M-convex fuzzy mappings and fuzzy integral mean[1]

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Abstract—We introduce the notions of m-convex fuzzy mapping and fuzzy integral mean. We study their properties and we give some applications.

Keywords—Fuzzy mappings, integration of fuzzy mappings, Hausdorff convergence.

1. INTRODUCTION

The notion of m-convexity of functions and integral mean was studied by several authors, including Bruckner and Ostrow [3], Toader [15], [16].

The set-valued versions of some of the above results were, recently, given by Ślepak [14].

Our purpose here, roughly speaking, is to extend the previous works to the fuzzy context as well to give some convergence results for fuzzy integral means.

The structure of this paper is as follows. In Section 2 we give the basic concepts and results that will be used in the article and, in Section 3, we study the m-convexity and integral mean of fuzzy mappings and its properties. Finally, in Section 4 we show some results on convergence of mean integral for fuzzy mappings. Furthermore, some examples and applications are presented.

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2. BASIC CONCEPTS AND RESULTS

Let X be a real separable Banach space with dual X^* and $\mathcal{K}(X)$, $K_c(X)$, respectively, the class of all nonempty and compact subsets of X, and the class of all nonempty compact and convex subsets of X.
The Hausdorff metric H on $\mathcal{K}(X)$ is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

and it is known that $(\mathcal{K}(X), H)$ is a complete and separable metric space and $\mathcal{K}_c(X)$ is a closed subspace of $\mathcal{K}(X)$. See [4,7,8,9,10,12].
Also, a linear structure of convex cone in $\mathcal{K}(X)$ is defined by

$$A + B = \{a + b/\ a \in A, b \in B\} \text{ and } \ \lambda A = \{\lambda a/\ a \in A\}$$

for all $A, B \in \mathcal{K}(X)$, $\lambda \in \mathbb{R}$.
DEFINITION 2.1. *A set-valued function $F : [0,b] \to \mathcal{K}(X)$ is called Borel measurable, if its graph, i.e., the set $\{(t,x)/\ x \in F(t)\}$, is a Borel subset of $[0,b] \times X$.*
REMARK 2.2. Because Lebesgue measure is complete, the Borel measurability of the set-valued mapping F is equivalent to the following condition : for every Borel set B, $F^{-1}(B) = \{t \in [0,b]/\ F(t) \cap B \neq \emptyset\}$ where $\mathcal{L}$ denotes the σ-algebra of all Lebesgue-measurable subsets of interval $[0,b]$. Also, a measurable set-valued function $F : [0,b] \to \mathcal{K}(X)$ is called a **random set** (r.s.).
DEFINITION 2.3. *The integral of the set-valued function $F : [0,b] \to \mathcal{K}(X)$ is defined by*

$$\int_0^b F dt = \left\{\int_0^b f(t)dt/\ f \in S(F)\right\},$$

where $\int_0^b f(t)dt$ is the Bochner-integral and $S(F)$ is the set of all integrable selectors of F, i.e.,

$$S(F) = \left\{f \in L^1([0,b],X)/\ f(t) \in F(t)\ a.e.\right\}.$$

This definition was introduced by Aumann [1] as a natural generalization of the integration of single-valued functions.

Definition 2.4. *A set-valued function* $F : [0, b] \to \mathcal{K}(X)$ *is said to be integrably bounded, if there exists a single-valued integrable function* $h : [0, b] \to X$ such that $||x|| \leq h(t)$ *for all* x *and* t *such that* $x \in F(t)$.
Remark 2.5. If $F : [0, b] \to \mathcal{K}(X)$ is an integrably bounded r.s., then the Aumann integral of F is a nonempty subset of X.
Proposition 2.6. *If* $\lambda \in \mathbb{R}$ *and* $F, F_1, F_2 : [0, b] \to \mathcal{K}_c(X)$ *are integrably bounded r.s., then*

a) $\int_0^b F dt \in \mathcal{K}_c(X)$

b) $\int_0^b (\lambda F_1 + F_2) dt = \lambda \int_0^b F_1 dt + \int_0^b F_2 dt$.

For details see Hiai&Umegaki [5].
The following result is a generalization of Lebesgue's dominated convergence theorem:

Theorem 2.7. *If* $F_p : \Omega \to \mathcal{K}(\mathcal{X})$ *are r.s and there is* $f \in L^1(\Omega, I\!R)$ *such that* $\sup\limits_{p \geq 1} ||g_p(w)|| \leq f(w)$ *for all* $g_p \in S(F_p)$, *then if* $F_p(w) \overset{H}{\to} F(w)$ *a.e we have*

$$\int F_p \overset{H}{\to} \int F \quad \text{as} \quad p \to \infty \ .$$

See [7, Theor. 17.2.6, p. 192].Also, see [8].
Now, we will give the extensions of the above results to the fuzzy context. A fuzzy subset of X is a function $u : X \to [0, 1]$ and, for $0 < \alpha \leq 1$, we denote by $L_\alpha u = \{x \in X / \ u(x) \geq \alpha\}$ the α-level of u, and $L_0 u = supp(u) = \overline{\{x \in x : u(x) > 0\}}$ is called the support of u.
As an extension of $\mathcal{K}(X)$ ($\mathcal{K}_c(X)$, respectively) we define the space $\mathcal{F}(X)$ of all fuzzy compact subsets $u : X \to [0, 1]$ (the space $\mathcal{F}_c(X)$ of all fuzzy compact and convex subsets of X, respectively) such that

$$L_\alpha u \in \mathcal{K}(X) \ (L_\alpha u \in \mathcal{K}_c(X), \text{respectively}), \ \forall \alpha \in [0, 1].$$

Remark 2.8. If $u \in \mathcal{F}(X)$, then the family $\{L_\alpha u / \ \alpha \in [0, 1]\}$ satisfies the following properties :

a) $L_0 u \supseteq L_\alpha u \supseteq L_\beta u$ for all $0 \leq \alpha \leq \beta$,

b) If $\alpha_n \nearrow \alpha$ then $L_\alpha u = \cap_{n=1}^{\infty} L_{\alpha_n} u$ (i.e., the level-application is left-continuous),

c) $u = v \Leftrightarrow L_\alpha u = L_\alpha v, \ \forall \alpha \in [0,1]$

d) $L_\alpha u \neq \emptyset$ for all $\alpha \in [0,1]$ is equivalent to $u(x) = 1$ for some $x \in X$.

e) If $u \in \mathcal{F}(X)$, then $u(x) = sup\{\alpha / \ x \in L_\alpha u\}$ (see [9]).

f) We can to define a partial order $\subseteq$ on $\mathcal{F}(X)$ by setting

$$u \subseteq v \Leftrightarrow u(x) \leq v(x), \forall x \in X \Leftrightarrow L_\alpha u \subseteq L_\alpha v, \ \forall \alpha \in [0,1].$$

Also, we can define a metric on $\mathcal{F}(X)$ by using the Hausdorff metric H as follows:

$$D(u,v) = \sup_{\alpha \in [0,1]} H(L_\alpha u, L_\alpha v), \ \forall u, v \in \mathcal{F}(X).$$

It is known that $(\mathcal{F}(X), D)$ is a complete but non-separable metric space (see [12]).

A linear structure of convex cone in $\mathcal{F}(X)$ is defined by

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\} \quad \text{and} \quad (\lambda u)(x) = \begin{cases} u(\frac{x}{\lambda}) & if \quad \lambda \neq 0 \\ \chi_{\{0\}}(x) & if \quad \lambda = 0, \end{cases}$$

where $u, v \in \mathcal{F}(X)$, $\lambda \in \mathbb{R}$ and χ_A denote the characteristic function of A.

With these definitions we obtain $L_\alpha(u+v) = L_\alpha u + L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$, for all $u, v \in \mathcal{F}(X)$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$ (see [9], [10], [11]).

Definition 2.9. *Let* $\Gamma : [0,b] \to \mathcal{F}(X)$ *be a fuzzy mapping and define* $\Gamma_\alpha : [0,b] \to \mathcal{K}(X)$ *by* $\Gamma_\alpha(t) = L_\alpha \Gamma(t), \ \forall \alpha \in [0,1]$. Then Γ *is called measurable if* Γ_α *is measurable for all* $\alpha \in [0,1]$. *Also,* Γ *is called integrably bounded if* Γ_α *is an integrably bounded set-valued function for every* $\alpha \in [0,1]$.

If Γ is a measurable fuzzy mapping, then Γ is called a **fuzzy random variable** (f.r.v.). See [9].

PROPOSITION 2.10.([9]) *If* $\Gamma : [0, b] \to \mathcal{F}(X)$ *is an integrably bounded f.r.v., then there exists a unique fuzzy set* $u \in \mathcal{F}(X)$ *such that* $L_\alpha u = \int_0^b \Gamma_\alpha dt,\ \forall \alpha \in [0, 1]$.
DEFINITION 2.11. *The element* $u \in \mathcal{F}(X)$ *obtained in Proposition 2.10 define the integral of the fuzzy random variable* Γ, *i.e.,* $\int_0^b \Gamma dt = u \Leftrightarrow L_\alpha u = \int_0^b \Gamma_\alpha dt$, *for every* $\alpha \in [0, 1]$.
PROPOSITION 2.12. *If* $\Gamma_1, \Gamma_2 : [0, b] \to \mathcal{F}_c(X)$ *are integrably bounded f.r.v. and* $\lambda \in \mathbb{R}$, *then*

$$\int_0^b (\lambda \Gamma_1 + \Gamma_2) dt = \lambda \int_0^b \Gamma_1 dt + \int_0^b \Gamma_2 dt.$$

For more details on properties of the integral for f.r.v., see [4,6,8,9].

3. FUZZY M-CONVEXITY

DEFINITION 3.1. *Let* $m \in [0, 1]$ *arbitrary. A fuzzy mapping* $\Gamma : [0, b] \to \mathcal{F}(X)$ *is called m-convex if*

$$t\Gamma(x) + m(1-t)\Gamma(y) \subseteq \Gamma(tx + m(1-t)y)$$

for all $x, y \in [0, b]$ *and* $t \in [0, 1]$. Also, *if* $m = 0$, *then we say that* Γ *is starshaped and if* $m = 1$, *we say that* Γ *is convex.*
REMARK 3.2. We observe that, Γ is m-convex if only if for each $\alpha \in [0, 1]$ the set-valued mapping Γ_α is m-convex, i.e.,

$$t\Gamma_\alpha(x) + m(1-t)\Gamma_\alpha(y) \subseteq \Gamma_\alpha(tx + m(1-t)y)$$

for all $x, y \in [0, b]$, $t \in [0, 1]$.
DEFINITION 3.3. *A fuzzy mapping* $\Lambda : [0, b] \to \mathcal{F}(X)$ *is called decreasing if for all* $x, y \in [0, b]$

$$x \leq y \Rightarrow \Lambda(x) \supseteq \Lambda(y)$$

or equivalently

$$x \leq y \Rightarrow \Lambda_\alpha(x) \supseteq \Lambda_\alpha(y),\ \forall \alpha \in [0, 1].$$

We consider the following sets of fuzzy mappings:

$$K_m = \{\Gamma : [0,b] \to \mathcal{F}(X) / \ \Gamma \ is \ m - convex \ and \ \chi_{\{\mathbf{0}\}} \subseteq \Gamma(0)\}$$

$$S = \{\Gamma : [0,b] \to \mathcal{F}(X) / \ x, y, x+y \in [0,b] \Rightarrow \Gamma(x+y) \subseteq \Gamma(x) + \Gamma(y)\}.$$

If $\Gamma \in S$ we say that Γ is **subadditive**.

We observe that, due Remark 2.8, we have

$$\chi_{\{\mathbf{0}\}} \subseteq \Gamma(0) \Leftrightarrow \mathbf{0} \in \Gamma_\alpha(0), \ \forall \alpha \in [0,1] \Leftrightarrow \mathbf{0} \in \Gamma_1(0)$$

where $\mathbf{0}$ is the null element of X.
LEMMA 3.4. *If $m \in [0,1]$, $\Gamma : [0,b] \to \mathcal{F}(X)$ is m-convex and $\chi_{\{\mathbf{0}\}} \subseteq \Gamma(0)$, then Γ is starshaped.*
PROOF.
Let $\alpha \in [0,1]$ arbitrary. Then, by the m-convexity of Γ, follows that Γ_α is m-convex, hence

$$\begin{aligned}
\Gamma_\alpha(tx) &= \Gamma_\alpha(tx + m(1-t)0) \\
&\supseteq t\Gamma_\alpha(x) + m(1-t)\Gamma_\alpha(0) \\
&\supseteq t\Gamma_\alpha(x) + m(1-t)\{\mathbf{0}\} \\
&= t\Gamma_\alpha(x),
\end{aligned}$$

consequently, $\Gamma(tx) \supseteq t\Gamma(x)$ for all $t \in [0,1]$ and $x \in [0,b]$. ■
LEMMA 3.5. *Let $\Gamma : [0,b] \to \mathcal{F}(X)$ a fuzzy mapping. Then Γ is starshaped if and only if the fuzzy mapping $\Lambda : (0,b] \to \mathcal{F}(X)$, $\Lambda(x) = \frac{1}{x}\Gamma(x)$, is decreasing.*
PROOF.
$(\to)$. Since Γ is starshaped, we have $t\Gamma_\alpha(y) \subseteq \Gamma_\alpha(ty)$, for all $\alpha \in [0,1]$, $y \in [0,b]$ and $t \in [0,1]$. Let $x, y \in (0,b]$ arbitraries, such that $x \leq y$. Then, taking $t = \frac{x}{y} \in (0,1]$, we obtain $\frac{x}{y}\Gamma_\alpha(y) \subseteq \Gamma_\alpha(x)$, which implies $\frac{1}{y}\Gamma_\alpha(y) \subseteq \frac{1}{x}\Gamma_\alpha(x)$ for every $\alpha \in [0,1]$. Therefore, Λ is decreasing.
$(\leftarrow)$. If Λ is decreasing then, for each $\alpha \in [0,1]$, $t \in [0,1]$ and $x \in [0,b]$ we have $\frac{\Gamma_\alpha(tx)}{tx} \supseteq \frac{\Gamma_\alpha(x)}{x}$, i.e., $\Gamma_\alpha(tx) \supseteq t\Gamma_\alpha(x)$. Therefore, $\Gamma(tx) \supseteq t\Gamma(x)$ and, consequently, Γ is starshaped. ■

Now, we will show that the class K_m, $m \in [0,1]$, is monotone (decreasing) with respect to m.

THEOREMA 3.6. *If* $0 \le n \le m \le 1$ *then* $K_1 \subseteq K_m \subseteq K_n \subseteq K_0 \subseteq S$.

PROOF.

If $\Gamma \in K_m$, $m > 0$, then Γ is m-convex and $\chi_{\{\mathbf{0}\}} \subseteq \Gamma(0)$. Thus, due Lemma 3.4, Γ is starshaped (i.e., $\Gamma \in K_0$). Therefore, for $n \le m$, we have

$$\begin{aligned} \Gamma(tx + n(1-t)y) &= \Gamma(tx + m(1-t)\frac{n}{m}y) \\ &\supseteq t\Gamma(x) + m(1-t)\Gamma(\frac{n}{m}y) \\ &\supseteq t\Gamma(x) + n(1-t)\Gamma(y) \end{aligned}$$

for all $x, y \in [0,b]$ and $t \in [0,1]$. This prove that $K_m \subseteq K_n$.

It remains to show that $K_0 \subset S$. Take $x, y \in [0,b]$ such that $x + y \in [0,b]$ and $\Gamma \in K_0$. If either $x = 0$ or $y = 0$ the proof is trivial. Assume that $x, y > 0$. Then, due Lemma 3.5 and the starshapeness of Γ, we have

$$\begin{aligned} \Gamma(x+y) &= (x+y)\frac{\Gamma(x+y)}{x+y} \\ &\subseteq x\frac{\Gamma(x+y)}{x+y} + y\frac{\Gamma(x+y)}{x+y} \\ &\subseteq x\frac{\Gamma(x)}{x} + y\frac{\Gamma(y)}{y} \\ &= \Gamma(x) + \Gamma(y). \end{aligned}$$

This prove that $\Gamma \in S$. $\blacksquare$

DEFINITION 3.7. *Let* $\Gamma : [0,b] \to \mathcal{F}(X)$ *an integrably bounded f.r.v., then the fuzzy mapping* $M_\Gamma : (0,b] \to \mathcal{F}(X)$ *given by*

$$M_\Gamma(x) = \frac{1}{x}\int_0^x \Gamma(t)dt \ , \ \forall x \in (0,b],$$

is called the fuzzy integral mean of Γ.

LEMA 3.8. *The* α*-level of* $M_\Gamma(x)$ *is given by* $L_\alpha M_\Gamma(x) = \frac{1}{x}\int_0^x \Gamma_\alpha(t)dt$, *for each* $x \in (0,b]$.

PROOF.

$L_\alpha\left(\frac{1}{x}\int_0^x \Gamma dt\right) = \frac{1}{x}L_\alpha\left(\int_0^x \Gamma dt\right) = \frac{1}{x}\int_0^x L_\alpha \Gamma dt = \frac{1}{x}\int_0^x \Gamma_\alpha dt.$ $\blacksquare$

REMARK 3.9. We observe that making the change of variable $t = xs$, the α-level of M_Γ can be writte as $L_\alpha M_\Gamma(x) = \int_0^1 \Gamma_\alpha(xs)ds$, i.e., $M_\Gamma(x) = \int_0^1 \Gamma(xs)ds$.

THEOREM 3.10. *Let $\Gamma : [0,b] \to \mathcal{F}_c(X)$ an integrably bounded f.r.v.. If Γ is m-convex, then so is M_Γ.*

PROOF.

Let Γ be m-convex and $x, y \in [0,b]$, and $t \in [0,1]$. Then, by using Remark 3.2 and Lemma 3.8, we obtain

$$\begin{aligned} L_\alpha M_\Gamma(tx + m(1-t)y) &= \int_0^1 \Gamma_\alpha(txs + m(1-t)ys)ds \\ &\supseteq \int_0^1 (t\Gamma_\alpha(xs) + m(1-t)\Gamma_\alpha(ys))ds \\ &= t\int_0^1 \Gamma_\alpha(xs)ds + m(1-t)\int_0^1 \Gamma_\alpha(ys)ds \\ &= tL_\alpha M_\Gamma(x) + m(1-t)L_\alpha M_\Gamma(y), \end{aligned}$$

thus, M_Γ is m-convex. ■

COROLLARY 3.11. *If Γ is an starshaped and integrably bounded f.r.v., then so is M_Γ.*

COROLLARY 3.12. *If Γ is a convex and integrably bounded f.r.v., then so is M_Γ.*

REMARK 3.13. Let $F : [0,b] \to \mathcal{K}(\mathbb{R})$ a set-valued function such that F has the form $F(x) = [f(x), g(x)]$, where $f, g : [0,b] \to \mathbb{R}$ and $f \le g$. Then, the measurability of F implies the measurability of f and g. Moreover, if F is integrably bounded then f and g are integrable functions (in the Lebesgue sense). Also, in this case, the subadditivity of F implies the subadditivity of $M_F(x) = \frac{1}{x}\int_0^x Fdt$, i.e., $M_F(x+y) \subseteq M_F(x) + M_F(y)$, $\forall x, y \in (0,b]$ such that $x + y \in (0,b]$ (see [14]).

THEOREM 3.14. *If $\Gamma : [0,b] \to \mathcal{F}_c(X)$ is an integrably bounded f.r.v., then the subadditivity of Γ implies the subadditivity of M_Γ.*

PROOF.

From subadditivity of Γ follows that Γ_α is subadditive for each $\alpha \in [0,1]$. Let $\alpha \in [0,1]$ and $y^* : X \to \mathbb{R}$ an arbitrary continuous linear functional in X^*, and consider the set-valued function $y^* \circ \Gamma_\alpha : [0,b] \to \mathcal{K}(\mathbb{R})$ defined by

$$(y^* \circ \Gamma_\alpha)(x) = y^*(\Gamma_\alpha(x)),\ \forall x \in [0,b].$$

Since for every $x \in [0, b]$, $\Gamma_\alpha(x)$ is a nonempty, compact and convex subset of X, due continuity of y^*, $y^* \circ \Gamma_\alpha$ has the form $(y^* \circ \Gamma_\alpha)(x) = [f(x), g(x)]$, where $f, g : [0, b] \to \mathbb{R}$ and $f \leq g$. Now, since that Γ_α is an integrably bounded r.s., then so is $y^* \circ \Gamma_\alpha$. Consequently, from Remark 3.13, follows that f and g are integrable functions. Thus, by Remark 3.13, implies that $M_{y^* \circ \Gamma_\alpha}$is also subadditive, i.e.,

$$(y^* \circ M_{\Gamma_\alpha})(x) = \frac{1}{x} \int_0^x y^*(\Gamma_\alpha(t))dt = y^*(\int_0^1 \Gamma_\alpha(xs)ds) = \int_0^1 y^*(\Gamma_\alpha(xs))ds$$

is subadditive, that is,

$$x, y, x+y \in (0, b] \Rightarrow (y^* \circ M_{\Gamma_\alpha})(x+y) \subseteq (y^* \circ M_{\Gamma_\alpha})(x) + (y^* \circ M_{\Gamma_\alpha})(y). \quad (1)$$

Since Γ_α is an integrably bounded r.s. taking compact values, follows that for each $x \in (0, b]$, $M_{\Gamma_\alpha}(x)$ is a compact set (see [5]). Consequently, $(y^* \circ M_{\Gamma_\alpha})(x) \in \mathcal{K}(X), \forall\ x \in (0, b]$. So, from (1) and linearity of y^* we obtain

$$\begin{aligned} (y^* \circ M_{\Gamma_\alpha})(x+y) &\subseteq (y^* \circ M_{\Gamma_\alpha})(x) + (y^* \circ M_{\Gamma_\alpha})(y) \\ &\subseteq y^*(M_{\Gamma_\alpha}(x) + M_{\Gamma_\alpha}(y)) \end{aligned}$$

for all $x, y \in (0, b]$ and $x + y \in (0, b]$.

We will prove that the set-valued function M_{Γ_α} must be subadditive. By contradiction, assume that there are $x, y \in (0, b]$, such that $x + y \in (0, b]$ and

$$M_{\Gamma_\alpha}(x+y) \nsubseteq M_{\Gamma_\alpha}(x) + M_{\Gamma_\alpha}(y).$$

Then there exist $p \in M_{\Gamma_\alpha}(x+y)$ such that $p \notin M_{\Gamma_\alpha}(x) + M_{\Gamma_\alpha}(y)$. So, by using the separation Hahn-Banach Theorem, there exist $c \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$y^*(p) \geq c + \varepsilon \quad \text{and} \quad \sup_{q \in M_{\Gamma_\alpha}(x) + M_{\Gamma_\alpha}(y)} y^*(q) \leq c.$$

But this contradicts the inclusion (1), and the proof is completed. ■

4. CONVERGENCE OF INTEGRAL MEAN

In this section we consider the space $B([0,b],\mathcal{F}(X))$ of all f.r.v. $\Gamma : [0,b] \to \mathcal{F}(X)$, endowed of the uniform metric D_∞ defined by $D_\infty(\Gamma,\Lambda) = \sup\limits_{x\in[0,b]} D(\Gamma(x),\Lambda(x))$.

It is known that $(B([0,b],\mathcal{F}(X)), D_\infty)$ is a complete metric space.

PROPOSITION 4.1. *Let $\Gamma_n, \Gamma \in B([0,b],\mathcal{F}(X))$ such that $\Gamma_n \overset{D_\infty}{\to} \Gamma$. Then $M_{\Gamma_n} \overset{D_\infty}{\to} M_\Gamma$ on $(0,b]$.*

PROOF.

If $\Gamma_n \overset{D_\infty}{\to} \Gamma$, then given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$D(\Gamma_n(x),\Gamma(x)) < \epsilon \ \ , \ \forall n \geq n_0, \ \forall x \in [0,b].$$

Now, for $n \geq n_0$, we have

$$\begin{aligned} D(M_{\Gamma_n}(x), M_\Gamma(x)) &= D\left(\frac{1}{x}\int_0^x \Gamma_n(t)dt, \frac{1}{x}\int_0^x \Gamma(t)dt\right) \\ &= \frac{1}{x} D\left(\int_0^x \Gamma_n(t)dt, \int_0^x \Gamma(t)dt\right) \\ &\leq \frac{1}{x}\int_0^x D(\Gamma_n(t),\Gamma(t))dt < \frac{1}{x}\int_0^x \epsilon dt = \epsilon. \end{aligned}$$

Therefore $M_{\Gamma_n} \overset{D_\infty}{\to} M_\Gamma$. ■

We recall that if $A \in \mathcal{K}(X)$, the support function $\sigma_A : X^* \to \mathbb{R}$ is defined as

$$\sigma_A(x^*) = \max_{a\in A} < x^*, a >, \ \forall x^* \in X^*.$$

It is important to remark that if $A, B \in \mathcal{K}_c(X)$ then, as a direct consequence of the separation Hahn-Banach theorem, we obtain: $\sigma_A = \sigma_B \Leftrightarrow A = B$.

In the fuzzy context, the support function $s_u : [0,1] \times X^* \to \mathbb{R}$ of a fuzzy set $u \in \mathcal{F}(X)$, is defined by $s_u(\alpha, x^*) = \sigma_{L_\alpha u}(x^*), \ \forall(\alpha, x^*) \in [0,1] \times X^*$.

We observe that if $u = \chi_A, \ A \in \mathcal{K}(X)$, then

$$s_u(\alpha, x^*) = \sigma_A(x^*), \ \forall \alpha \in [0,1].$$

For details on support functions see [2], [4], [10], [11].

A beautiful result due Aumann [1] (see also [2]), is the following

THEOREM 4.2. ([1]). *If F is a measurable, closed, nonempty and integrably bounded set-valued function on $[a,b]$, then*

$$\sigma_{\int_a^b F(t)dt}(x^*) = \int_a^b \sigma_{F(t)}(x^*)dt.$$

PROPOSITION 4.3. *If $\Gamma : [0,b] \to \mathcal{F}(X)$ is an integrably bounded f.r.v., then*

$$s_{M_\Gamma(x)}(\cdot,\cdot) = \int_0^1 s_{\Gamma(xs)}(\cdot,\cdot)ds \quad \forall x \in (0,b].$$

PROOF.
Let $x \in (0,b]$ arbitrary and $(\alpha, y) \in [0,1] \times S^{n-1}$. Then, Theorem 4.2 and Remark 3.9, we have

$$\begin{aligned} s_{M_\Gamma(x)}(\alpha,y) &= \sigma_{L_\alpha M_\Gamma(x)}(y) = \sigma_{\int_0^1 \Gamma_\alpha(xs)ds}(y) \\ &= \int_0^1 \sigma_{\Gamma_\alpha(xs)}(y)ds = \int_0^1 s_{\Gamma(xs)}(\alpha,y)ds. \end{aligned}$$

Therefore $s_{M_\Gamma(x)}(\alpha,y) = \int_0^1 \sigma_{\Gamma(xs)}(\alpha,y)ds. \quad \forall x \in (0,b].$ ■

PROPOSITION 4.4. *Let $\Gamma : [0,b] \to \mathcal{F}(X)$ a D-continuous fuzzy mapping. If $x_n \to x$ in $(0,b]$, then $M_\Gamma(x_n) \xrightarrow{D} M_\Gamma(x)$.*

PROOF.
Since Γ is D-continuous on $[0,b]$, then Γ is uniformly continuous. Therefore given $\epsilon > 0$, there exist $\delta > 0$ such that

$$\mid z - w \mid < \delta \text{ implies } D(\Gamma(z), \Gamma(w)) < \epsilon. \tag{2}$$

Also, due $x_n \to x$, there exist $n_0 \in \mathbb{N}$ such that $\mid x_n - x \mid < \delta$, $\forall n \geq n_0$. Now, we note that if $\mid x_n - x \mid < \delta$ and $s \in [0,1]$, then $\delta > \mid x_n - x \mid \geq \mid x_n s - xs \mid$. Thus, for $n \geq n_0$, by Remark 3.9 and (2) we have

$$\begin{aligned} D(M_\Gamma(x_n), M_\Gamma(x)) &= D(\int_0^1 \Gamma(x_n s)ds, \int_0^1 \Gamma(xs)ds) \\ &\leq \int_0^1 D(\Gamma(x_n s), \Gamma(xs))ds < \epsilon. \end{aligned}$$

Therefore, $M_\Gamma(x_n) \overset{D}{\to} M_\Gamma(x)$. ■

Let $u \in \mathcal{F}(X)$ be and consider the level-application $\alpha \mapsto L_\alpha u$ which is a set-valued function taking compact valued. Define

$$\mathcal{F}_C(X) = \{u \in \mathcal{F}(X)/ \ \alpha \mapsto L_\alpha u \ H - continuous \ on \ [0,1]\},$$

the space of all compact and level-continuous fuzzy set on X.

It is known that $(\mathcal{F}_C(X), D)$ is a complete and separable metric space (see [11], [12]). Also, for more details on level-continuity of functions, see [13].

THEOREM 4.5. *If* $\Gamma : [0,b] \to \mathcal{F}_C(X)$ *is an integrably bounded f.r.v., then* $M_\Gamma(x) \in \mathcal{F}_C(X), \ \forall x \in [0,1]$.

PROOF.

Let $x \in [0,b]$ and $\alpha_n, \alpha \in [0,1]$ such that $\alpha_n \to \alpha$. Since Γ_0 is an integrably r.s., then there exist $f \in L^1([0,b], X)$ such that $||w|| \le f(t)$ for all w and t such that $w \in \Gamma_0(t)$. Because $\Gamma_\beta(t) \subseteq \Gamma_0(t), \ \forall \beta \in [0,1]$, then $\sup\limits_{n\ge1} \| \ g_n(t) \ \| \le f(t)$ for all $g_n \in S(\Gamma_{\alpha_n})$. Thus, due $\Gamma_{\alpha_n}(y) = L_{\alpha_n}\Gamma(y) \underset{n\to\infty}{\overset{H}{\longrightarrow}} L_\alpha\Gamma(y) = \Gamma_\alpha(y)$,for each $y \in [0,b]$, by Remark 3.9 and Th.2.7, we obtain

$$L_{\alpha_n} M_\Gamma(x) = \int_0^1 \Gamma_{\alpha_n}(xs)ds \underset{n\to\infty}{\longrightarrow} \int_0^1 \Gamma_\alpha(xs)ds = L_\alpha M_\Gamma(x).$$

Therefore, $M_\Gamma(x)$ is level-continuous. ■

EXAMPLE 4.6. Consider $\Gamma : [0,1] \to \mathcal{F}(\mathbb{R})$ defined by

$$\Gamma(x)(t) = \begin{cases} \begin{cases} \frac{2t}{x} & if \quad 0 \le t \le \frac{x}{2} \\ 1 & if \quad \frac{x}{2} \le t \le 1 \\ 0 & if \quad t \notin [0,1] \end{cases} & if \quad x \ne 0 \\ \chi_{[0,1]} & if \quad x = 0. \end{cases}$$

Then, for $x \in (0,1]$, we want to calculate $M_\Gamma(x)$. For this, firstly we observe that $L_\alpha\Gamma(x) = \left[\frac{\alpha x}{2}, 1\right]$ for each $\alpha \in (0,1]$, therefore $\Gamma_\alpha(xs) = \left[\frac{\alpha xs}{2}, 1\right]$. Thus, for every $r \in \mathbb{R}$,

$$\sigma_{\Gamma_\alpha(xs)}(r) = \begin{cases} r & if \quad r \ge 0 \\ \frac{\alpha xsr}{2} & if \quad r < 0. \end{cases}$$

Consequently, due Th. 4.2, we have

$$\sigma_{\int_0^1 \Gamma_\alpha(xs)ds}(r) = \left\{ \begin{array}{lll} r & if & r \geq 0 \\ \frac{\alpha x r}{4} & if & r < 0. \end{array} \right. \quad = \quad \sigma_{\left[\frac{\alpha x}{4},1\right]}(r)$$

Therefore, $\int_0^1 \Gamma_\alpha(xs)ds = L_\alpha \int_0^1 \Gamma(xs)ds = \left[\frac{\alpha x}{4}, 1\right]$, $\forall \alpha \in [0,1]$.
Now, if $u = \int_0^1 \Gamma(xs)ds$ then, $L_0 u = [0,1]$ and $L_1 u = [\frac{x}{4}, 1]$, and by Remark 2.8, $u(r) = sup\{\alpha / \ r \in [\frac{\alpha x}{4}, 1]\} = \frac{4r}{x}$, i.e.,

$$M_\Gamma(x)(r) = u_x(r) = \left\{ \begin{array}{lll} \frac{4r}{x} & if & 0 \leq r \leq \frac{x}{4} \\ 1 & if & \frac{x}{4} \leq r \leq 1 \\ 0 & if & r \notin [0,1]. \end{array} \right.$$

Finally, the reader can be check that Γ is a convex, subadditve and continuous (actually, Γ is an $\frac{1}{2}$-contraction) fuzzy mapping taking values in $\mathcal{F}_C(X)$, and so is M_Γ on $(0,1]$.$\blacktriangle$
The next example shows a discontinuous fuzzy mapping Γ on $[0,1]$ taking values in $\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}_C(\mathbb{R})$ *a.e.*, and whose integral mean M_Γ is a D–continuous fuzzy mapping on $(0,1]$ taking values in $\mathcal{F}_C(\mathbb{R})$.
Example 4.7. Consider $\Gamma : [0,1] \to \mathcal{F}(\mathbb{R})$ defined by

$$\Gamma(x)(t) = \left\{ \begin{array}{ll} \left\{ \begin{array}{lll} x & if & 0 \leq t < 1 \\ 1 & if & t = 1 \\ 0 & if & t \notin [0,1] \end{array} \right. & if \quad x \neq 1 \\ \chi_{[0,1]} & if \quad x = 1. \end{array} \right.$$

We observe that if $x \in (0,1)$ then $\Gamma(x) \notin \mathcal{F}_C(\mathbb{R})$, i.e., $\Gamma(x)$ is not level-continuous. In fact, a straightforwad calculus shows that, in this case, $\Gamma(x)$ is level-discontinuous in $\alpha = x$.
Nevertheless, the reader can be check that

$$M_\Gamma(x)(r) = u_x(r) = \left\{ \begin{array}{lll} rx & if & 0 \leq r < 1 \\ 1 & if & r = 1 \\ 0 & if & r \notin [0,1]. \end{array} \right.$$

Thus,

$$L_\alpha M_\Gamma(x) = L_\alpha u_x(r) = \left\{ \begin{array}{lll} \left[\frac{\alpha}{x}, 1\right] & if & 0 \leq \alpha \leq x \\ \{1\} & if & x \leq \alpha \leq 1, \end{array} \right.$$

and it is easy to see that $M_\Gamma(x) \in \mathcal{F}_C(\mathbb{R})$, $\forall x \in (0,1]$. .

Finally, because $D(\Gamma(x),\Gamma(y)) = 1$, $\forall x \neq y$, we obtain that Γ is not $D-$continuous on $[0,1]$. Now, if $0 < x \leq y \leq 1$ and $y \leq \alpha \leq 1$, then $H(L_\alpha u_x, L_\alpha u_y) = H(\{1\},\{1\}) = 0$. Therefore,

$$\begin{aligned}
D(M_\Gamma(x), M_\Gamma(y)) &= \sup_{\alpha\in[0,1]} H(L_\alpha u_x, L_\alpha u_y) \\
&= \sup_{\alpha\in[0,x]} (HL_\alpha u_x, L_\alpha u_y) \vee \sup_{\alpha\in[x,y]} H(L_\alpha u_x, L_\alpha u_y) \\
&= \sup_{\alpha\in[0,x]} H([\frac{\alpha}{x},1],[\frac{\alpha}{y},1]) \vee \sup_{\alpha\in[x,y]} H(\{1\},[\frac{\alpha}{y},1]) \\
&= \sup_{\alpha\in[0,x]} \left|\frac{\alpha}{x} - \frac{\alpha}{y}\right| \vee \sup_{\alpha\in[x,y]} \left|1 - \frac{\alpha}{y}\right| \\
&= 1 - \frac{x}{y}\,.
\end{aligned}$$

Thus, $D(M_\Gamma(x_n), M_\Gamma(x)) \to 0$ as $x_n \to x$, *i.e.*, Γ is $D-$continuous “in mean” on $(0,1]$.▲

References

[1] R.J. Aumann, Integrals of set-valued functions, *J. Math. Anal. Appl.* **12**, 1-11, (1965).

[2] J.P. Aubin and H. Frankowska, *Set-valued analysis*, Birkahuser, (1990).

[3] A.M. Bruckner and E. Ostrow, Some functions classes related to the class of convex functions, *Pacific J. Math.* **12**, 1203-1215, (1962).

[4] Z. Ding, M. Ma and A. Kandel, Existence of the solutions of fuzzy differential equations with parameters, *Information Sciences* **99**, 205-217, (1997).

[5] F. Hiai and H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivar. Anal.* **7**, 149-182, (1977).

[6] O. Kaleva, Fuzzy differential equations, *Fuzzy sets and Systems* **24**, 301-307, (1987).

[7] E. Klein and A.C. Thompson, *Theory of correspondences*, Wiley-Interscience, New York, 1984.

[8] B-S Lee, S.J. Cho, G.M. Lee and D.S. Kim, Integration of generalized fuzzy number-valued mappings, *J. Fuzzy Math.* **7**, 209-219, (1999).

[9] M.L. Puri and D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114**, 301-317, (1987).

[10] M.A. Rojas-Medar, R.C. Bassanezi and H. Román-Flores, A generalization of the Minkowski embedding theorem and applications, *Fuzzy Sets and Systems* **102**, 263-269, (1999).

[11] H. Román-Flores and M. Rojas-Medar**,** Embedding of level-continuous fuzzy sets on Banach spaces, Technical Report **46/99**, IMECC-UNICAMP, (1999).

[12] H.Román-Flores, The compactness of E(X), *Appl. Math. Lett.* **11** (2), 13-17, (1998).

[13] H. Román-Flores and M. Rojas-Medar, Level-continuity of functions and applications, *Computers Math. Applic.* **38**, 143-149, (1999).

[14] W. Ślepak, On multivalued integral mean, *Demonstratio Mathematica* **31**, 339-344, (1998).

[15] G.H. Toader, On the hierarchy of convexity of functions, *L'analyse Num. Théorie L'Approx.* **15**, 167-172, (1986).

[16] G.H. Toader, On a generalization of the convexity, *L'analyse Num. Théorie L'Approx.* **30**, 83-87, (1988).