to the differential operator* ∇_a. *It is convenient to use the short notations* $\nabla_a b = a \cdot \partial b + \boldsymbol{\gamma}_a(b)$ *and* $\boldsymbol{\gamma}_a(b) = b \cdot \partial_n \nabla_a n$ *for* $\nabla_a b(x) = a \cdot \partial b(x) + \boldsymbol{\gamma}_a|_x (b(x))$ *and* $\boldsymbol{\gamma}_a|_x (b) = b \cdot \partial_{n(x)} \nabla_a n(x)$.

Definition 2.3. *The* $(1,1)$*-extensor field* $\boldsymbol{\gamma}_a$ *will be called first connection extensor field associated to* ∇_a.

Proposition 2.4. *There exists a unique smooth* $(1,2)$*-extensor field* $a \mapsto \Omega(a)$,

$$\Omega(a) = -\frac{1}{2} \partial_n \wedge \nabla_a n \tag{2.8}$$

such that the skew-symmetric part *of* $\boldsymbol{\gamma}_a$ *(i.e.,* $\boldsymbol{\gamma}_{a-} = \frac{1}{2}(\boldsymbol{\gamma}_a - \boldsymbol{\gamma}_a^\dagger)$ *and* $\boldsymbol{\gamma}_a^\dagger$ *is the adjoint*[8]*) of* $\boldsymbol{\gamma}_a$ *can be factorized by*

$$\boldsymbol{\gamma}_{a-}(b) = \Omega(a) \times b, \tag{2.9}$$

for any form field b*.*

Proof. Recall that for any $(1,1)$-extensor, say $v \mapsto t(v)$, the *skew-symmetric part of* t (i.e., $t_-(v) = \frac{1}{2}[t(v) - t^\dagger(v)]$) can be factorized as $t_-(v) = \frac{1}{2}\mathrm{bif}(t) \times v$, where $\mathrm{bif}(t) = -\partial_n \wedge t(n)$ is the so-called *biform of* t.

Thus, for the skew-symmetric part of $\boldsymbol{\gamma}_a$, taking into account the eq.(2.2), we have

$$\boldsymbol{\gamma}_{a-}(b) = \frac{1}{2}\mathrm{bif}(\gamma_a) \times b = (-\frac{1}{2}\partial_n \wedge \boldsymbol{\gamma}_a(n)) \times b = (-\frac{1}{2}\partial_n \wedge \nabla_a n) \times b. \tag{2.10}$$

This implies the existence of a $(1,2)$-extensor field, defined as

$$a \mapsto \Omega(a) = -\frac{1}{2}\partial_n \wedge \nabla_a n, \tag{2.11}$$

(note the linearity with respect to $a \in \Lambda^1(\mathcal{M})$), such that for any form field b, it holds

$$\boldsymbol{\gamma}_{a-}(b) = \Omega(a) \times b.$$

$\square$

Remark 12. Ω *is indeed a smooth* $(1,2)$*-extensor field over the Minkowski manifold, associated to* ∇_a*. Convenient short notations for* $\Omega_x(a) = -\frac{1}{2}\partial_{n(x)} \wedge \nabla_a n(x)$ *and* $\boldsymbol{\gamma}_{a-}\big|_x(b) = \Omega_x(a) \times b$ *are* $\Omega(a) = -\frac{1}{2}\partial_n \wedge \nabla_a n$ *and* $\boldsymbol{\gamma}_{a-}(b) = \Omega(a) \times b$*.*

Definition 2.5. *The* $(1,2)$*-extensor field* Ω *will be called second connection extensor field associated to* ∇_a*.*

3 Associated covariant derivatives

Definition 3.1. *Associated to a given directional covariant derivative operator* ∇_a*, we introduce two other directional covariant derivative operators* ∇_a^- *and* ∇_a^0*, by*

$$\nabla_a^- b = a \cdot \partial b - \gamma_a^\dagger(b). \tag{3.1}$$

$$\nabla_a^0 b = \frac{1}{2}(\nabla_a b + \nabla_a^- b). \tag{3.2}$$

Remark 13. ∇_a^- *and* ∇_a^0 *are in fact operators acting on the set of smooth form fields satisfying the axiomatic of def.(2.1).*

[8]Let m be and arbitrary (p,q)-extensor and let $A \in \Lambda^q(\mathcal{M})$, $B \in \Lambda^p(\mathcal{M})$. The adjoint of m, denoted $m^\dagger$ is the (q,p)-extensor such that $m^\dagger(A) \cdot B = A \cdot m(B)$

Proposition 3.2. *For any smooth form field b, it holds*

$$\nabla_a^0 b = a \cdot \partial b + \gamma_{a-}(b) = a \cdot \partial b + \Omega(a) \times b. \tag{3.3}$$

Proof. By using the eq.(2.3), def.(3.1) and eq.(2.9), we obtain the required result. □

Proposition 3.3. *For any smooth form field b, it holds*

$$\begin{aligned} \nabla_a b &= \nabla_a^0 b + \gamma_{a+}(b). & (3.4)\\ \nabla_a^- b &= \nabla_a^0 b - \gamma_{a+}(b). & (3.5) \end{aligned}$$

Proof. The proof of formulas (3.4) and (3.5) follows directly from eq.(2.3), def.(3.1) and eq.(3.3). □

Proposition 3.4. *For any smooth form fields b, c it holds*

$$\begin{aligned} a \cdot \partial(b \cdot c) &= (\nabla_a b) \cdot c + b \cdot (\nabla_a^- c). & (3.6)\\ a \cdot \partial(b \cdot c) &= (\nabla_a^0 b) \cdot c + b \cdot (\nabla_a^0 c). & (3.7) \end{aligned}$$

Proof. In order to prove the identity (3.6) we must use eq.(2.3) and def.(3.1).

The proof of the identity (3.7) is left to the reader. □

4 Covariant derivative of (1,1)-extensor fields

The differential operator ∇_a acting on smooth form fields can be extended in order to act on smooth multiform fields and on extensor fields. The general theory concerning these extensions is given in (Fernández, Moya and Rodrigues, 1999b)[9]. Here we need only the action of ∇_a on $(1,1)$-extensor fields.

Definition 4.1. *If t is a smooth $(1,1)$-extensor field, then $\nabla_a t$ is another smooth $(1,1)$-extensor field such that, for any smooth form field b, it holds*

$$(\nabla_a t)(b) = \nabla_a^- t(b) - t(\nabla_a b). \tag{4.1}$$

Proposition 4.2. *The extended covariant derivative $t \mapsto \nabla_a t$ satisfies the following fundamental properties:*

For all scalars α, α' and forms a, a' and for any smooth form field b, it holds

$$(\nabla_{\alpha a + \alpha' a'} t)(b) = \alpha(\nabla_a t)(b) + \alpha'(\nabla_{a'} t)(b), \tag{4.2}$$

For all smooth scalar fields f, f' and for any form a and form fields b, b' it holds

$$(\nabla_a t)(fb + f'b') = f(\nabla_a t)(b) + f'(\nabla_a t)(b'), \tag{4.3}$$

[9] In (Fernández, Moya and Rodrigues, 1999b) we introduce also a generalization of the concept of Cartan's differential operator which acts on the set of completely skew-symmetric extensor fields (the so called exform fields).

Proof. It follows from the def.(2.1) and the linearity properties of the smooth $(1,1)$-extensor fields. $\square$

Proposition 4.3. *For any smooth form field b, it holds*

$$(\nabla_a t)(b) = a \cdot \partial t(b) - t(\nabla_a b) - \partial_n (t(b) \cdot \nabla_a n). \tag{4.4}$$

Proof. Let b be an arbitrary smooth form field. Taking into account def.(4.1), def.(3.1) and eq.(2.3) we have

$$\begin{aligned}
(\nabla_a t)(b) &= a \cdot \partial t(b) - \boldsymbol{\gamma}_a^\dagger t(b) - t(\nabla_a b) = a \cdot \partial t(b) - t(\nabla_a b) - \partial_n (\boldsymbol{\gamma}_a^\dagger t(b) \cdot n) \\
&= a \cdot \partial t(b) - t(\nabla_a b) - \partial_n (t(b) \cdot (a \cdot \partial n)) - \partial_n (t(b) \cdot \boldsymbol{\gamma}_a(n)), \\
(\nabla_a t)(b) &= a \cdot \partial t(b) - t(\nabla_a b) - \partial_n (t(b) \cdot \nabla_a n).
\end{aligned}$$

$\square$

Proposition 4.4. *For all smooth form fields b, c it holds*

$$(\nabla_a t)(b) \cdot c = a \cdot \partial (t(b) \cdot c) - t(\nabla_a b) \cdot c - t(b) \cdot (\nabla_a c). \tag{4.5}$$

Proof. Take two arbitrary smooth form fields b, c. Using eq.(4.4), eq.(2.2) and eq.(2.3) we get

$$\begin{aligned}
(\nabla_a t)(b) \cdot c &= (a \cdot \partial t(b)) \cdot c - t(\nabla_a b) \cdot c - \partial_n (t(b) \cdot \nabla_a n) \cdot c \\
&= (a \cdot \partial t(b)) \cdot c + t(b) \cdot (a \cdot \partial c) - t(\nabla_a b) \cdot c \\
&\quad - t(b) \cdot (a \cdot \partial c) - c \cdot \partial_n (t(b) \cdot \nabla_a n) \\
&= a \cdot \partial (t(b) \cdot c) - t(\nabla_a b) \cdot c - t(b) \cdot (a \cdot \partial c) - t(b) \cdot \gamma_a(c), \\
(\nabla_a t)(b) \cdot c &= a \cdot \partial (t(b) \cdot c) - t(\nabla_a b) \cdot c - t(b) \cdot \nabla_a c.
\end{aligned}$$

$\square$

Proposition 4.5. *For all smooth $(1,1)$-extensor field t, it holds*

$$\nabla_a t^\dagger = (\nabla_a t)^\dagger. \tag{4.6}$$

Proof. Let b, c be two arbitrary smooth form fields. Using eq.(4.5) and the fundamental scalar product property in the adjoint $t^\dagger$ of the extensor t, we have

$$\begin{aligned}
(\nabla_a t^\dagger)(b) \cdot c &= a \cdot \partial (t^\dagger(b) \cdot c) - t^\dagger(\nabla_a b) \cdot c - t^\dagger(b) \cdot \nabla_a c \\
&= a \cdot \partial (b \cdot t(c)) - \nabla_a b \cdot t(c) - b \cdot t(\nabla_a c) \\
&= a \cdot \partial (t(c) \cdot b) - t(\nabla_a c) \cdot b - t(c) \cdot \nabla_a b \\
&= (\nabla_a t)(c) \cdot b = c \cdot (\nabla_a t)^\dagger(b).
\end{aligned}$$

This implies that $(\nabla_a t^\dagger)(b) = (\nabla_a t)^\dagger(b)$, that is $\nabla_a t^\dagger = (\nabla_a t)^\dagger$. $\square$

Finally, we will present two very important properties: one for an *identity extensor field* i_d and another for the so-called *gauge metric extensor field* g. In the general gauge theory of gravitation, $g \equiv h^\dagger h$, where h is a smooth $(1,1)$-extensor field which has inverse h^{-1} (h is the so-called *gauge distortion extensor field*).

Proposition 4.6. *If $\boldsymbol{\gamma}_{a+}$ is the symmetric part of $\boldsymbol{\gamma}_a$ ($\boldsymbol{\gamma}_a$ is the first connection extensor field associated to any covariant derivative ∇_a), then*

$$\nabla_a i_d = -2\boldsymbol{\gamma}_{a+}. \tag{4.7}$$

Proof. ¿From def.(4.4) we can write

$$\begin{aligned}(\nabla_a i_d)(b) &= \nabla_a^- i_d(b) - i_d(\nabla_a b) \\ &= \nabla_a^-(b) - \nabla_a b.\end{aligned} \tag{4.8}$$

Now, using eq.(3.1) and eq.(2.3), we get

$$\begin{aligned}(\nabla_a i_d)(b) &= a \cdot \partial b - \boldsymbol{\gamma}_a^\dagger(b) - a \cdot \partial b - \boldsymbol{\gamma}_a(b) \\ (\nabla_a i_d)(b) &= -2\boldsymbol{\gamma}_{a+}(b).\end{aligned}$$

□

Corollary 4.7. *∇_a is i_d-compatible (i.e., $\nabla_a i_d = 0$) if and only if $\boldsymbol{\gamma}_{a+} = 0$ (i.e., $\boldsymbol{\gamma}_a = -\boldsymbol{\gamma}_a^\dagger$, $\boldsymbol{\gamma}_a$ is skew-symmetric).*

Remark 14. *It is important to have in mind that i_d is the $(1,1)$-extensor field that represents the Minkowski metric tensor $\boldsymbol{\eta} \in \sec T_2^0(M)$.*

Note that, given any covariant derivative $b \mapsto \nabla_a b$ it is possible (and convenient) to introduce a well-defined covariant derivative by $b \mapsto \widehat{\nabla}_a b = h^{-1}(\nabla_a h(b))$. In the general gauge theory of gravitation, $\widehat{\nabla}_a$ is the so-called *h-gauge of* ∇_a.

Proposition 4.8. *If $\boldsymbol{\gamma}_{a+}$ is the symmetric part of $\boldsymbol{\gamma}_a$ ($\boldsymbol{\gamma}_a$ is the first connection extensor field associated to ∇_a), then*

$$\widehat{\nabla}_a g = -2h^\dagger \boldsymbol{\gamma}_{a+} h. \tag{4.9}$$

Proof. Let b, c be smooth form fields; by eq.(4.5) and eq.(2.3) we have

$$\begin{aligned}(\widehat{\nabla}_a g)(b) \cdot c &= a \cdot \partial(g(b) \cdot c) - g(\widehat{\nabla}_a b) \cdot c - g(b) \cdot (\widehat{\nabla}_a c) \\ &= a \cdot \partial(h(b) \cdot h(c)) - \nabla_a h(b) \cdot h(c) - h(b) \cdot \nabla_a h(c) \\ &= -\boldsymbol{\gamma}_a h(b) \cdot h(c) - h(b) \cdot \boldsymbol{\gamma}_a h(c) \\ &= -(\boldsymbol{\gamma}_a h(b) + \boldsymbol{\gamma}_a^\dagger h(b)) \cdot h(c) \\ &= -2h^\dagger \boldsymbol{\gamma}_{a+} h(b) \cdot c,\end{aligned} \tag{4.10}$$

that is, $\widehat{\nabla}_a g = -2h^\dagger \boldsymbol{\gamma}_{a+} h$. □

Corollary 4.9. *$\widehat{\nabla}_a$ is g-compatible (i.e., $\widehat{\nabla}_a g = 0$) if and only if $\boldsymbol{\gamma}_{a+} = 0$ (i.e., $\boldsymbol{\gamma}_a = -\boldsymbol{\gamma}_a^\dagger$, $\boldsymbol{\gamma}_a$ is skew-symmetric).*

Remark 15. *Taking into account the first corollary above, the second corollary above set: $\widehat{\nabla}_a$ is g-compatible if and only if ∇_a is i_d-compatible.*

In (DLG, 1998) the authors introduce a covariant derivative $b \mapsto \mathcal{D}_a b = a \cdot \partial b + \Omega(a) \times b$. According to our theory of covariant derivation, $\mathcal{D}_a$ would be the most general covariant derivative with the property of being i_d-compatible (i.e., $\mathcal{D}_a i_d = 0$). However, it is not g-compatible as claimed by the mentioned authors.

Observe that eq.(4.5) implies a logical equivalence between the property $\mathcal{D}_a i_d = 0$ and the following property $a \cdot \partial(b \cdot c) = (\mathcal{D}_a b) \cdot c + b \cdot (\mathcal{D}_a c)$, where a, b, c are smooth form fields. Hence, by putting $a = \partial_\mu x$, $b = g_\nu$ and $c = g_\lambda$ into the last identity and taking into account the definitions and properties used in (DLG,1998), it is not difficult to get the differential equation $\partial_\mu g_{\nu\lambda} = \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu}$, where $\Gamma^\alpha_{\mu\nu} = (\mathcal{D}_{\partial_\mu x} g_\nu) \cdot g^\alpha$. That one is only a *coordinate expression for the i_d-compatibility of $\mathcal{D}_a$*, $\Gamma^\alpha_{\mu\nu}$ are *hybrid connection coefficients* among the natural basis and gauge basis!

A correct g-compatible gauge theory of gravitation should be formulated by taking into account the corollary from prop.(4.9). Such a theory is presented in (Fernández, Moya and Rodrigues, 1999a)

5 Structural extensor fields

5.1 Nonmetricity extensor field

Definition 5.1. *Given a symmetric $(1,1)$ extensor field g ($g = g^\dagger$) and an arbitrary covariant derivative operator ∇ on the Minkowski manifold. The smooth biextensor field, say $(a, b) \mapsto A(a, b)$ given by*

$$A(a, b) = \nabla_a g(b), \tag{5.1}$$

for any smooth form fields a, b is called nonmetricity of g relative to ∇.

Definition 5.2. *∇ is said to be g-compatible if and only if $A(a, b) = 0$, for any form fields a, b.*

5.2 Torsion extensor field

We show now, that there exists a well-defined smooth *bi-exform field* (i.e., a skew-symmetric bi-extensor field), associated to $\boldsymbol{\gamma}_a$, that measures the so-called *torsion* of ∇_a. After that, we introduce the *torsion extensor field.*

Proposition 5.3. *There exists a unique smooth bi-exform field $(a, b) \mapsto \tau(a, b)$,*

$$\tau(a, b) = \boldsymbol{\gamma}_a(b) - \boldsymbol{\gamma}_b(a), \tag{5.2}$$

such that for all smooth form fields a, b it holds

$$\nabla_a b - \nabla_b a = [a, b] + \tau(a, b). \tag{5.3}$$

Proof. Let a, b be two smooth form fields. Using eq.(2.3), we have

$$\begin{aligned} \nabla_a b - \nabla_b a &= a \cdot \partial b + \boldsymbol{\gamma}_a(b) - b \cdot \partial a - \boldsymbol{\gamma}_b(a) \\ &= [a, b] + \boldsymbol{\gamma}_a(b) - \boldsymbol{\gamma}_b(a), \end{aligned} \tag{5.4}$$

where $[a,b] = a \cdot \partial b - b \cdot \partial a$ is the so-called *Lie bracket* of the form fields a, b.

Eq.(5.4) implies the existence of a bi-exform field, defined as

$$(a,b) \mapsto \tau(a,b) = \boldsymbol{\gamma}_a(b) - \boldsymbol{\gamma}_b(a), \tag{5.5}$$

(note the linearity with respect to $a, b \in \Lambda^1(\mathcal{M})$ and the skew-symmetry under interchange of their variables), such that for all smooth form fields a, b

$$\nabla_a b - \nabla_b a = [a,b] + \tau(a,b).$$

$\square$

Remark 16. *τ is indeed a smooth bi-exform field, associated to ∇_a. ¿From eq.(2.2), it follows that $\tau(a,b) = b \cdot \partial_n \nabla_a n - a \cdot \partial_n \nabla_b n$.*

Definition 5.4. *The bi-exform field τ will be called torsion bi-exform field.*

Proposition 5.5. *For any form fields a, b it holds*

$$\tau(a,b) = \boldsymbol{\gamma}_{a+}(b) - \boldsymbol{\gamma}_{b+}(a) + \Omega(a) \times b - \Omega(b) \times a. \tag{5.6}$$

Proof. It is enough to use the decomposition of $\boldsymbol{\gamma}_a$ into symmetric and skew-symmetric parts and the factorization (2.9) into the def.(5.2). $\square$

Remark 17. *In eq.(5.6), the bi-exform field $\boldsymbol{\gamma}_{a+}(b) - \boldsymbol{\gamma}_{b+}(a)$ comes from the symmetric part of $\boldsymbol{\gamma}_a$ and the bi-exform field $\Omega(a) \times b - \Omega(b) \times a$ comes from the skew-symmetric part of $\boldsymbol{\gamma}_a$.*

Definition 5.6. *The smooth $(1,2)$-extensor field, say $n \mapsto T(n)$, such that for each $x \in \Lambda^1(\mathcal{M})$ its adjoint is the smooth $(2,1)$-extensor field given by, :*

$$\Lambda^2(\mathcal{M}) \ni B \mapsto T_x^\dagger(B) = \frac{1}{2!} B \cdot (\partial_a \wedge \partial_b) \tau_x(a,b) \in \Lambda^1(\mathcal{M}), \tag{5.7}$$

will be called torsion extensor field.

Remark 18. *We usually employ the short notation $T^\dagger(B) = \frac{1}{2} B \cdot (\partial_a \wedge \partial_b) \tau(a,b)$.*

Proposition 5.7. *For any forms a, b it holds*

$$T^\dagger(a \wedge b) = \tau(a,b). \tag{5.8}$$

Proof. Taking into account the skew-symmetry of τ, we have

$$\begin{aligned}
T^\dagger(a \wedge b) &= \frac{1}{2}(a \wedge b) \cdot (\partial_p \wedge \partial_q)\tau(p,q) = \frac{1}{2}(a \cdot \partial_p b \cdot \partial_q - a \cdot \partial_q b \cdot \partial_p)\tau(p,q) \\
&= \frac{1}{2}[a \cdot \partial_p b \cdot \partial_q \tau(p,q) - a \cdot \partial_q b \cdot \partial_p \tau(p,q)] = \frac{1}{2}[\tau(a,b) - \tau(b,a)], \\
T^\dagger(a \wedge b) &= \tau(a,b).
\end{aligned}$$

$\square$

5.3 Curvature extensor field

We show now the existence of two well-defined operators, associated to $\boldsymbol{\gamma}_a$, involved in the concept of *curvature* of the covariant derivative operator ∇. After that, we introduce the *curvature extensor field.*

Proposition 5.8. *There exists a unique operator acting on the set of smooth vector fields, say* $(a, b, c) \mapsto \widehat{\omega}_1(a, b, c)$,

$$\widehat{\omega}_1(a, b, c) = c \cdot \partial_n(a \cdot \partial\boldsymbol{\gamma}_b(n) - b \cdot \partial\boldsymbol{\gamma}_a(n) + [\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b](n)), \tag{5.9}$$

such that for all smooth form fields a, b, c *it holds*

$$[\nabla_a, \nabla_b]c = [a \cdot \partial, b \cdot \partial]c + \widehat{\omega}_1(a, b, c). \tag{5.10}$$

Proof. Let a, b, c be three smooth form fields; by using eq.(2.3) we have

$$\begin{aligned} \nabla_a(\nabla_b c) &= a \cdot \partial b \cdot \partial c + a \cdot \partial\boldsymbol{\gamma}_b(c) + \boldsymbol{\gamma}_a(b \cdot \partial c) + \boldsymbol{\gamma}_a\boldsymbol{\gamma}_b(c). \\ \nabla_b(\nabla_a c) &= b \cdot \partial a \cdot \partial c + b \cdot \partial\boldsymbol{\gamma}_a(c) + \boldsymbol{\gamma}_b(a \cdot \partial c) + \boldsymbol{\gamma}_b\boldsymbol{\gamma}_a(c). \end{aligned} \tag{5.11}$$

Subtracting we get

$$\begin{aligned} [\nabla_a, \nabla_b]c &= [a \cdot \partial, b \cdot \partial]c + a \cdot \partial\boldsymbol{\gamma}_b(c) - \boldsymbol{\gamma}_b(a \cdot \partial c) \\ &\quad - b \cdot \partial\boldsymbol{\gamma}_a(c) + \boldsymbol{\gamma}_a(b \cdot \partial c) + [\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b](c). \end{aligned} \tag{5.12}$$

Now, using the formula $b \cdot \partial_n a \cdot \partial t(n) = a \cdot \partial t(b) - t(a \cdot \partial b)$, valid for any smooth $(1, 1)$-extensor field, where a is a form and b, n are smooth form fields, we obtain

$$[\nabla_a, \nabla_b]c = [a \cdot \partial, b \cdot \partial]c + c \cdot \partial_n(a \cdot \partial\boldsymbol{\gamma}_b(n) - b \cdot \partial\boldsymbol{\gamma}_a(n) + [\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b](n)). \tag{5.13}$$

Eq.(5.13) implies the existence of an operator,

$$\underbrace{\mathrm{hom}[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})]}_{\text{3-copies}} \ni (a, b, c) \mapsto \widehat{\omega}_1(a, b, c) \in \mathrm{hom}[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})],$$

defined by

$$\widehat{\omega}_1(a, b, c) = c \cdot \partial_n(a \cdot \partial\boldsymbol{\gamma}_b(n) - b \cdot \partial\boldsymbol{\gamma}_a(n) + [\boldsymbol{\gamma}_a, \boldsymbol{\gamma}_b](n)), \tag{5.14}$$

such that for all smooth form fields a, b, c,

$$[\nabla_a, \nabla_b]c = [a \cdot \partial, b \cdot \partial]c + \widehat{\omega}_1(a, b, c).$$

$\square$

Remark 19. $\widehat{\omega}_1$ *is an operator, associated to the directional covariant derivative operator* ∇_a, *which is linear with respect to its third variable and skew-symmetric under interchange of first and second variables. In terms of* ∇_a, *it is obviously that* $\widehat{\omega}_1(a, b, c) = c \cdot \partial_n[\nabla_a, \nabla_b]n$.

Definition 5.9. *The operator $\widehat{\omega}_1$ will be called first curvature operator.*

Besides $\widehat{\omega}_1$, we introduce another operator related to curvature.

Definition 5.10. *The second curvature operator $\widehat{\omega}_2$ acting on the set of smooth form fields is given by*

$$\underbrace{\hom[\Lambda^1(\mathcal{M}),\Lambda^1(\mathcal{M})]}_{2\ copies} \ni (a,b) \mapsto \widehat{\omega}_2(a,b) \in \hom[\Lambda^1(\mathcal{M}),\Lambda^2(\mathcal{M})],$$

such that

$$\widehat{\omega}_2(a,b) = -\frac{1}{2}\partial_c \wedge \widehat{\omega}_1(a,b,c) = -\frac{1}{2}\partial_n \wedge (a\cdot\partial\boldsymbol{\gamma}_b(n) - b\cdot\partial\boldsymbol{\gamma}_a(n) + [\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b](n)). \tag{5.15}$$

Remark 20. *The operator $\widehat{\omega}_2$ is skew-symmetric under interchange of their variables. In terms of ∇_a, we have $\widehat{\omega}_2(a,b) = -\frac{1}{2}\partial_n \wedge [\nabla_a,\nabla_b]n$.*

Proposition 5.11. *For any smooth vector form fields a,b it holds*

$$\widehat{\omega}_2(a,b) = \frac{1}{2}\mathrm{bif}([\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}]) + a\cdot\partial\Omega(b) - b\cdot\partial\Omega(a) + \Omega(a)\times\Omega(b). \tag{5.16}$$

Proof. By straightforward calculation we have

$$\begin{aligned}
\widehat{\omega}_2(a,b) &= -\frac{1}{2}\partial_n \wedge (a\cdot\partial\boldsymbol{\gamma}_b(n) - b\cdot\partial\boldsymbol{\gamma}_a(n) + [\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b](n))\\
&= a\cdot\partial(-\frac{1}{2}\partial_n\wedge\boldsymbol{\gamma}_b(n)) - b\cdot\partial(-\frac{1}{2}\partial_n\wedge\boldsymbol{\gamma}_a(n)) - \frac{1}{2}\partial_n\wedge[\boldsymbol{\gamma}_a,\boldsymbol{\gamma}_b](n)\\
&= a\cdot\partial\Omega(b) - b\cdot\partial\Omega(a)\\
&\quad -\frac{1}{2}\partial_n\wedge([\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}] + [\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b-}] + [\boldsymbol{\gamma}_{a-},\boldsymbol{\gamma}_{b+}] + [\boldsymbol{\gamma}_{a-},\boldsymbol{\gamma}_{b-}])(n);\\
\widehat{\omega}_2(a,b) &= a\cdot\partial\Omega(b) - b\cdot\partial\Omega(a)\\
&\quad -\frac{1}{2}\partial_n\wedge[\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}](n) - \frac{1}{2}\partial_n\wedge[\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b-}](n)\\
&\quad -\frac{1}{2}\partial_n\wedge[\boldsymbol{\gamma}_{a-},\boldsymbol{\gamma}_{b+}](n) - \frac{1}{2}\partial_n\wedge[\boldsymbol{\gamma}_{a-},\boldsymbol{\gamma}_{b-}](n).
\end{aligned} \tag{5.17}$$

Now, using eq.(2.8), eq.(2.9) and Jacobi's identity $A\times(B\times C) = (A\times B)\times C + B\times(A\times C)$, we get

$$\begin{aligned}
\widehat{\omega}_2(a,b) &= a\cdot\partial\Omega(b) - b\cdot\partial\Omega(a) + \frac{1}{2}\mathrm{bif}([\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}]) + \frac{1}{2}\mathrm{bif}([\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b-}])\\
&\quad +\frac{1}{2}\mathrm{bif}([\boldsymbol{\gamma}_{a-},\boldsymbol{\gamma}_{b+}]) - \frac{1}{2}\partial_n\wedge([\Omega(a)\times\Omega(b)]\times n).
\end{aligned} \tag{5.18}$$

Since $[\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}]$ and $[\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b-}]$ are symmetric extensors, their biforms vanish. Using the formula $\partial_n\wedge(B\times n) = -2B$, where B is a biform and n is a form we get

$$\widehat{\omega}_2(a,b) = \frac{1}{2}\mathrm{bif}([\boldsymbol{\gamma}_{a+},\boldsymbol{\gamma}_{b+}]) + a\cdot\partial\Omega(b) - b\cdot\partial\Omega(a) + \Omega(a)\times\Omega(b).$$

$\square$

Remark 21. *In eq.(5.16), the bi-exform field* $\frac{1}{2}\mathrm{bif}([\gamma_{a+},\gamma_{b+}])$ *comes from the symmetric part of* γ_a *and the operator* $(a,b)\mapsto a\cdot\partial\Omega(b)-b\cdot\partial\Omega(a)+\Omega(a)\times\Omega(b)$ *comes from the skew-symmetric part of* γ_a*.*

Definition 5.12. *The smooth* $(2,2)$*-extensor field, say* $B\mapsto R(B)$*, for each* $x\in\Lambda^1(\mathcal{M})$ *:*

$$\Lambda^2(\mathcal{M})\ni B\mapsto R_x(B)=\frac{1}{2!}B\cdot(\partial_{a(x)}\wedge\partial_{b(x)})\widehat{\omega}_2(a,b)(x)\in\Lambda^2(\mathcal{M}), \tag{5.19}$$

will be called curvature extensor field.

Remark 22. *We usually employ the short notation* $R(B)=\frac{1}{2}B\cdot(\partial_a\wedge\partial_b)\widehat{\omega}_2(a,b)$*.*

Proposition 5.13. *For all smooth form fields* a,b *it holds*

$$R(a\wedge b)=\widehat{\omega}_2(a,b)-\Omega([a,b]). \tag{5.20}$$

Proof. Due the skew-symmetry of $\widehat{\omega}_2(a,b)$, we have

$$\begin{aligned} R(a\wedge b) &= \frac{1}{2}(a\wedge b)\cdot(\partial_p\wedge\partial_q)\widehat{\omega}_2(a,b)\\ &= \frac{1}{2}[a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(p,q)-a\cdot\partial_q b\cdot\partial_p\widehat{\omega}_2(p,q)]\\ &= \frac{1}{2}[a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(p,q)-a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(q,p)]\\ &= \frac{1}{2}[a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(p,q)+a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(p,q)],\\ R(a\wedge b) &= a\cdot\partial_p b\cdot\partial_q\widehat{\omega}_2(p,q). \end{aligned} \tag{5.21}$$

Taking into account eq.(5.16) and using the general formula for a smooth $(1,k)$-extensor fields $b\cdot\partial_n a\cdot\partial t(n)=a\cdot\partial t(b)-t(a\cdot\partial b)$,where a is a form and b,n are smooth form fields, we obtain

$$\begin{aligned} R(a\wedge b) &= a\cdot\partial_p b\cdot\partial_q(\frac{1}{2}\mathrm{bif}([\gamma_{p+},\gamma_{q+}])+p\cdot\partial\Omega(q)-q\cdot\partial\Omega(p)+\Omega(p)\times\Omega(q))\\ &= \frac{1}{2}\mathrm{bif}([\gamma_{a+},\gamma_{b+}])+a\cdot\partial\Omega(b)-\Omega(a\cdot\partial b)\\ &\quad -b\cdot\partial\Omega(a)+\Omega(b\cdot\partial a)+\Omega(a)\times\Omega(b),\\ R(a\wedge b) &= \widehat{\omega}_2(a,b)-\Omega([a,b]). \end{aligned}$$

□

6 Examples

6.1 Levi-Civita derivative

We suppose the existence of a smooth $(1,1)$-extensor field on the Minkowski manifold (modeled by $(M,\mathcal{M}^*)$), say $n\mapsto g(n)$, which is symmetric (i.e., $g=g^\dagger$) and has inverse. It will be called *gauge metric extensor field.*

We introduce two fundamental operators acting on the set of smooth form fields on $(M, \mathcal{M}^*)$:

(a) The *Christoffel operator of the first kind*,

$$\underbrace{\hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})]}_{3 \text{ copies}} \ni (a, b, c) \mapsto [a, b, c] \in \hom[\Lambda^1(\mathcal{M}), R], \tag{6.1}$$

where

$$\begin{aligned} 2[a, b, c] &= a \cdot \partial(g(b) \cdot c) \\ &= b \cdot \partial(g(c) \cdot a) - c \cdot \partial(g(a) \cdot b) \\ &\quad + g(c) \cdot [a, b] + g(b) \cdot [c, a] - g(a) \cdot [b, c]. \end{aligned} \tag{6.2}$$

and where $[a, b]$ is the *Lie bracket* of the form fields a, b.

(b) The *Christoffel operator of the second kind,*

$$\underbrace{\hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})]}_{3 \text{ copies}} \ni (a, b, c) \mapsto \left\{ \begin{array}{c} c \\ a, b \end{array} \right\} \in \hom[\Lambda^1(\mathcal{M}), R], \tag{6.3}$$

such that

$$\left\{ \begin{array}{c} c \\ a, b \end{array} \right\} = [a, b, g^{-1}(c)]. \tag{6.4}$$

The Christoffel operator of the first kind has several useful properties, namely for any form fields a, a', b, c and any scalar field f,we have:

$$\begin{aligned} [a + a', b, c] &= [a, b, c] + [a', b, c] \\ [fa, b, c] &= f[a, b, c]. && (6.5) \\ [a, b + b', c] &= [a, b, c] + [a, b', c] \\ [a, fb, c] &= f[a, b, c] + (a \cdot \partial f) g(b) \cdot c. && (6.6) \\ [a, b, c + c'] &= [a, b, c] + [a, b, c'] \\ [a, b, fc] &= f[a, b, c]. && (6.7) \end{aligned}$$

Now, we can define the Levi-Civita covariant derivative D as follows:

$$\hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})] \ni b \mapsto D_a b \in \hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})], \tag{6.8}$$

such that if $a \in \hom[\Lambda^1(\mathcal{M}), \Lambda^1(\mathcal{M})]$, then $D_a b \equiv \partial_n \left\{ \begin{array}{c} n \\ a, b \end{array} \right\} = \partial_n [a, b, g^{-1}(n)]$.

Observe that for any smooth form fields a, b, c we have

$$(D_a b) \cdot c = \left\{ \begin{array}{c} c \\ a, b \end{array} \right\}. \tag{6.9}$$

D_a is a well-defined directional covariant derivative. In fact, the linearity with respect to the direction a is consequence of eq.(6.5). Also, if f is a scalar field and b, b' are form fields. Using eq.(6.7), we have

$$\begin{aligned} D_a(b+b') &= \partial_n[a, b+b', g^{-1}(n)] \\ &= \partial_n[a,b,g^{-1}(n)] + \partial_n[a,b',g^{-1}(n)], \\ &= D_a b + D_a b'. \qquad (6.10) \\ D_a(fb) &= \partial_n[a, fb, g^{-1}(n)] \\ &= f\partial_n[a,b,g^{-1}(n)] + \partial_n(a\cdot\partial f)(g(b)\cdot g^{-1}(n)), \\ &= (a\cdot\partial f)b + fD_a b. \qquad (6.11) \end{aligned}$$

Thus, the ax.(ii) into def.(2.1) is satisfied.

Theorem 6.1. . *Let a, b, c be arbitrary smooth form fields, then*

$$g(D_a b)\cdot c + g(b)\cdot D_a c = a\cdot\partial(g(b)\cdot c). \qquad (6.12)$$

Proof.

$$\begin{aligned} g(D_a b)\cdot c + g(b)\cdot D_a c &= \partial_n[a,b,g^{-1}(n)]\cdot g(c) + g(b)\cdot\partial_n[a,c,g^{-1}(n)] \\ &= [a,b,g^{-1}g(c)] + [a,c,g^{-1}g(b)], \\ g(D_a b)\cdot c + g(b)\cdot D_a c &= a\cdot\partial(g(b)\cdot c). \end{aligned}$$

We have used def.(6.8) and in the last line an important identity for the Christoffel operator of the first kind, namely, $[a,b,c]+[a,c,b] = a\cdot\partial(g(b)\cdot c)$. □

Remark 23. *The above theorem is known as Ricci theorem for the Levi-Civita derivative.*

Corollary 6.1. *Let a, b be arbitrary smooth form fields,*

$$D_a^- = g(D_a g^{-1}(b)). \qquad (6.13)$$

Proof. Once again, let a, b, c be arbitrary smooth form fields. Using the Ricci theorem (eq.(6.12)) we have

$$\begin{aligned} a\cdot\partial(b\cdot c) &= a\cdot\partial(g(b)\cdot g^{-1}(c)) \\ &= g(D_a b)\cdot g^{-1}(c) + g(b)\cdot D_a g^{-1}(c) \\ &= (D_a b)\cdot c + b\cdot g(D_a g^{-1}(c)). \qquad (6.14) \end{aligned}$$

Comparing eq.(6.14) with the identity eq.(3.6), it follows

$$D_a^- b = g(D_a g^{-1}(b)). \qquad (6.15)$$

□

Proposition 6.2. *g is compatible with D, i.e., for any form field a, the extension of the Levi-Civita derivative acting on the smooth $(1,1)$-extensor field g (the gauge metric extensor field) vanishes,*

$$D_a g = 0. \qquad (6.16)$$

Proof.

$$\begin{aligned}(D_a g)(b) &= D_a^- g(b) - g(D_a b)\\ &= g(D_a g^{-1} g(b)) - g(D_a b),\\ (D_a g)(b) &= 0.\end{aligned}$$

□

Finally, we will see that Levi-Civita derivative is torsionless.

In fact, according to eq.(5.3), we can write

$$\begin{aligned}\tau(a,b) &= D_a b - D_b a - [a,b]\\ &= \partial_n(D_a b\cdot n - D_b a\cdot n) - [a,b]\\ &= \partial_n(\left\{\begin{array}{c} n\\ a,b\end{array}\right\} - \left\{\begin{array}{c} n\\ b,a\end{array}\right\}) - [a,b]\\ &= \partial_n([a,b,g^{-1}(n)] - [b,a,g^{-1}(n)]) - [a,b],\end{aligned} \tag{6.17}$$

and using the remarkable identity $[a,b,c]-[b,a,c]=g(c)\cdot[a,b]$ yields

$$\tau(a,b) = \partial_n g g^{-1}(n)\cdot[a,b] - [a,b] = 0, \tag{6.18}$$

that is, $T^\dagger(a\wedge b)=0$.

6.2 Hestenes derivative

Take a smooth even multiform field over the Minkowski manifold modeled by $(M,\mathcal{M}^*)$, say $\Lambda^1(\mathcal{M})\ni x\mapsto R(x)\in\Lambda^+(\mathcal{M})$ (where $\Lambda^+(\mathcal{M})\subset\mathcal{C}\ell(\mathcal{M})$ is the set of even multiforms), such that $R(x)\widetilde{R}(x)=1$. It is called a *Lorentz rotator,* since for any $x\in\forall\Lambda^1(\mathcal{M}), R(x)\in\mathbf{Spin}_+(1,3)\simeq SL(2,\mathbf{C})$, which is the universal covering group of $\mathfrak{L}_+^\uparrow$, the restrict orthochronous Lorentz group (Lounesto, 1997).

Definition 6.3. *A Lorentz extensor field l_x is a smooth $(1,1)$-extensor field over $\mathcal{M}^*$ such that for each $x\in\Lambda^1(\mathcal{M})$:*

$$\begin{aligned}\Lambda^1(\mathcal{M}) &\ni n\mapsto l_x(n)\in\Lambda^1(\mathcal{M}),\\ l_x(n) &= R(x)n\widetilde{R}(x).\end{aligned} \tag{6.19}$$

Some important properties for Lorentz extensor field $n\mapsto l(n)=Rn\widetilde{R}$, are: For any multiform field X,the extension[10] of l, denoted $\underline{l}$ is such that $\underline{l}(X)=RX\widetilde{R}$. Also $\det(l)=1$, $l^\dagger(n)=\widetilde{R}nR$ (where $l^\dagger$ is the adjoint of l), and $l^{-1}=l^\dagger$.

Definition 6.4. *Let $(\langle\varepsilon_\mu\rangle,\langle\varepsilon^\mu\rangle)$ be any pair of reciprocal frames (i.e., $\varepsilon_\mu\cdot\varepsilon^\nu=\delta_\mu^\nu$), then the pair of reciprocal frames $(\langle l(\varepsilon_\mu)\rangle,\langle l^\star(\varepsilon^\mu)\rangle)$, where $l^\star\equiv(l^{-1})^\dagger=(l^\dagger)^{-1}$ (in this case, it holds $l^\star=l$) are called the l-gauge frames of $(\langle\varepsilon_\mu\rangle,\langle\varepsilon^\mu\rangle)$.*

[10] The extension of a $(1,1)$-extensor t is the general extensor $\underline{t}$ defined by the properties: $\underline{t}(\alpha)=\alpha$, $\alpha\in R$ and $\underline{t}(a_1\wedge\ldots\wedge a_k)=t(a_1)\wedge\ldots\wedge t(a_k)$, $a_1,\ldots,a_k\in\Lambda^1(\mathcal{M})$.

The existence of a multiform field R, with the properties $R(x) \in \Lambda^{+}(\mathcal{M})$ (i.e., $R(x) = \overline{R}(x)$) and $R(x)\widetilde{R}(x) = 1$, implies the existence of a smooth $(1,2)$-extensor field. For each $x \in \Lambda^1(\mathcal{M})$:

$$\Lambda^1(\mathcal{M}) \ni a \mapsto \omega_x(a) \in \Lambda^2(\mathcal{M}), \tag{6.20}$$

such that $\omega_x(a) = (-2a \cdot \partial R(x))\widetilde{R}(x)$.

Observe that the canonical biforms $\omega(\gamma_\mu) = (-2\partial_\mu R)\widetilde{R}$ are the so called Darboux biform fields. This suggests calling ω the *Darboux extensor field* (Rodrigues et al, 1996)

We prove now two important properties involving the Darboux extensor field.

Proposition 6.5. *For the l-gauge frames of the fundamental frames $(\langle\gamma_\mu\rangle, \langle\gamma^\mu\rangle)$, we have*

$$\begin{aligned} a \cdot \partial l(\gamma_\mu) &= l(\gamma_\mu) \times \omega(a), \\ a \cdot \partial l(\gamma^\mu) &= l(\gamma^\mu) \times \omega(a) \end{aligned} \tag{6.21}$$

Proof. Using the known property $R\widetilde{R} = \widetilde{R}R = 1$ and the def.(6.20),

$$\begin{aligned} a \cdot \partial l(\gamma_\mu) &= (a \cdot \partial R)\gamma_\mu\widetilde{R} + R\gamma_\mu(a \cdot \partial\widetilde{R}) \\ &= (a \cdot \partial R)\widetilde{R}R\gamma_\mu\widetilde{R} + R\gamma_\mu\widetilde{R}R(a \cdot \partial\widetilde{R}) \\ &= -\frac{1}{2}\omega(a)l(\gamma_\mu) + l(\gamma_\mu)\frac{1}{2}\omega(a), \\ a \cdot \partial l(\gamma_\mu) &= l(\gamma_\mu) \times \omega(a). \end{aligned}$$

In analogous way, $a \cdot \partial l(\gamma^\mu) = l(\gamma^\mu) \times \omega(a)$. □

For the Darboux extensor field we have a remarkable identity.

Proposition 6.6. *For any smooth form fields a, b it holds*

$$a \cdot \partial\omega(b) - b \cdot \partial\omega(a) + \omega(a) \times \omega(b) = \omega([a,b]). \tag{6.22}$$

Proof. We have,

$$a \cdot \partial\omega(b) = -2((a \cdot \partial b \cdot \partial R)\widetilde{R} + (b \cdot \partial R)(a \cdot \partial\widetilde{R})) \tag{6.23}$$

$$b \cdot \partial\omega(a) = -2((b \cdot \partial a \cdot \partial R)\widetilde{R} + (a \cdot \partial R)(b \cdot \partial\widetilde{R})). \tag{6.24}$$

Thus,

$$\begin{aligned} a \cdot \partial\omega(b) - b \cdot \partial\omega(a) &= -2(([a \cdot \partial, b \cdot \partial]R)\widetilde{R} + (b \cdot \partial R)(a \cdot \partial\widetilde{R}) - (a \cdot \partial R)(b \cdot \partial\widetilde{R})) \\ &= -2(([a,b] \cdot \partial R)\widetilde{R} + (b \cdot \partial R)(a \cdot \partial\widetilde{R}) - (a \cdot \partial R)(b \cdot \partial\widetilde{R})) \\ &= \omega([a,b]) - 2((b \cdot \partial R)(a \cdot \partial\widetilde{R}) - (a \cdot \partial R)(b \cdot \partial\widetilde{R})), \end{aligned} \tag{6.25}$$

where we used the identity $[a \cdot \partial, b \cdot \partial]X = [a,b] \cdot \partial X$ where a, b are smooth form fields and X is a smooth multiform field.

Also,

$$\begin{aligned}\omega(a)\omega(b) &= 4(a\cdot\partial R)\widetilde{R}(b\cdot\partial R)\widetilde{R} \\ &= -4(a\cdot\partial R)(b\cdot\partial\widetilde{R}), \qquad (6.26)\\ \omega(b)\omega(a) &= -4(b\cdot\partial R)(a\cdot\partial\widetilde{R}), \qquad (6.27)\end{aligned}$$

which implies that

$$\omega(a)\times\omega(b) = -2((a\cdot\partial R)(b\cdot\partial\widetilde{R}) - (b\cdot\partial R)(a\cdot\partial\widetilde{R})). \qquad (6.28)$$

Comparing eq.(6.25) with eq.(6.28), we finally obtain

$$a\cdot\partial\omega(b) - b\cdot\partial\omega(a) + \omega(a)\times\omega(b) = \omega([a,b]).$$

□

Definition 6.7. *The Hestenes directional covariant derivative (Hestenes derivative, for short)* d_a, *is defined by*

$$\hom[\Lambda^1(\mathcal{M}),\Lambda^1(\mathcal{M})] \ni b \mapsto d_a b \in \hom[\Lambda^1(\mathcal{M}),\Lambda^1(\mathcal{M})], \qquad (6.29)$$

such that $d_a b \equiv l(a\cdot\partial l^{-1}(b)) = l(a\cdot\partial l^{\dagger}(b))$.

As the reader can easily prove it is indeed a well-defined directional covariant derivative, since it satisfies the axiomatic in def.(2.1).

Proposition 6.8. *The associated derivatives* d_a^- *and* d_a^0 *coincide with* d_a, *i.e., for any smooth form field* b, *we have*

$$d_a^- b = d_a b = d_a^0 b. \qquad (6.30)$$

Proof. Let a, b, c be arbitrary smooth form fields, then

$$\begin{aligned}a\cdot\partial(b\cdot c) &= a\cdot\partial(l^{-1}(b)\cdot l^{\dagger}(c)) \\ &= a\cdot\partial l^{-1}(b)\cdot l^{\dagger}(c) + l^{-1}(b)\cdot a\cdot\partial l^{\dagger}(c) \\ &= l(a\cdot\partial l^{-1}(b))\cdot c + b\cdot l^{\star}(a\cdot\partial l^{\dagger}(c)) \\ &= l(a\cdot\partial l^{-1}(b))\cdot c + b\cdot l(a\cdot\partial l^{-1}(c)) \\ &= (d_a b)\cdot c + b\cdot(d_a c). \qquad (6.31)\end{aligned}$$

Comparing the eq.(6.31) with the identity (3.6) and employing the def.(3.2), it follows

$$d_a^- b = d_a b = d_a^0 b. \qquad (6.32)$$

□

The result above, eq.(6.32), means that Hestenes derivative is i_d-compatible.

Finally, we will find the second connection extensor field associated to Hestenes derivative.

Proposition 6.9. *The second connection extensor field associated to Hestenes derivative coincides with the Darboux extensor field,*

$$\Omega(a) = \omega(a). \tag{6.33}$$

Proof.

$$\Omega(a) = -\frac{1}{2}\partial_n \wedge d_a n = -\frac{1}{2}\partial_n \wedge l(a \cdot \partial l^{-1}(n)) = -\frac{1}{2}\gamma^\mu \wedge l(a \cdot \partial l^{-1}(\gamma_\mu)). \tag{6.34}$$

hence, using the remarkable identity $\gamma^\mu \wedge l(a \cdot \partial l^{-1}(\gamma_\mu)) + l^\star(\gamma^\mu) \wedge a \cdot \partial l(\gamma_\mu) = O$ and the eq.(6.21), we have

$$\begin{aligned}
\Omega(a) &= \frac{1}{2} l^\star(\gamma^\mu) \wedge a \cdot \partial l(\gamma_\mu) \\
&= \frac{1}{2} l^\star(\gamma^\mu) \wedge (l(\gamma_\mu) \times \omega(a)) \\
&= \frac{1}{2}\partial_n \wedge (n \times \omega(a)) \\
\Omega(a) &= \omega(a).
\end{aligned}$$

we have employed the invariant representation property for curl operator $\partial_n \wedge$ and the formula $\partial_n \wedge (n \times B) = 2B$. □

Remark 24. *Eq.(6.30) together with the eq.(6.21) imply that the curvature extensor field of the Hestenes derivative is null.*

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