Periodic Strong Solutions of the Magnetohydrodynamic Type Equations[1]

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Abstract

Existence and uniqueness of periodic strong solutions for the magnetohydrodynamic type equations are studied in this work

1. Introduction

In several situations the motion of incompressible electrical conducting fluid can be modelled by the magnetohydrodynamics equation , which correspond to the Navier-Stokes equations coupled with the Maxwell equations. In presence of a free motion of heavy ions, not directly due to the electrical field (see Schlüter [21], and Pikelner [15]), the magnetohydrodynamics equation can be reduced to

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \frac{\eta}{\rho_m}\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{\mu}{\rho_m}\mathbf{h}\cdot\nabla\mathbf{h} &= \mathbf{f} - \frac{1}{\rho_m}\nabla(p^* + \frac{\mu}{2}\mathbf{h}^2) \\ \frac{\partial \mathbf{h}}{\partial t} - \frac{1}{\mu\sigma}\Delta \mathbf{h} + \mathbf{u}\cdot\nabla\mathbf{h} - \mathbf{h}\cdot\nabla\mathbf{u} &= -\ \mathrm{grad}\ w \qquad (1.1) \\ \mathrm{div}\ \mathbf{u} &= 0 \\ \mathrm{div}\ \mathbf{h} &= 0. \end{aligned}$$

Here, $\mathbf{u}$ and $\mathbf{h}$ are respectively the unknown velocity and magnetic fields; p^* is the unknown hydrostatic pressure; w is an unknown function related to the motion of heavy ions (in such way that the density of electric current, j_0, generated by this motion satisfies the relation rot

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$j_0 = -\sigma\nabla w$); ρ_m is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; $\mathbf{f}$ is an given external force field.

We append to equation (1.1) the following boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \mathbf{h}|_{\partial\Omega} = 0. \tag{1.2}$$

In this paper, we will consider the problem of the existence and uniqueness of the periodic strong solutions in a bounded domain $\Omega \subset I\!R^N, N = 3$ or 4 : the given external force $\mathbf{f}$ be periodic in t with some period τ. Then we will prove the existence and uniqueness of periodic strong solution $(\mathbf{u}, \mathbf{h})$ of the magnetohydrodynamic type equations (1.1) with the same period τ

$$\mathbf{u}(x, t+\tau) = \mathbf{u}(x, t); \qquad \mathbf{h}(x, t+\tau) = \mathbf{h}(x, t) \tag{1.3}$$

The initial value problem associated to the system (1.1) has been studied by several authors. Lassner [13], by using the semigoup results of Kato and Fujita [9], proved the existence and uniqueness of strong solutions. Boldrini and Rojas-Medar , [5], [18] improved this results to global solutions by using the spectral Galerkin method. Damásio and Rojas-Medar [8] studied the regularity of weak solutions, Notte-Cuello and Rojas-Medar [16] using an iterative approach to show the existence and uniqueness of strong solutions. The initial value problem in a time dependent domains was studied by Rojas-Medar and Beltrán-Barrios [17] and Berselli and Ferreira [4].

The periodic problem to the classical Navier-Stokes equatios, was studied by Serrin [19] using the perturbation method and recently by Kato [12] using the spectral Galerkin method. In this work we follow [12].

Finally, we would like to say that, as it usual in this context, to simplicity the notation in the expressions we will denote by $C, C_1, \ldots$,generic positive constants depending only on the fixed data of the problem.

2. Preliminaries and Results

We begin by recalling certain definitions and facts to be used later in this paper.

The $L^2(\Omega)$-product and norm are denoted by $(,)$ and $|\ \ |$, respectively; the $L^p(\Omega)$-norm by $|\ \ |_{L^p}, 1 \le p \le \infty$; the $H^m(\Omega)$- norm are denoted by $\|\ \ \|_{H^m}$ and the $W^{k,p}(\Omega)$-norm by $|\ \ |_{W^{k,p}}$.

Here $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the H^1- norm.

If B is a Banach space, we denote $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval (0, T) that are L^q-integrable in the sense of Bochner.

Let $C_{0,\sigma}^\infty(\Omega) = \{v \in (C_0^\infty(\Omega))^N;\ \text{div}\ v = 0\}, V =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $(H_0^1(\Omega))^N$ and $H =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $(L^2(\Omega))^N$.

Let P be the orthogonal projection from $(L^2(\Omega))^N$ onto H obtained by the usual Helmholtz decomposition. Then, the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = (H^2(\Omega))^N \cap V$ is called the Stokes operator.

In order to obtain regularity properties of the Stokes operator we will assume that Ω is of class $C^{1,1}$ [2]. This assumption implies, in particular, that when $A\mathbf{u} \in L^2(\Omega)$, then $\mathbf{u} \in H^2(\Omega)$ and $\|\mathbf{u}\|_{H^2}$ and $|A\mathbf{u}|$ are equivalent norms.

Now, let us introduce some functions spaces consisting of τ-periodic functions. For $k \ge 0$, $k \in I\!N$, we denote by

$$C^k(\tau;B) = \{f : I\!R \to B\ /\ f \text{ is } \tau\text{- periodic and } D_t^i f \in C(I\!R;B) \text{ para todo } i \le k\}.$$

Then, let us define the norm

$$\|f\|_{C^k(\tau;B)} = \sup_{0\le t\le\tau} \sum_{i=1}^{k} \|D_t^i f(t)\|_B.$$

We denote for $1 \le p \le \infty$, the spaces

$$L^p(\tau;B) = \{f : I\!R \to B\ /\ f \text{ is measurable,} \tau\text{- periodic and } \|f\|_{L^p(\tau;B)} < \infty\},$$

where

$$\|f\|_{L^p(\tau;B)} = \left(\int_0^\tau \|f(t)\|_B^p\right)^{\frac{1}{p}} \text{ para } 1 \le p < \infty$$

and

$$\|f\|_{L^\infty(\tau;B)} = \sup_{0\le t\le\tau} \|f(t)\|_B.$$

Similary, we denote by

$$W^{k,p}(\tau;B)=\{f\in L^p(\tau;B)\ /\ D_t^i f\in L^p(\tau;B) \text{ para todo } i\le k\}.$$

In particular, $H^k(\tau;B)=W^{k,2}(\tau;B)$, when B is a Hilbert space.

The following results will be using in this paper.

Proposition 2.1. *(Giga e Miyakawa [10]). If $0\le\delta<\frac{1}{2}+\frac{N}{4}$, the following estimate is valid with a constant $C_1=C_1(\delta,\theta,\rho)$,*

$$|A^{-\delta}P\mathbf{u}\cdot\nabla\mathbf{v}|\le C_1|A^\theta\mathbf{u}||A^\rho\mathbf{v}| \text{ for any } \mathbf{u}\in D(A^\theta) \text{ and } \mathbf{v}\in D(A^\rho), \quad (2.1)$$

with $\delta+\theta+\rho\ge\frac{N}{4}+\frac{1}{2}$, $\rho+\delta>\frac{1}{2}$, and $\theta,\rho>0$.

We consider too the Sobolev inequality [10],

$$|\mathbf{u}|_{L^r(\Omega)}\le C_2||\mathbf{u}||_{H^\beta}, \text{ if } \frac{1}{r}\ge\frac{1}{2}-\frac{\beta}{N}>0, \quad (2.2)$$

and the inequality due to Giga and Miyakawa [10]

$$|\mathbf{u}|_{L^r(\Omega)}\le C_3|A^\gamma\mathbf{u}|, \text{ if } \frac{1}{r}\ge\frac{1}{2}-\frac{2\gamma}{N}>0. \quad (2.3)$$

Here, we note that if $r=N$ in (2.3) it follows

$$|\mathbf{u}|_{L^N(\Omega)}\le C_3|A^\gamma\mathbf{u}|, \text{ with } \gamma=\frac{N}{4}-\frac{1}{2}. \quad (2.4)$$

Lemma 2.2. *If $\mathbf{u}\in D(A^\theta)$ and $0\le\theta<\beta$, then*

$$|A^\theta\mathbf{u}(x)|\le\mu^{\theta-\beta}|A^\beta\mathbf{u}(x)| \quad (2.5)$$

where $\mu=\min\lambda_j>0$.

Lemma 2.3. *(Simon [20])Let X, B and Y Banach spaces such that $X\hookrightarrow B\hookrightarrow Y$, where the firts embedding is compact and the second is continuous. Then, if $T>0$ is finite, we have that the following embedding is compact*

$$L^\infty(0,T;X)\cap\{\phi:\phi_t\in L^r(0,T;Y\}\hookrightarrow C(0,T;B), \text{ se } 1<r\leq\infty.$$

Ours results are the following.

Theorem 2.4. *(Existence). Let* $\mathbf{f}\in H^1(\tau;H)$ $(\tau>0)$. *Then there exists a constant* $K_0=K_0(N)>0$ *such that if*

$$M=\sup_{0\leq t\leq\tau}|\mathbf{f}|_{L^{\frac{N}{2}}(\Omega)}\leq K_0,$$

the problem (1.1)- (1.3) has an τ*- periodic strong solution* $(\mathbf{u}(t),\mathbf{h}(t))$ *satisfying*

$$(\mathbf{u},\mathbf{h})\in(H^2(\tau;H))^2\cap(H^1(\tau;D(A)))^2\cap(L^\infty(\tau;D(A)))^2\cap(W^{1,\infty}(\tau;V)))^2.$$

Theorem 2.5. *(Uniqueness). The solution of (1.1)- (1.3) given in Theorem 1.4 is unique.*

The ideia of the proof is use the spectral Galerkin method together with compactness arguments. The principal problem is obtain the uniform boundedness of certains norms of $\mathbf{u}^n(t)$ and $\mathbf{h}^n(t)$ in some point t^*. This difficult was early treated by Heywood [11] to prove the regulatity of the classical solutions for Navier-Stokes equations.

3. Approximate problem and a priori estimates

By using the operator P, the periodic problem (1.1)- (1.3) is formulated as a system of ordinary differential equations

$$\begin{aligned}\frac{d}{dt}\mathbf{u}(t)+\nu A\mathbf{u}(t)+\alpha P(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t))-P(\mathbf{h}(t)\cdot\nabla\mathbf{h}(t)) &= \alpha P\mathbf{f}(t),\\ \frac{d}{dt}\mathbf{h}(t)+\chi A\mathbf{h}(t)+P(\mathbf{u}(t)\cdot\nabla\mathbf{h}(t))-P(\mathbf{h}(t)\cdot\nabla\mathbf{u}(t)) &= 0,\end{aligned}$$

$$\mathbf{u}(x,t+\tau)=\mathbf{u}(x,t);\qquad \mathbf{h}(x,t+\tau)=\mathbf{h}(x,t).$$

Where,

$$\alpha=\frac{\rho_m}{\mu},\quad \nu=\frac{\eta}{\mu},\quad \chi=\frac{1}{\mu\sigma}.$$

We consider $V_n = span\{w_1(x), w_2(x), ..., w_n(x)\}$ and the approximations $\mathbf{u}^n(t) = \sum_{j=1}^n c_{jn}(t)w_j(x)$ and $\mathbf{h}^n(t) = \sum_{j=1}^n d_{jn}(t)w_j(x)$, of $\mathbf{u}$ and $\mathbf{h}$, respectively satisfying the following system of ordinary differential equations

$$\begin{aligned}
(\alpha\mathbf{u}_t^n + \nu A\mathbf{u}^n + \alpha P\mathbf{u}^n \cdot \nabla\mathbf{u}^n - P\mathbf{h}^n \cdot \nabla\mathbf{h}^n, \omega^j) &= (\alpha\mathbf{f}, \omega^j), \\
(\mathbf{h}_t^n + \chi A\mathbf{h}^n + P\mathbf{u}^n \cdot \nabla\mathbf{h}^n - P\mathbf{h}^n \cdot \nabla\mathbf{u}^n, \omega^j) &= 0, \\
\mathbf{u}^n(t+\tau) &= \mathbf{u}^n(t), \\
\mathbf{h}^n(t+\tau) &= \mathbf{h}^n(t).
\end{aligned} \tag{3.1}$$

To show that the system (3.1) has an unique $\tau-$periodic solution, we consider the following linearized problem:

$$\begin{aligned}
(\alpha\mathbf{u}_t^n + \nu A\mathbf{u}^n, \omega^j) &= (\alpha\mathbf{f} - \alpha P\mathbf{v}^n \cdot \nabla\mathbf{v}^n + P\mathbf{b}^n \cdot \nabla\mathbf{b}^n, \omega^j), \\
(\mathbf{h}_t^n + \chi A\mathbf{h}^n, \omega^j) &= (-P\mathbf{u}^n \cdot \nabla\mathbf{h}^n + P\mathbf{h}^n \cdot \nabla\mathbf{u}^n, \omega^j),
\end{aligned} \tag{3.2}$$

where $\mathbf{v}^n(t) = \sum_{j=1}^n e_{jn}(t)\omega^j(x)$ and $\mathbf{b}^n(t) = \sum_{j=1}^n g_{jn}(t)\omega^j(x)$ are functions given in $C^1(\tau; V_n)$.

It is well know that the linearized system (3.2) has an unique $\tau-$periodic solution $(\mathbf{u}^n(t), \mathbf{h}^n(t)) \in (C^1(\tau; V_n))^2$ (see for instance, [1], [6]). On the other hand, it is easily checked that the map: $(\mathbf{v}^n, \mathbf{b}^n) \to (\mathbf{u}^n, \mathbf{h}^n)$ is continuous and compact in $(C^1(\tau; V_n))^2$.

By using the Leray-Schauder principle is sufficient to show the boundedness

$$\sup_{0\le t\le\tau} |\mathbf{u}^n(t)| \le C,$$
$$\sup_{0\le t\le\tau} |\mathbf{h}^n(t)| \le C,$$

where C is a positive constant independent of λ , for all solutions of (1.1) replacing $P\mathbf{u}^n\cdot\nabla\mathbf{u}^n$ by $\lambda P\mathbf{u}^n\cdot\nabla\mathbf{u}^n$, $P\mathbf{h}^n\cdot\nabla\mathbf{h}^n$ by $\lambda P\mathbf{h}^n\cdot\nabla\mathbf{h}^n$, $P\mathbf{u}^n\cdot\nabla\mathbf{h}^n$ by $\lambda P\mathbf{u}^n \cdot \nabla\mathbf{h}^n$ and $P\mathbf{h}^n \cdot \nabla\mathbf{u}^n$ by λ $P\mathbf{h}^n \cdot \nabla\mathbf{u}^n$ $(0 \le \lambda \le 1)$ (see [3]).

Then, multiplying $(3.2)_i$ and $(3.2)_{ii}$ by $e_{jn}(t)$ and $g_{jn}(t)$ respectively, and adding in j, we obtain

$$\frac{\alpha}{2}\frac{d}{dt}|\mathbf{u}^n|^2 + \nu|\nabla\mathbf{u}^n|^2 = (\alpha\mathbf{f}, \mathbf{u}^n) + \lambda(\mathbf{h}^n \cdot \nabla\mathbf{h}^n, \mathbf{u}^n),$$
$$\frac{1}{2}\frac{d}{dt}|\mathbf{h}^n|^2 + \chi|\nabla\mathbf{h}^n|^2 = \lambda(\mathbf{h}^n \cdot \nabla\mathbf{u}^n, \mathbf{h}^n)$$

since $\alpha(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \mathbf{u}^n) = (\mathbf{u}^n \cdot \nabla \mathbf{h}^n, \mathbf{h}^n) = 0$.

Adding the above inequalities and using (2.3), we get

$$\begin{aligned} & \frac{d}{dt}(\alpha|\mathbf{u}^n|^2 + |\mathbf{h}^n|^2) + 2\nu|\nabla \mathbf{u}^n|^2 + 2\chi|\nabla \mathbf{h}^n|^2 = 2(\alpha \mathbf{f}, \mathbf{u}^n) \\ \leq \;& 2\alpha \|\mathbf{f}\|_{L^{2N/(N+2)}(\Omega)} \|\mathbf{u}^n\|_{L^{2N/(N-2)}(\Omega)} \\ \leq \;& 2C_3 C(N) \alpha \, |f|_{L^{N/2}(\Omega)} \, |\nabla \mathbf{u}^n| \end{aligned}$$

where $C(N) = |\Omega|^{\frac{N-2}{2N}}$ and $|\Omega| \equiv$ the volume of Ω. By using the Young inequality, we obtain

$$\frac{d}{dt}(\alpha|\mathbf{u}^n|^2 + |\mathbf{h}^n|^2) + \nu|\nabla \mathbf{u}^n|^2 + 2\chi|\nabla \mathbf{h}^n|^2 \leq C_4 C(N)^2 M^2 \tag{3.3}$$

where $C_4 = C_3^2 \alpha^2/\nu$ and M is defined as in theorem 2.4. Moreover, since $(\mathbf{u}^n, \mathbf{h}^n)$ are $\tau-$periodic functions, we have

$$\int_0^\tau \frac{d}{dt}(\alpha|\mathbf{u}^n|^2 + |\mathbf{h}^n|^2) dt = 0.$$

and consequently, from (3.3), we obtain

$$\int_0^\tau (\nu|\nabla \mathbf{u}^n|^2 + 2\chi|\nabla \mathbf{h}^n|^2) dt \leq C_4 C(N)^2 M^2 \tau.$$

It follows by Mean Value Theorem for integral, that there exists $t^* \in [0, \tau]$ such that

$$\nu|\nabla \mathbf{u}^n(t^*)|^2 + 2\chi|\nabla \mathbf{h}^n(t^*)|^2 \leq C_4 C(N)^2 M^2 \tau.$$

Now, using the Lemma 2.2, with $\theta = 0$, $\beta = 1/2$, we get

$$|\mathbf{u}^n(t^*)| \leq \mu^{-1/2} |\nabla \mathbf{u}^n(t^*)|. \tag{3.4}$$

and consequently

$$|\mathbf{u}^n(t^*)|^2 \leq \mu^{-1} |\nabla \mathbf{u}^n(t^*)|^2 \leq C_4 C(N)^2 M^2/(\nu\mu). \tag{3.5}$$

Analogously,

$$|\mathbf{h}^n(t^*)|^2 \leq \mu^{-1} |\nabla \mathbf{h}^n(t^*)|^2 \leq C_4 C(N)^2 M^2/(2\chi\mu). \tag{3.6}$$

Integrating (3.3) from t^* to $t+\tau$ ($t \in [0,\tau]$), we obtain

$$
\begin{aligned}
& |\mathbf{u}^n(t+\tau)|^2 + |\mathbf{h}^n(t+\tau)|^2 \\
\leq\ & \alpha|\mathbf{u}^n(t^*)|^2 + |\mathbf{h}^n(t^*)|^2 + C_4 C(N)^2 M^2 (t+\tau-t^*) \\
\leq\ & \alpha C_4 C(N)^2 M^2 (\mu\nu)^{-1} + C_4 C(N)^2 M^2 (2\mu\chi)^{-1} + 2\tau C_4 C(N)^2 M^2 \\
\equiv\ & K_1.
\end{aligned}
$$

Consequently,

$$
\sup_{0\leq t\leq\tau} |\mathbf{u}^n(t)|^2 \leq K_1, \qquad \sup_{0\leq t\leq\tau} |\mathbf{h}^n(t)|^2 \leq K_1 \tag{3.7}
$$

where K_1 is independent of λ and n. Thus, we have proved the existence of the solution $(\mathbf{u}^n, \mathbf{h}^n) \in (C^1(\tau; V_n))^2$.

Lemma 3.1. *Let $(\mathbf{u}^n(t), \mathbf{h}^n(t))$ be the solution of (3.1) given above. Suppose that*

$$
M < \min\{K_2^{-2}, K_3^{-3}, 1\}
$$

where

$$
\begin{aligned}
K_2 &= \nu^{-1}(\alpha C_1 C_5 C(N)\mu^{\gamma-\frac{1}{2}} + C_1 C_5 C(N)\mu^{\gamma-\frac{1}{2}} + \frac{3}{2}\alpha C_3 C_5^{-1} C(N)^{-1}\mu^{-\gamma}) \\
K_3 &= \chi^{-1}(2C_1 C_5 C(N)\mu^{\gamma-\frac{1}{2}} + \frac{1}{2}\alpha C_3 C_5^{-1} C(N)^{-1}\mu^{-\gamma}).
\end{aligned}
$$

Then, we have

$$
|A^\gamma \mathbf{u}^n(t)|^2 + |A^\gamma \mathbf{h}^n(t)|^2 \leq C_5 C(N)\mu^{\gamma-1/2} M^{1/2} \text{ for any } t \in (-\infty, +\infty).
$$

Proof. Taking $A^{2\gamma}\mathbf{u}^n$ and $A^{2\gamma}\mathbf{h}^n$ as test functions in $(3.1)_i$ and $(3.1)_{ii}$ repectively, we get

$$
\begin{aligned}
&\frac{\alpha}{2}\frac{d}{dt}|A^\gamma \mathbf{u}^n|^2 + \nu|A^{\frac{1+2\gamma}{2}}\mathbf{u}^n|^2 = (\alpha\mathbf{f}, A^{2\gamma}\mathbf{u}^n) - \alpha(P\mathbf{u}^n \cdot \nabla\mathbf{u}^n, A^{2\gamma}\mathbf{u}^n) \\
&+(P\mathbf{h}^n \cdot \nabla\mathbf{h}^n, A^{2\gamma}\mathbf{u}^n),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|A^\gamma \mathbf{h}^n|^2 + \chi|A^{\frac{1+2\gamma}{2}}\mathbf{h}^n|^2 = (P\mathbf{h}^n \cdot \nabla\mathbf{u}^n, A^{2\gamma}\mathbf{h}^n) \\
&+(P\mathbf{u}^n \cdot \nabla\mathbf{h}^n, A^{2\gamma}\mathbf{h}^n).
\end{aligned}
$$

Now, we estimate the right-hand sides of the above equalities as follows:

$$\begin{aligned} |(\alpha\mathbf{f}, A^{2\gamma}\mathbf{u}^n)| &\leq \alpha|\mathbf{f}|_{L^{N/2}(\Omega)}|A^{2\gamma}\mathbf{u}^n|_{L^{N/(N-2)}(\Omega)} \\ &\leq \alpha C_3 M|A^{\frac{1+2\gamma}{2}}\mathbf{u}^n|, \end{aligned}$$

here we use the Hölder's inequality and the estimate (1.1).

$$\begin{aligned} |(P\mathbf{v}\cdot\nabla\mathbf{b}, A^{2\gamma}\phi)| &= (A^{\frac{2\gamma-1}{2}}P\mathbf{v}\cdot\nabla\mathbf{b}, A^{\frac{1+2\gamma}{2}}\phi) \\ &\leq C_1|A^{\gamma}\mathbf{v}||A^{\frac{2+\gamma}{2}}\mathbf{b}||A^{\frac{1+2\gamma}{2}}\phi| \end{aligned}$$

where we use the Giga-Miyakawa estimate with $\theta = \gamma$ and $\rho = \frac{2\gamma+1}{2}$.

Now, adding the above estimates, we get

$$\begin{aligned} &\frac{\alpha}{2}\frac{d}{dt}|A^{\gamma}\mathbf{u}^n|^2 + \frac{d}{dt}|A^{\gamma}\mathbf{h}^n|^2 + \nu|A^{\frac{2\gamma+1}{2}}\mathbf{u}^n|^2 + \chi|A^{\frac{2\gamma+1}{2}}\mathbf{h}^n|^2 \\ \leq\ &\alpha C_3 M|A^{\frac{1+2\gamma}{2}}\mathbf{u}^n| + C_1\alpha|A^{\gamma}\mathbf{u}^n||A^{\frac{1+2\gamma}{2}}\mathbf{u}^n|^2 \\ &+2C_1|A^{\gamma}\mathbf{h}^n||A^{\frac{1+2\gamma}{2}}\mathbf{h}^n||A^{\frac{1+2\gamma}{2}}\mathbf{u}^n| + C_1|A^{\gamma}\mathbf{u}^n||A^{\frac{1+2\gamma}{2}}\mathbf{h}^n|^2. \end{aligned} \tag{3.8}$$

By using the Lemma 2.2 with $\theta = \gamma$ and $\beta = \frac{1}{2}$, we obtain from (2.5) and (3.5)

$$\begin{aligned} |A^{\gamma}\mathbf{u}^n(t^*)|^2 &\leq \mu^{\gamma-1/2}|\nabla\mathbf{u}^n(t^*)| \\ &\leq \mu^{\gamma-1/2}(\frac{C_4}{\nu})^{1/2}C(N)M \end{aligned}$$

and

$$\begin{aligned} |A^{\gamma}\mathbf{h}^n(t^*)|^2 &\leq \mu^{-1/2}|\nabla\mathbf{h}^n(t^*)| \\ &\leq \mu^{\gamma-1/2}(\frac{C_4}{2\chi})^{1/2}C(N)M. \end{aligned}$$

Consequently, if $M < 1$, we have

$$|A^{\gamma}\mathbf{u}^n(t^*)|^2 \leq \mu^{\gamma-1/2}(\frac{C_4}{\nu})^{1/2}C(N)M^{1/2},$$

$$|A^{\gamma}\mathbf{h}^n(t^*)|^2 \leq \mu^{\gamma-1/2}(\frac{C_4}{2\chi})^{1/2}C(N)M^{1/2}.$$

Thus,

$$|A^{\gamma}\mathbf{u}^n(t^*)|^2 + |A^{\gamma}\mathbf{h}^n(t^*)|^2 \leq C_5 C(N)\mu^{\gamma-1/2}M^{1/2}$$

where $C_5 = (\frac{C_4}{\nu})^{1/2} + (\frac{C_4}{2\chi})^{1/2}$.

Let $T^* = \sup\{T \ / \ |A^\gamma \mathbf{u}^n(t)|^2 + |A^\gamma \mathbf{h}^n(t)|^2 \leq C_5 C(N)\mu^{\gamma-1/2}M^{1/2}$ for any $t \in [t^*, T)\}$.

We will prove by contradiction that $T^* = \infty$. In fact, if T^* $(t^* < T^*)$ is finite it should follow that

$$|A^\gamma \mathbf{u}^n(t)|^2 + |A^\gamma \mathbf{h}^n(t)|^2 \leq C_5 C(N)\mu^{\gamma-1/2}M^{1/2} \text{ for any } t \in [t^*, T^*) \text{ and}$$

$$|A^\gamma \mathbf{u}^n(T^*)|^2 + |A^\gamma \mathbf{h}^n(T^*)|^2 = C_5 C(N)\mu^{\gamma-1/2}M^{1/2}.$$

Therefore, for such a value $t = T^*$, the estimates of the right-hand side of (3.8) are

$$\begin{aligned}
& \alpha C_3 M |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)| \\
\leq\ & \alpha C_3 C_5^{-1} C(N)^{-1} M^{1/2} \{|A^\gamma \mathbf{u}^n(t)|^2 + |A^\gamma \mathbf{h}^n(t)|^2\} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)| \\
\leq\ & \alpha C_3 C_5^{-1} C(N)^{-1} \mu^{-\gamma} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)|^2 \\
& + \alpha C_3 C_5^{-1} C(N)^{-1} \mu^{-\gamma} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)| \\
\leq\ & \frac{3}{2} \alpha C_3 C_5^{-1} C(N)^{-1} \mu^{-\gamma} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)|^2 \\
& + \frac{1}{2} \alpha C_3 C_5^{-1} C(N)^{-1} \mu^{-\gamma} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)|^2,
\end{aligned}$$

$$\alpha C_1 |A^\gamma \mathbf{u}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)|^2 \leq \alpha C_1 C_5 C(N) \mu^{\gamma-1/2} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)|^2,$$

$$\begin{aligned}
& 2C_1 |A^\gamma \mathbf{h}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)| \\
\leq\ & 2C_1 C_5 C(N) \mu^{\gamma-1/2} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)| \\
\leq\ & C_1 C_5 C(N) \mu^{\gamma-1/2} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{u}^n(t)|^2 \\
& + C_1 C_5 C(N) \mu^{\gamma-1/2} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)|^2,
\end{aligned}$$

$$C_1 |A^\gamma \mathbf{u}^n(t)| |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)|^2 \leq C_1 C_5 C(N) \mu^{\gamma-1/2} M^{1/2} |A^{\frac{1+2\gamma}{2}} \mathbf{h}^n(t)|^2.$$

Consequently, the above estimate and (3.8) implies

$$\begin{aligned}
& \frac{\alpha}{2} \frac{d}{dt} |A^\gamma \mathbf{u}^n(t)|^2 + \frac{d}{dt} |A^\gamma \mathbf{h}^n(t)|^2 + \nu |A^{\frac{2\gamma+1}{2}} \mathbf{u}^n(t)|^2 + \chi |A^{\frac{2\gamma+1}{2}} \mathbf{h}^n(t)|^2 \\
\leq\ & (\alpha C_1 C_5 C(N) \mu^{\gamma-1/2} + C_1 C_5 C(N) \mu^{\gamma-1/2}
\end{aligned}$$

$$\begin{aligned}
&+\frac{3}{2}\alpha C_3 C_5^{-1} C(N)^{-1}\mu^{-\gamma})M^{1/2}|A^{\frac{1+2\gamma}{2}}\mathbf{u}^n(t)| \\
&+(C_1C_5C(N)\mu^{\gamma-1/2}+C_1C_5C(N)\mu^{\gamma-1/2} \\
&+\frac{1}{2}\alpha C_3 C_5^{-1} C(N)^{-1}\mu^{-\gamma})M^{1/2}|A^{\frac{1+2\gamma}{2}}\mathbf{h}^n(t)|^2 \\
= \quad & K_2 M^{1/2}\nu|A^{\frac{1+2\gamma}{2}}\mathbf{u}^n(t)|^2 + K_3 M^{1/2}\chi|A^{\frac{1+2\gamma}{2}}\mathbf{h}^n(t)|,
\end{aligned}$$

where

$$K_2 = \nu^{-1}(\alpha C_1C_5C(N)\mu^{\gamma-1/2} + C_1C_5C(N)\mu^{\gamma-1/2} + \frac{3}{2}\alpha C_3C_5^{-1}C(N)^{-1}\mu^{-\gamma}$$

$$K_3 = \chi^{-1}(2C_1C_5C(N)\mu^{\gamma-1/2} + \frac{1}{2}\alpha C_3C_5^{-1}C(N)^{-1}\mu^{-\gamma}.$$

If $M < \min\{K_1, K_2, 1\}$, we have

$$\frac{\alpha}{2}\frac{d}{dt}|A^\gamma\mathbf{u}^n(t)|^2 + \frac{d}{dt}|A^\gamma\mathbf{h}^n(t)|^2 < 0 \text{ at } t = T^*.$$

Thus, in a neighborhood of $t = T^*$ it follows

$$|A^\gamma\mathbf{u}^n(t)|^2 + |A^\gamma\mathbf{h}^n(t)|^2 \le C_5C(N)\mu^{\gamma-1/2}M^{1/2} \text{ for any } t \in [T^*, T^*+\delta)$$

which implies $T^* = \infty$. Therefore, this given

$$|A^\gamma\mathbf{u}^n(t)|^2 \le C_5C(N)\mu^{\gamma-1/2}M^{1/2} \text{ for any } t \in (-\infty, \infty)$$

$$|A^\gamma\mathbf{h}^n(t)|^2 \le C_5C(N)\mu^{\gamma-1/2}M^{1/2} \text{ for any } t \in (-\infty, \infty)$$

since $\mathbf{u}^n(t)$ and $\mathbf{h}^n(t)$ are periodics.

4. Estimates of derivatives of higher order

To show the convergence of the approximate solutions we shall derive estimates of derivatives of higher order. By Lemma 3.1, if M is sufficiently small the approximate solutions satisfy

$$\sup_t |A^\gamma\mathbf{u}^n(t)| \le C(M), \quad \sup_t |A^\gamma\mathbf{h}^n(t)| \le C(M) \tag{4.1}$$

with $\gamma = \frac{N}{4} - \frac{1}{2}$, where $C(M)$ denotes a constant depending on M and independent of n.

Lemma 4.1. *Let* $(\mathbf{u}^n(t), \mathbf{h}^n(t))$ *be the solution of (3.1) given above. Set*

$$M_0 = (\int_0^\tau |\mathbf{f}|^2 dt)^{\frac{1}{2}},\ M_1 = (\int_0^\tau |\mathbf{f}_t|^2 dt)^{\frac{1}{2}}.$$

Then, we have

$$\sup_{0\le t\le \tau} |\nabla \mathbf{u}^n(t)|^2 \le C(M_0, M),\quad \sup_{0\le t\le \tau} |\nabla \mathbf{h}^n(t)|^2 \le C(M_0, M),$$

and

$$\sup_t (\alpha |\mathbf{u}_t^n(t)|^2 + |\mathbf{h}_t^n(t)|^2) \le C(M_0, M_1, M),$$

where $C(M_0, M)$ and $C(M_0, M_1, M)$ denote constants depending on M_0, M_1, M and independing of n.

Proof. From (3.1) and using (2.1), we have

$$\begin{aligned} & \frac{\alpha}{2}\frac{d}{dt}|\nabla \mathbf{u}^n|^2 + \nu |A\mathbf{u}^n|^2 \\ \le\ & \alpha |\mathbf{f}||A\mathbf{u}^n| + C_1 \alpha |A^\gamma \mathbf{u}^n||A\mathbf{u}^n|^2 + 2C_1 |A^\gamma \mathbf{h}^n||A\mathbf{h}^n||A\mathbf{u}^n|, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\nabla \mathbf{h}^n|^2 + \chi |A\mathbf{h}^n|^2 \\ \le\ & C_1 |A^\gamma \mathbf{u}^n||A\mathbf{h}^n|^2 + C_1 |A^\gamma \mathbf{h}^n||A\mathbf{h}^n||A\mathbf{u}^n|. \end{aligned}$$

Adding the above inequalities and using the estimate (4.1), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\alpha |\nabla \mathbf{u}^n|^2 + |\nabla \mathbf{h}^n|^2) + \nu |A\mathbf{u}^n|^2 + \chi |A\mathbf{h}^n|^2 \qquad (4.2) \\ \le\ & \alpha |\mathbf{f}||A\mathbf{u}^n| + C_1 C(M) |A\mathbf{u}^n|^2 + 2C_1 C(M)|A\mathbf{h}^n||A\mathbf{u}^n| + C_1 C(M)|A\mathbf{h}^n|^2 \\ \le\ & \alpha |\mathbf{f}||A\mathbf{u}^n| + C_1 C(M)(\alpha+1)|A\mathbf{u}^n|^2 + 2C_1 C(M)|A\mathbf{h}^n|^2. \end{aligned}$$

Recalling the perioricity of $\nabla \mathbf{u}^n(t)$ and $\nabla \mathbf{h}^n(t)$, we deduce from the above inequality

$$\begin{aligned} & \int_0^\tau (\nu |A\mathbf{u}^n|^2 + \chi |A\mathbf{h}^n|^2) dt \\ \le\ & \alpha M_0 (\int_0^\tau \nu |A\mathbf{u}^n|^2)^{1/2} dt + C_1 C(M)(\alpha+1) \int_0^\tau \nu |A\mathbf{u}^n|^2 dt \\ & + 2C_1 C(M) \int_0^\tau \nu |A\mathbf{h}^n|^2 dt. \end{aligned}$$

or, if $d_0 = \min\{\nu, \chi\} > 0$,

$$\begin{aligned}
&\int_0^\tau (|A\mathbf{u}^n|^2 + |A\mathbf{h}^n|^2)dt \\
\leq\;& \alpha M_0 d_0^{-1}(\int_0^\tau \nu|A\mathbf{u}^n|^2)^{1/2}dt + C_1C(M)d_0^{-1}(\alpha+2)\int_0^\tau \nu|A\mathbf{u}^n|^2 dt.
\end{aligned}$$

Seeing that $d_0^{-1}C_1C(M)(\alpha+3) < 1$, we obtain

$$\int_0^\tau (|A\mathbf{u}^n|^2 + |A\mathbf{h}^n|^2)dt \leq C(M_0, M).$$

Newly, applying the Mean Value Theorem for integrals, we have that there exists $t^* \in [0, \tau]$ such that

$$|A\mathbf{u}^n(t^*)|^2 + |A\mathbf{h}^n(t^*)|^2 \leq \tau^{-1}C(M_0, M).$$

By using the Lemma 2.2 , with $\theta = \frac{1}{2}$, $\beta = 1$, we have

$$|\nabla\mathbf{u}^n(t^*)|^2 \leq \mu^{-1}|A\mathbf{u}^n(t^*)| \leq \mu^{-1}\tau^{-1}C(M_0, M)$$

and

$$|\nabla\mathbf{h}^n(t^*)|^2 \leq \mu^{-1}|A\mathbf{h}^n(t^*)| \leq \mu^{-1}\tau^{-1}C(M_0, M).$$

Now, integrating the inequality (4.2) from t^* to $t + \tau$ ($t \in [0, \tau]$), we deduce easily

$$\begin{aligned}
\sup_{0\leq t\leq\tau} |\nabla\mathbf{u}^n(t)|^2 &\leq C(M_0, M), \qquad (4.3)\\
\sup_{0\leq t\leq\tau} |\nabla\mathbf{h}^n(t)|^2 &\leq C(M_0, M)
\end{aligned}$$

where $C(M_0, M)$ is independent of n.

By other hand, from equations (3.1) we have

$$\begin{aligned}
(\alpha\mathbf{u}_t^n + \nu A\mathbf{u}^n, \mathbf{u}_t^n) &= (\alpha\mathbf{f} - \alpha P\mathbf{u}^n\cdot\nabla\mathbf{u}^n + P\mathbf{h}^n\cdot\nabla\mathbf{h}^n, \mathbf{u}_t^n),\\
(\mathbf{h}_t^n + \chi A\mathbf{h}^n, \mathbf{h}_t^n) &= (-P\mathbf{u}^n\cdot\nabla\mathbf{h}^n + P\mathbf{h}^n\cdot\nabla\mathbf{u}^n, \mathbf{h}_t^n)
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}|\nabla\mathbf{u}^n|^2 + \alpha|\mathbf{u}_t^n|^2 = (\alpha\mathbf{f}, \mathbf{u}_t^n) + \alpha(P\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \mathbf{u}_t^n) + (P\mathbf{h}^n\cdot\nabla\mathbf{h}^n, \mathbf{u}_t^n),\\
&\frac{\chi}{2}\frac{d}{dt}|\nabla\mathbf{h}^n|^2 + |\mathbf{h}_t^n|^2 = (P\mathbf{h}^n\cdot\nabla\mathbf{u}^n, \mathbf{h}_t^n) - (P\mathbf{u}^n\cdot\nabla\mathbf{h}^n, \mathbf{h}_t^n). \qquad (4.4)
\end{aligned}$$

Now, we estimate the right-hand sides of the above inequalities, we have

$$|(\alpha \mathbf{f}, \mathbf{u}_t^n)| \le \frac{\alpha}{2}|\mathbf{f}|^2 + \frac{\alpha}{2}|\mathbf{u}_t^n|^2$$

and

$$\begin{aligned} |(\mathbf{v}\cdot\nabla\mathbf{b}, \phi_t)| &\le |\mathbf{v}|_{L^N}|\nabla\mathbf{b}||\phi_t|_{L^{\frac{2N}{N-2}}} \\ &\le C_2C_3|A^\gamma\mathbf{v}||\nabla\mathbf{b}||\nabla\phi_t^n| \end{aligned}$$

where we use the Hölder' inequality, the estimate (2.3) and Sobolev's embedding with $r = \frac{2N}{N-2}$ and $\beta = 1$.

Thus, by using the above estimates and the equalities (4.4), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\nu|\nabla\mathbf{u}^n|^2 + \chi|\nabla\mathbf{h}^n|^2) + \alpha|\mathbf{u}_t^n|^2 + |\mathbf{h}_t^n|^2 \\ \le\ & \frac{\alpha}{2}|\mathbf{f}|^2 + \frac{\alpha}{2}|\mathbf{u}_t^n| + C_2C_3|A^\gamma\mathbf{u}^n||\nabla\mathbf{u}^n||\nabla\mathbf{u}_t^n| \\ &+C_2C_3|A^\gamma\mathbf{h}^n||\nabla\mathbf{h}^n||\nabla\mathbf{u}_t^n| + C_2C_3|A^\gamma\mathbf{u}^n||\nabla\mathbf{h}^n||\nabla\mathbf{h}_t^n| \\ &+C_2C_3|A^\gamma\mathbf{h}^n||\nabla\mathbf{u}^n||\nabla\mathbf{h}_t^n| \\ \le\ & \frac{\alpha}{2}|\mathbf{f}|^2 + \frac{\alpha}{2}|\mathbf{u}_t^n| + C_2C_3C(M_0, M)|\nabla\mathbf{u}_t^n| + C(M_0, M)|\nabla\mathbf{h}_t^n|. \end{aligned}$$

Integrating from 0 to τ, we have

$$\int_0^\tau \alpha|\mathbf{u}_t^n|^2 + |\mathbf{h}_t^n|^2 \le \frac{\alpha}{2}M_0^2 + C(M_0, M)\int_0^\tau(|\nabla\mathbf{u}_t^n| + |\nabla\mathbf{h}_t^n|)ds. \qquad (4.5)$$

By other hand, differentiating with respect to t the equalities (3.1), we obtain

$$\begin{aligned} &(\alpha\mathbf{u}_{tt}^n + \nu A\mathbf{u}_t^n, \omega^j) \\ =\ & (\alpha\mathbf{f}_t - \alpha P\mathbf{u}_t^n\cdot\nabla\mathbf{u}^n + \alpha P\mathbf{u}^n\cdot\nabla\mathbf{u}_t^n + P\mathbf{h}_t^n\cdot\nabla\mathbf{h}^n + P\mathbf{h}^n\cdot\nabla\mathbf{h}_t^n, \omega^j), \\ &(\mathbf{h}_{tt}^n + \chi A\mathbf{h}_t^n, \omega^j) \qquad (4.6) \\ =\ & (-P\mathbf{u}_t^n\cdot\nabla\mathbf{h}^n - P\mathbf{u}^n\cdot\nabla\mathbf{h}_t^n + P\mathbf{h}_t^n\cdot\nabla\mathbf{u}^n + P\mathbf{h}^n\cdot\nabla\mathbf{u}_t^n, \omega^j). \end{aligned}$$

Multiplying the first equality by $c'_{jn}(t)$ and the second equality by $d'_{jn}(t)$, and summing the result for $j = 1, ..., n$, we obtain

$$\frac{\alpha}{2}\frac{d}{dt}|\mathbf{u}_t^n|^2 + \nu|\nabla\mathbf{u}_t^n|^2$$

$$
\begin{aligned}
&= (\alpha\mathbf{f}_t,\mathbf{u}_t^n)-\alpha(P\mathbf{u}_t^n\cdot\nabla\mathbf{u}^n,\mathbf{u}_t^n)+(P\mathbf{h}_t^n\cdot\nabla\mathbf{h}^n,\mathbf{u}_t^n)+(P\mathbf{h}^n\cdot\nabla\mathbf{h}_t^n,\mathbf{u}_t^n),\\
&\quad \frac{1}{2}\frac{d}{dt}|\mathbf{h}_t^n|^2+\chi|\mathbf{h}_t^n|^2 \qquad (4.7)\\
&= (P\mathbf{h}_t^n\cdot\nabla\mathbf{u}^n,\mathbf{h}_t^n)+(P\mathbf{h}^n\cdot\nabla\mathbf{u}_t^n,\mathbf{h}_t^n)-(P\mathbf{u}_t^n\cdot\nabla\mathbf{h}^n,\mathbf{h}_t^n),
\end{aligned}
$$

since $(P\mathbf{u}^n\cdot\nabla\mathbf{u}_t^n,\mathbf{u}_t^n)=(P\mathbf{u}^n\cdot\nabla\mathbf{h}_t^n,\mathbf{h}_t^n)=0$.

We observe that using the Hölder's inequality, we obtain

$$
|(P\phi_t\cdot\nabla\mathbf{v},\mathbf{b}_t)|\leq|\phi_t|_{L^N}|\nabla\mathbf{v}||\mathbf{b}_t|_{L^{\frac{2N}{N-2}}}
$$

and using the inequality (2.2) and (2.3) with $r=3$ and $\beta=1$, we infer of the above inequality

$$
|(P\phi_t\cdot\nabla\mathbf{v},\mathbf{b}_t)|\leq C_2C_3|A^\gamma\phi_t||\nabla\mathbf{v}||\nabla\mathbf{b}_t| \qquad (4.8)
$$

for any ϕ, $\mathbf{v}$, $\mathbf{b}\in V$. Analogously,

$$
|(P\phi\cdot\nabla\mathbf{v}_t,\mathbf{b}_t)|\leq C_2C_3|A^\gamma\mathbf{b}_t||\nabla\phi||\nabla\mathbf{v}_t|. \qquad (4.9)
$$

If, we use the estimate (4.8) for second and third terms in equality $(4.7)_i$ and the estimate (4.9) in the last term of this equality, we get

$$
\begin{aligned}
&\frac{\alpha}{2}\frac{d}{dt}|\mathbf{u}_t^n|^2+\nu|\nabla\mathbf{u}_t^n|^2\leq\alpha\mu^{-1}|\mathbf{f}_t||\nabla\mathbf{u}_t^n|+\alpha C_2C_3|A^\gamma\mathbf{u}_t^n|\ |\nabla\mathbf{u}^n||\nabla\mathbf{u}_t^n|\\
&+C_2C_3|A^\gamma\mathbf{h}_t^n||\nabla\mathbf{h}^n||\nabla\mathbf{u}_t^n|+C_2C_3|A^\gamma\mathbf{u}_t^n||\nabla\mathbf{h}^n||\nabla\mathbf{h}_t^n| \qquad (4.10)
\end{aligned}
$$

similarly, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathbf{h}_t^n|^2+\chi|\nabla\mathbf{h}_t^n|^2\leq C_2C_3|A^\gamma\mathbf{h}_t^n|\ |\nabla\mathbf{u}^n||\nabla\mathbf{h}_t^n| \qquad (4.11)\\
&+C_2C_3|A^\gamma\mathbf{h}_t^n||\nabla\mathbf{h}^n||\nabla\mathbf{u}_t^n|+C_2C_3|A^\gamma\mathbf{u}_t^n||\nabla\mathbf{h}^n||\nabla\mathbf{h}_t^n|.
\end{aligned}
$$

Adding the inequalities (4.10) and (4.11), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\mathbf{u}_t^n|^2+|\mathbf{h}_t^n|^2)+\nu|\nabla\mathbf{u}_t^n|^2+\chi|\nabla\mathbf{h}_t^n|^2\\
\leq\ &\alpha\mu^{-1}|\mathbf{f}_t||\nabla\mathbf{u}_t^n|+\alpha C_2C_3C(M_0,M)|A^\gamma\mathbf{u}_t^n||\nabla\mathbf{u}_t^n|\\
&+2C_2C_3C(M_0,M)|A^\gamma\mathbf{h}_t^n||\nabla\mathbf{u}_t^n|+2C_2C_3C(M_0,M)|A^\gamma\mathbf{u}_t^n||\nabla\mathbf{h}_t^n|\\
&+C_2C_3C(M_0,M)|A^\gamma\mathbf{h}_t^n||\nabla\mathbf{h}_t^n|
\end{aligned}
$$

where we use the estimates (4.3).

By using the interpolation inequality ($\alpha = 0, \beta = 1/2$)

$$|A^\gamma v| \leq C_4 |A^\beta v|^{\frac{\gamma-\delta}{\beta-\delta}} |A^\gamma v|^{\frac{\beta-\gamma}{\beta-\delta}}$$

where $0 \leq \delta < \gamma < \beta$; $C_4 = C(\delta, \beta, \gamma)$ and $v \in D(A^\beta)$, we have

$$|A^\gamma \mathbf{u}_t^n| \leq C(0, \frac{1}{2}, \gamma) |A^{\frac{1}{2}} \mathbf{u}_t^n|^{2\gamma} |A^0 \mathbf{u}_t^n|^{1-2\gamma}.$$

we observe

$$|A^{\frac{1}{2}} \mathbf{v}| = |\nabla \mathbf{v}|,$$

and

$$|A^0 \mathbf{v}| = |\mathbf{v}|.$$

thus,

$$|A^\gamma \mathbf{u}_t^n| \leq C(0, \frac{1}{2}, \gamma) |\nabla \mathbf{u}_t^n|^{2\gamma} |\mathbf{u}_t^n|^{1-2\gamma}.$$

Then, the first term in (**??**) is bounded as follows

$$\begin{aligned} \alpha C_2 C_3 C(M_0, M) |A^\gamma \mathbf{u}_t^n| |\nabla \mathbf{u}_t^n| &\leq \alpha C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{u}_t^n|^{2\gamma} |\mathbf{u}_t^n|^{1-2\gamma} |\nabla \mathbf{u}_t^n| \\ &\leq \alpha C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{u}_t^n|^{2\gamma+1} |\mathbf{u}_t^n|^{1-2\gamma}. \end{aligned}$$

Analogously, we have

$$|A^\gamma \mathbf{h}_t^n| \leq C_4 |\nabla \mathbf{h}_t^n|^{2\gamma} |\mathbf{h}_t^n|^{1-2\gamma}.$$

and thus the last term of (**??**) is bounded of the following manner

$$\begin{aligned} C_2 C_3 C(M_0, M) |A^\gamma \mathbf{h}_t^n| |\nabla \mathbf{h}_t^n| &\leq C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{h}_t^n|^{2\gamma} |\mathbf{h}_t^n|^{1-2\gamma} |\nabla \mathbf{h}_t^n| \\ &\leq C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{h}_t^n|^{2\gamma+1} |\mathbf{h}_t^n|^{1-2\gamma}. \end{aligned}$$

Now, we look the mixtes terms of (**??**)

$$2C_2 C_3 C(M_0, M) |A^\gamma \mathbf{h}_t^n| |\nabla \mathbf{u}_t^n| \leq 2C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{h}_t^n|^{2\gamma} |\mathbf{h}_t^n|^{1-2\gamma} |\nabla \mathbf{u}_t^n|$$

similarly

$$2C_2 C_3 C(M_0, M) |A^\gamma \mathbf{u}_t^n| |\nabla \mathbf{h}_t^n| \leq 2C_2 C_3 C_4 C(M_0, M) |\nabla \mathbf{u}_t^n|^{2\gamma} |\mathbf{u}_t^n|^{1-2\gamma} |\nabla \mathbf{h}_t^n|$$

The above estimates imply the following differential inequality

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\alpha|\mathbf{u}_t^n|^2 + |\mathbf{h}_t^n|^2) + \nu|\nabla\mathbf{u}_t^n|^2 + \chi|\nabla\mathbf{h}_t^n|^2 \\
\leq\ & \alpha\mu^{-1}|\mathbf{f}_t||\nabla\mathbf{u}_t^n| + \alpha C_2C_3C_4C(M_0,M)|\nabla\mathbf{u}_t^n|^{2\gamma+1}|\mathbf{u}_t^n|^{1-2\gamma} \\
& + C_2C_3C_4C(M_0,M)|\nabla\mathbf{h}_t^n|^{2\gamma+1}|\mathbf{h}_t^n|^{1-2\gamma} \\
& + 2C_2C_3C_4C(M_0,M)|\nabla\mathbf{h}_t^n|^{2\gamma}|\mathbf{h}_t^n|^{1-2\gamma}|\nabla\mathbf{u}_t^n| \\
& + 2C_2C_3C_4C(M_0,M)|\nabla\mathbf{u}_t^n|^{2\gamma}|\mathbf{u}_t^n|^{1-2\gamma}|\nabla\mathbf{h}_t^n|.
\end{aligned} \tag{4.12}$$

Integrating over $[0,\tau]$ we have, since $\mathbf{u}$ and $\mathbf{h}$ are periodics,

$$\begin{aligned}
\int_0^\tau(\nu|\nabla\mathbf{u}_t^n|^2 + \chi|\nabla\mathbf{h}_t^n|^2) \leq\ & C(M_1) + \alpha C_2C_3C_4C(M_0,M)\int_0^\tau|\nabla\mathbf{u}_t^n|^{2\gamma+1}|\mathbf{u}_t^n|^{1-2\gamma} \\
& + C_2C_3C_4C(M_0,M)\int_0^\tau|\nabla\mathbf{h}_t^n|^{2\gamma+1}|\mathbf{h}_t^n|^{1-2\gamma} \\
& + 2C_2C_3C_4C(M_0,M)\int_0^\tau|\nabla\mathbf{h}_t^n|^{2\gamma}|\mathbf{h}_t^n|^{1-2\gamma}|\nabla\mathbf{u}_t^n| \\
& + 2C_2C_3C_4C(M_0,M)\int_0^\tau|\nabla\mathbf{u}_t^n|^{2\gamma}|\mathbf{u}_t^n|^{1-2\gamma}|\nabla\mathbf{h}_t^n|.
\end{aligned}$$

We will prove the following estimate

$$\int_0^\tau(\nu|\nabla\mathbf{u}_t^n|^2 + \chi|\nabla\mathbf{h}_t^n|^2) \leq C(M_1,M_0,M). \tag{4.13}$$

If $N=3$, we have $\gamma = \frac{1}{4}$, thus the last inequality implies

$$\begin{aligned}
\int_0^\tau(\nu|\nabla\mathbf{u}_t^n|^2 + \chi|\nabla\mathbf{u}_t^n|^2) \leq\ & C(M_1) + C(M_0,M)\int_0^\tau|\nabla\mathbf{u}_t^n|^{\frac{3}{2}}|\mathbf{u}_t^n|^{\frac{1}{2}} \\
& + C(M_0,M)\int_0^\tau|\nabla\mathbf{h}_t^n|^{\frac{3}{2}}|\mathbf{h}_t^n|^{\frac{1}{2}} \\
& + C(M_0,M)\int_0^\tau|\nabla\mathbf{h}_t^n|^{\frac{1}{2}}|\mathbf{h}_t^n|^{\frac{1}{2}}|\nabla\mathbf{u}_t^n| \\
& + C(M_0,M)\int_0^\tau|\nabla\mathbf{u}_t^n|^{\frac{1}{2}}|\mathbf{u}_t^n|^{\frac{1}{2}}|\nabla\mathbf{h}_t^n|.
\end{aligned} \tag{4.14}$$

By using the Hölder inequality, we get

$$\int_0^\tau|\nabla\mathbf{u}_t^n|^{\frac{3}{2}}|\mathbf{u}_t^n|^{\frac{1}{2}} \leq (\int_0^\tau|\mathbf{u}_t^n|^2dt)^{1/4}(\int_0^\tau|\nabla\mathbf{u}_t^n|^2)^{3/4}$$

analogously

$$\int_0^\tau|\nabla\mathbf{h}_t^n|^{\frac{3}{2}}|\mathbf{h}_t^n|^{\frac{1}{2}} \leq (\int_0^\tau|\mathbf{h}_t^n|^2dt)^{1/4}(\int_0^\tau|\nabla\mathbf{h}_t^n|^2)^{3/4}$$

and

$$\begin{aligned}\int_0^\tau |\nabla \mathbf{u}_t^n|^{\frac{1}{2}}|\mathbf{u}_t^n|^{\frac{1}{2}}|\nabla \mathbf{h}_t^n| &\le (\int_0^\tau |\mathbf{u}_t^n|^2 dt)^{1/4}(\int_0^\tau |\nabla \mathbf{u}_t^n|^{\frac{2}{3}}|\nabla \mathbf{h}_t^n|^{\frac{4}{3}})^{3/4} \\ &\le (\int_0^\tau |\mathbf{u}_t^n|^2 dt)^{1/4}(\int_0^\tau |\nabla \mathbf{u}_t^n|^2)^{\frac{1}{4}}(\int_0^\tau |\nabla \mathbf{h}_t^n|^2)^{1/2}\end{aligned}$$

analogously

$$\int_0^\tau |\mathbf{h}_t^n|^{\frac{1}{2}}|\nabla \mathbf{h}_t^n|^{\frac{1}{2}}|\nabla \mathbf{u}_t^n| \le (\int_0^\tau |\mathbf{h}_t^n|^2 dt)^{1/4}(\int_0^\tau |\nabla \mathbf{h}_t^n|^2)^{\frac{1}{4}}(\int_0^\tau |\nabla \mathbf{u}_t^n|^2)^{1/2}.$$

¿From the inequality (4.12) and by using the above inequality together with (4.5), we obtain

$$\int_0^\tau (\nu|\nabla \mathbf{u}_t^n|^2 + \chi|\nabla \mathbf{h}_t^n|^2) \le C(M_1, M_0, M)(\int_0^\tau \nu|\nabla \mathbf{u}_t^n|^2 + \chi|\nabla \mathbf{h}_t^n|^2)^{\frac{7}{8}}$$

which implies the boundeness for $N = 3$,

$$\int_0^\tau (\nu|\nabla \mathbf{u}_t^n|^2 + \chi|\nabla \mathbf{h}_t^n|^2) \le C(M_1, M_0, M).$$

If $N = 4$, we have $\gamma = \frac{1}{2}$ and from the inequality (4.12), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\alpha|\mathbf{u}_t^n|^2 + |\mathbf{h}_t^n|^2) + \nu|\nabla \mathbf{u}_t^n|^2 + \chi|\nabla \mathbf{h}_t^n|^2 \\ \le\ & \alpha\mu^{-1}|\mathbf{f}_t||\nabla \mathbf{u}_t^n| + C(M_0, M)|\nabla \mathbf{u}_t^n|^2 + C(M_0, M)|\nabla \mathbf{h}_t^n||\nabla \mathbf{u}_t^n| \\ &+C(M_0, M)|\nabla \mathbf{h}_t^n|^2.\end{aligned}$$

¿From this, we find

$$\frac{1}{2}\frac{d}{dt}(\alpha|\mathbf{u}_t^n|^2+|\mathbf{h}_t^n|^2)+\nu|\nabla \mathbf{u}_t^n|^2+\chi|\nabla \mathbf{h}_t^n|^2 \le C(M_1)+C(M_0, M)(\nu|\nabla \mathbf{u}_t^n|^2+\chi|\nabla \mathbf{h}_t^n|^2).$$

thus,

$$\int_0^\tau (\nu|\nabla \mathbf{u}_t^n|^2+\chi|\nabla \mathbf{h}_t^n|^2)dt \le C(M_1)+C(M_0, M)\int_0^\tau (\nu|\nabla \mathbf{u}_t^n|^2+\chi|\nabla \mathbf{h}_t^n|^2)dt.$$

Since, we can take $(1 - C(M_0, M_1, M)) > 0$, we obtain the desired result for $N = 4$.

Newly, by using the mean-value theorem for integral, we have that there exists $t^* \in [0, \tau]$, such that

$$\nu|\nabla \mathbf{u}_t^n(t^*)|^2 + \chi|\nabla \mathbf{h}_t^n(t^*)|^2 \le \tau^{-1} C(M_0, M_1, M). \tag{4.15}$$

Consequently,

$$\begin{aligned} |\mathbf{u}_t^n(t^*)|^2 &\le \mu^{-1}|\nabla \mathbf{u}_t^n(t^*)|^2 \\ &\le \nu^{-1}\mu^{-1}\tau^{-1}C(M_0, M_1, M) \end{aligned}$$

and

$$\begin{aligned} |\mathbf{h}_t^n(t^*)|^2 &\le \mu^{-1}|\nabla \mathbf{h}_t^n(t^*)|^2 \\ &\le \chi^{-1}\mu^{-1}\tau^{-1}C(M_0, M_1, M) \end{aligned}$$

Integrating (4.12) from t^* to $t+\tau$ with $t \in [0, \tau]$, and using the above estimates, we get

$$\sup_t (\alpha|\mathbf{u}_t^n(t)|^2 + |\mathbf{h}_t^n(t)|^2) \le C(M_0, M_1, M).$$

Lemma 4.2. *Let $(\mathbf{u}^n(t), \mathbf{h}^n(t))$ be the approximate solution of (3.1) given above. Then, we have*

$$\begin{aligned} \sup_t |A\mathbf{u}^n(t)| &\le C(M_0, M_1, M), \\ \sup_t |A\mathbf{h}^n(t)| &\le C(M_0, M_1, M), \end{aligned} \tag{4.16}$$

$$\int_0^\tau (|A\mathbf{u}_t^n(t)|^2 + |A\mathbf{h}_t^n(t)|^2)dt \le C(M_0, M_1, M), \tag{4.17}$$

$$\int_0^\tau (|\mathbf{u}_{tt}^n(t)|^2 + |\mathbf{h}_{tt}^n(t)|^2)dt \le C(M_0, M_1, M) \tag{4.18}$$

Proof. From the equalities (3.1), we obtain easily

$$\begin{aligned} (\alpha\mathbf{u}_t^n + \nu A\mathbf{u}^n, A\mathbf{u}^n) &= (\alpha\mathbf{f} - \alpha P\mathbf{u}^n \cdot \nabla \mathbf{u}^n + P\mathbf{h}^n \cdot \nabla \mathbf{h}^n, A\mathbf{u}^n), \\ (\mathbf{h}_t^n + \chi A\mathbf{h}^n, A\mathbf{h}^n) &= (-P\mathbf{u}^n \cdot \nabla \mathbf{h}^n + P\mathbf{h}^n \cdot \nabla \mathbf{u}^n, A\mathbf{h}^n), \end{aligned}$$

consequently

$$\nu|A\mathbf{u}^n|^2 \le \alpha|f||A\mathbf{u}^n| + \alpha|\mathbf{u}_t^n||A\mathbf{u}^n| + \alpha|P\mathbf{u}^n \cdot \nabla \mathbf{u}^n||A\mathbf{u}^n| + |P\mathbf{h}^n \cdot \nabla \mathbf{h}^n||A\mathbf{u}^n|, \tag{4.19}$$

$$\chi|A\mathbf{h}^n|^2 \leq |\mathbf{h}_t^n||A\mathbf{h}^n| + |P\mathbf{h}^n\cdot\nabla\mathbf{u}^n||A\mathbf{h}^n| + |P\mathbf{u}^n\cdot\nabla\mathbf{h}^n||A\mathbf{h}^n|. \quad (4.20)$$

Now, we use the Proposition 2.1 with $\theta = \gamma$ and $\rho = 1$, to obtain

$$|P\phi\cdot\nabla v| \leq C_1C(M)|A^\gamma\phi||Av| \leq C_1C(M)|Av| \quad (4.21)$$

where we use the estimate (4.21) , here we set $\phi = \mathbf{u}^n$ or $\mathbf{h}^n$ and $v = \mathbf{u}^n$ or $\mathbf{h}^n$.

The inequalities (4.19) and (4.20) together with (4.21) imply

$$\begin{aligned}\nu|A\mathbf{u}^n|^2 + \chi|A\mathbf{h}^n|^2 &\leq \alpha|\mathbf{f}||A\mathbf{u}^n| + \alpha|\mathbf{u}_t^n||A\mathbf{u}^n| + |\mathbf{h}_t^n||A\mathbf{h}^n| \\ &\quad +2C_1C(M)|A\mathbf{h}^n|^2 + C_1C(M)(1+\alpha)|A\mathbf{u}^n|^2.\end{aligned}$$

Using the Young inequality to the third first terms and the fact that $C(M)$ is small, we obtain

$$|A\mathbf{u}^n(t)| \leq C(M_0, M_1, M), \qquad |A\mathbf{h}^n(t)| \leq C(M_0, M_1, M). \quad (4.22)$$

¿From of the inequality (3.1), we have

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}|\nabla\mathbf{u}_t^n|^2 + \nu|A\mathbf{u}_t^n|^2 &= (\alpha\mathbf{f}_t, A\mathbf{u}_t^n) - \alpha(P\mathbf{u}_t^n\cdot\nabla\mathbf{u}^n, A\mathbf{u}_t^n) + \alpha(P\mathbf{u}^n\cdot\nabla\mathbf{u}_t^n, A\mathbf{u}_t^n) \\ &\quad +(P\mathbf{h}_t^n\cdot\nabla\mathbf{h}^n, A\mathbf{u}_t^n) + (P\mathbf{h}^n\cdot\nabla\mathbf{h}_t^n, A\mathbf{u}_t^n),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla\mathbf{h}_t^n|^2 + \chi|A\mathbf{h}_t^n|^2 &= -(P\mathbf{u}_t^n\cdot\nabla\mathbf{h}^n, A\mathbf{h}_t^n) - (P\mathbf{u}^n\cdot\nabla\mathbf{h}_t^n, A\mathbf{h}_t^n) \\ &\quad +(P\mathbf{h}_t^n\cdot\nabla\mathbf{u}^n, A\mathbf{h}_t^n) + (P\mathbf{h}^n\cdot\nabla\mathbf{u}_t^n, A\mathbf{h}_t^n).\end{aligned}$$

Now, we estimate the right-hand side as is usual, for example

$$|(P\phi_t\cdot\nabla\mathbf{v}, \mathbf{b}_t)| \leq |\phi_t|_{L^{\frac{2N}{N-2}}}|\nabla\mathbf{v}|_{L^N}|A\mathbf{b}_t|$$

and

$$|(P\phi\cdot\nabla\mathbf{v}_t, \mathbf{b}_t)| \leq |\phi|_{L^{\frac{2N}{N-2}}}|\nabla\mathbf{v}_t|_{L^n}|A\mathbf{b}_t|$$

for any $\phi, \mathbf{v}, \mathbf{b} \in V_n$.

Consequently, by using the lemmas 2.1, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2)+\nu|A\mathbf{u}_t^n|^2+\chi|A\mathbf{h}_t^n|^2\\
\leq\ &\frac{\alpha^2}{2\nu}|\mathbf{f}_t|^2+\alpha C_1C_3|\nabla \mathbf{u}_t^n||A^{\gamma+\frac{1}{2}}\mathbf{u}^n||A\mathbf{u}_t^n|+\alpha C_2C_3|\nabla \mathbf{u}^n||A^{\gamma+\frac{1}{2}}\mathbf{u}_t^n||A\mathbf{u}_t^n|\\
&+C_2C_3|\nabla \mathbf{h}_t^n||A^{\gamma+\frac{1}{2}}\mathbf{h}^n||A\mathbf{u}_t^n|+C_2C_3|\nabla \mathbf{h}^n||A^{\gamma+\frac{1}{2}}\mathbf{h}_t^n||A\mathbf{u}_t^n| \qquad (4.23)\\
&+C_2C_3|\nabla \mathbf{u}_t^n||A^{\gamma+\frac{1}{2}}\mathbf{h}^n||A\mathbf{h}_t^n|+C_2C_3|\nabla \mathbf{u}^n||A^{\gamma+\frac{1}{2}}\mathbf{h}_t^n||A\mathbf{h}_t^n|\\
&+C_2C_3|\nabla \mathbf{h}_t^n||A^{\gamma+\frac{1}{2}}\mathbf{u}^n||A\mathbf{h}_t^n|+C_2C_3|\nabla \mathbf{h}^n||A^{\gamma+\frac{1}{2}}\mathbf{u}_t^n||A\mathbf{h}_t^n|.
\end{aligned}
$$

If $N=3$, then $\gamma=\frac{1}{4}$, thus the lemma 2.1, implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2)+\frac{\nu}{2}|A\mathbf{u}_t^n|^2+\chi|A\mathbf{h}_t^n|^2\\
\leq\ &\frac{\alpha^2}{2\nu}|\mathbf{f}_t|^2+C(M_0,M_1,M)|\nabla \mathbf{u}_t^n||A\mathbf{u}_t^n|+C(M_0,M_1,M)|A^{\frac{3}{4}}\mathbf{u}_t^n||A\mathbf{u}_t^n|\\
&+C(M_0,M_1,M)|\nabla \mathbf{h}_t^n||A\mathbf{u}_t^n|+C(M_0,M_1,M)|A^{\frac{3}{4}}\mathbf{h}_t^n||A\mathbf{u}_t^n| \qquad (4.24)\\
&+C(M_0,M_1,M)|\nabla \mathbf{u}_t^n||A\mathbf{h}_t^n|+C(M_0,M_1,M)|A^{\frac{3}{4}}\mathbf{h}_t^n||A\mathbf{h}_t^n|\\
&+C(M_0,M_1,M)|\nabla \mathbf{h}_t^n||A\mathbf{h}_t^n|+C(M_0,M_1,M)|A^{\frac{3}{4}}\mathbf{u}_t^n||A\mathbf{h}_t^n|.
\end{aligned}
$$

By using the momentum inequality with $\gamma=\frac{3}{4}, \alpha=\frac{1}{2}$ and $\beta=1$, we obtain

$$|A^{\frac{3}{4}}\mathbf{h}_t^n||A\mathbf{u}_t^n|\leq C|\nabla \mathbf{h}_t^n|^{\frac{1}{2}}|A\mathbf{h}_t^n|^{\frac{1}{2}}|A\mathbf{u}_t^n|.$$

Analogously

$$|A^{\frac{3}{4}}\mathbf{u}_t^n||A\mathbf{u}_t^n|\leq C|\nabla \mathbf{u}_t^n|^{\frac{1}{2}}|A\mathbf{u}_t^n|^{\frac{3}{2}},$$

$$|A^{\frac{3}{4}}\mathbf{h}_t^n||A\mathbf{h}_t^n|\leq C|\nabla \mathbf{h}_t^n|^{\frac{1}{2}}|A\mathbf{h}_t^n|^{\frac{3}{2}},$$

$$|A^{\frac{3}{4}}\mathbf{u}_t^n||A\mathbf{h}_t^n|\leq C|\nabla \mathbf{u}_t^n|^{\frac{1}{2}}|A\mathbf{u}_t^n|^{\frac{1}{2}}|A\mathbf{h}_t^n|.$$

Consequently, by using the Young inequality, we have in (4.24)

$$\begin{aligned}
&\frac{d}{dt}(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2)+\frac{\nu}{2}|A\mathbf{u}_t^n|^2+\chi|A\mathbf{h}_t^n|^2\\
\leq\ & C|\mathbf{f}_t|^2+C(M_0,M_1,M)(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2).
\end{aligned}$$

Integrating from t^* to $t\in[0,\tau]$, we get

$$\begin{aligned}
&\alpha|\nabla \mathbf{u}_t^n(t)|^2+|\nabla \mathbf{h}_t^n(t)|^2+\int_{t^*}^t(\frac{\nu}{2}|A\mathbf{u}_t^n(s)|^2+\chi|A\mathbf{h}_t^n(s)|^2)ds\\
\leq\ & C(M_1)+C(M_0,M_1,M)+\alpha|\nabla \mathbf{u}_t^n(t^*)|^2+|\nabla \mathbf{h}_t^n(t^*)|^2\\
\leq\ & C(M_0,M_1,M),
\end{aligned}$$

where we use the estimates of the lemma 2.1.
If $N=4$, then $\gamma=\frac{1}{2}$, this in (4.23), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2)+\frac{\nu}{2}|A\mathbf{u}_t^n|^2+\chi|A\mathbf{h}_t^n|^2\\
\leq\ & C|\mathbf{f}_t|^2+C(M_0,M_1,M)|\nabla \mathbf{u}_t^n||A\mathbf{u}_t^n|+C(M_0,M_1,M)|\nabla \mathbf{h}_t^n||A\mathbf{u}_t^n|\\
&+C(M_0,M_1,M)|A\mathbf{h}_t^n||A\mathbf{u}_t^n|+C(M_0,M_1,M)|\nabla \mathbf{u}_t^n||A\mathbf{h}_t^n|\\
&+C(M_0,M_1,M)|A\mathbf{h}_t^n|^2+C(M_0,M_1,M)|\nabla \mathbf{h}_t^n||A\mathbf{h}_t^n|\\
&+C(M_0,M_1,M)|A\mathbf{u}_t^n|^2.
\end{aligned}$$

The Young inequality implies

$$\begin{aligned}
&\frac{d}{dt}(\alpha|\nabla \mathbf{u}_t^n|^2+|\nabla \mathbf{h}_t^n|^2)+\nu|A\mathbf{u}_t^n|^2+\chi|A\mathbf{h}_t^n|^2\\
\leq\ & C|\mathbf{f}_t|^2+C(M_0,M_1,M)|\nabla \mathbf{u}_t^n|^2+C(M_0,M_1,M)|\nabla \mathbf{h}_t^n|^2\\
&+C(M_0,M_1,M)|A\mathbf{h}_t^n|^2+C(M_0,M_1,M)|A\mathbf{u}_t^n|^2.
\end{aligned}$$

Since, we can consider $(\min\{\nu,\chi\}-C(M_0,M_1,M))>0$, integrating from t^* to $t+\tau$,we obtain

$$\begin{aligned}
&\alpha|\nabla \mathbf{u}_t^n(t+\tau)|^2+|\nabla \mathbf{h}_t^n(t+\tau)|^2+\int_{t^*}^{t+\tau}(|A\mathbf{u}_t^n(s)|^2+|A\mathbf{h}_t^n(s)|^2)ds\\
\leq\ & C(M_1)+C(M_0,M_1,M)\int_{t^*}^{t+\tau}(|\nabla \mathbf{u}_t^n(s)|^2ds+|\nabla \mathbf{h}_t^n(s)|^2)ds \quad (4.25)\\
&+\alpha|\nabla \mathbf{u}_t^n(t^*)|^2+|\nabla \mathbf{h}_t^n(t^*)|^2.
\end{aligned}$$

Therefore, using the estimates (4.13) and (4.15), we obtain the desired estimative.

Equalities (4.6) imply

$$\begin{aligned}\alpha|\mathbf{u}^n_{tt}|^2 &= \alpha(\mathbf{f}_t,\mathbf{u}^n_{tt})-\nu(A\mathbf{u}^n_t,\mathbf{u}^n_{tt})-\alpha(P\mathbf{u}^n_t\cdot\nabla\mathbf{u}^n,\mathbf{u}^n_{tt})-\alpha(P\mathbf{u}^n\cdot\nabla\mathbf{u}^n_t,\mathbf{u}^n_{tt})\\ &\quad+(P\mathbf{h}^n_t\cdot\nabla\mathbf{h}^n,\mathbf{u}^n_{tt})+(P\mathbf{h}^n\cdot\nabla\mathbf{h}^n_t,\mathbf{u}^n_{tt}),\end{aligned}$$

$$\begin{aligned}|\mathbf{h}^n_{tt}|^2 &= -\chi(A\mathbf{h}^n_t,\mathbf{h}^n_{tt})-(P\mathbf{u}^n_t\cdot\nabla\mathbf{h}^n,\mathbf{h}^n_{tt})-(P\mathbf{u}^n\cdot\nabla\mathbf{h}^n_t,\mathbf{h}^n_{tt})\\ &\quad+(P\mathbf{h}^n_t\cdot\nabla\mathbf{u}^n,\mathbf{h}^n_{tt})+(P\mathbf{h}^n\cdot\nabla\mathbf{u}^n_t,\mathbf{h}^n_{tt}).\end{aligned}$$

Consequently,

$$\begin{aligned}\alpha|\mathbf{u}^n_{tt}|^2 &\le C(|\mathbf{f}_t|^2+|A\mathbf{u}^n_t|^2+|\nabla\mathbf{u}^n_t|^2|A^\gamma\mathbf{u}^n|^2+|A\mathbf{u}^n|^2|\nabla\mathbf{u}^n_t|^2\\ &\quad+|\nabla\mathbf{h}^n_t|^2|A^\gamma\mathbf{h}^n|^2+|A\mathbf{h}^n|^2|\nabla\mathbf{h}^n_t|^2),\end{aligned}$$

$$\begin{aligned}|\mathbf{h}^n_{tt}|^2 &\le C(|A\mathbf{h}^n_t|^2+|\nabla\mathbf{u}^n_t|^2|A^\gamma\mathbf{h}^n|^2+|A\mathbf{u}^n|^2|\nabla\mathbf{h}^n_t|^2\\ &\quad+|\nabla\mathbf{h}^n_t|^2|A^\gamma\mathbf{u}^n|^2+|A\mathbf{h}^n|^2|\nabla\mathbf{u}^n_t|^2.\end{aligned}$$

By using the estimates (4.25), (4.16) and (4.17), we obtain the estimate (4.18).

5. Proof of Theorems

By the Aubin-Lions theorem, we have from estimates (3.7) that there exist a subsequences $\mathbf{u}^n(t)$ and $\mathbf{h}^n(t)$ such that

$$\mathbf{u}^n\to\mathbf{u},\ \mathbf{h}^n\to\mathbf{h},\ \text{strongly in } L^\infty(\tau;V).$$

Also, by the estimates (4.1), we have

$$\mathbf{u}^n\to\mathbf{u},\ \mathbf{h}^n\to\mathbf{h},\ w^*\ \text{in } L^\infty(\tau;D(A)).$$

and

$$\mathbf{u}_t^n \to \mathbf{u}_t,\ \mathbf{h}_t^n \to \mathbf{h}_t,\ w^* \text{ in } L^\infty(\tau; V).$$

and the functions $\mathbf{u}(t)$ and $\mathbf{h}(t)$ satisfies

$$\mathbf{u}, \mathbf{h} \in H^2(\tau; H) \cap H^1(\tau; D(A)) \cap L^\infty(\tau; D(A)) \cap W^{1,\infty}(\tau; V).$$

We will show that

$$\mathbf{u}_t^n \to \mathbf{u}_t,\ \mathbf{h}_t^n \to \mathbf{h}_t, \text{ strongly in } L^\infty(\tau; H).$$

Taking $\phi = \mathbf{u}_t$ and $\phi = \mathbf{h}_t$ in Lemma 2.3, with $X = V, Y = B = H$, we obtain the desired convergences.

Once these later convergences are established, it is a standard procedure take the limit along the previous subsequences in (3.1) to conclude that $(\mathbf{u}, \mathbf{h})$ is a periodic strong solution of (1.1)-(1.3).

To prove the uniqueness, we consider that $(\mathbf{u}_1, \mathbf{h}_1)$ and $(\mathbf{u}_2, \mathbf{h}_2)$ are two solutions of problem (1.1)- (1.3). By defining differences

$$\theta = \mathbf{u}_1 - \mathbf{u}_2,\ \xi = \mathbf{h}_1 - \mathbf{h}_2.$$

They satisfy

$$\begin{aligned}
(\theta_t + \nu A\theta, \phi) &= (\xi \cdot \nabla \mathbf{h}_1 + \mathbf{h}_2 \cdot \nabla \xi, \phi) - \alpha(\theta \cdot \nabla \mathbf{u}_2, \phi) + \alpha(\mathbf{u}_2 \cdot \nabla \theta, \phi), \\
(\xi_t + \chi A\xi, \psi) &= (\xi \cdot \nabla \mathbf{u}_1 + \mathbf{h}_2 \cdot \nabla \theta, \psi) - (\theta \cdot \nabla \mathbf{h}_1, \psi) - (\mathbf{u}_2 \cdot \nabla \xi, \psi)
\end{aligned}$$

for all $\phi, \psi \in V$.

Setting $\phi = \theta$ and $\psi = \xi$ in the above inequalities, after of adding, we get

$$\frac{1}{2}\frac{d}{dt}(\alpha|\theta|^2 + |\xi|^2) + \nu|\nabla\theta|^2 + \chi|\nabla\xi|^2 = (\theta \cdot \nabla\xi, \mathbf{h}_1) - (\xi \cdot \nabla\xi, \mathbf{u}_1) + \alpha(\theta \cdot \nabla\theta, \mathbf{u}_2).$$

By other hand, using the Giga-Miyakawa result with $\delta = \gamma$, $\alpha = \rho = \frac{1}{2}$, we get

$$\begin{aligned}
(\theta \cdot \nabla\xi, \mathbf{h}_1) &= (A^{-\gamma} P\theta \cdot \nabla\xi, A^\gamma \mathbf{h}_1) \\
&\leq C_1 |A^{\frac{1}{2}}\theta||A^{\frac{1}{2}}\xi||A^\gamma \mathbf{h}_1| \\
&\leq C_1 C(M) |\nabla\theta||\nabla\xi|
\end{aligned}$$

where, we use the estimate given in the lemma 1.1. Analogously

$$\alpha(\theta \cdot \nabla\theta, \mathbf{u}_2) \le C_1 C(M)|\nabla\theta|^2,$$

$$(\xi \cdot \nabla\xi, \mathbf{u}_1) \le C_1 C(M)|\nabla\xi|^2.$$

Consequently, we obtain

$$\frac{1}{2}\frac{d}{dt}(\alpha|\theta|^2 + |\xi|^2) + \nu|\nabla\theta|^2 + \chi|\nabla\xi|^2 \le C(M)(\nu|\nabla\theta|^2 + \chi|\nabla\xi|^2). \quad (5.1)$$

Since $C(M) < 1$, we have

$$\frac{d}{dt}(\alpha|\theta|^2 + |\xi|^2) + L(\nu|\nabla\theta|^2 + \chi|\nabla\xi|^2) \le 0.$$

We recall that

$$|\theta|^2 \le \mu^{-1}|\nabla\theta|^2,\ |\xi|^2 \le \mu^{-1}|\nabla\xi|^2.$$

Consequently,

$$L(\nu\mu|\theta|^2 + \chi\mu|\xi|^2) \le L(\nu|\nabla\theta|^2 + \chi|\nabla\xi|^2).$$

Thus, we have in (5.1)

$$\frac{d}{dt}(\alpha|\theta|^2 + |\xi|^2) + L_1(\alpha|\theta|^2 + |\xi|^2) \le 0,$$

where $L_1 = L\mu\min\{\nu, \chi\}(\frac{1}{\alpha} + 1) > 0$.

Finally,

$$\alpha|\theta(t)|^2 + |\xi(t)|^2 \le (\alpha|\theta(0)|^2 + |\xi(0)|^2)\exp(-L_1 t)$$

for any $t \in (0, \infty)$.

Since $\theta(t)$ and $\xi(t)$ are periodic in t, for any $t \in (-\infty, +\infty)$ there exists a positive integer n_0 such that $t + n_0\tau > 0$ and

$$\alpha|\theta(t)|^2 + |\xi(t)|^2 = \alpha|\theta(t + n_0\tau)|^2 + |\xi(t + n_0\tau)|^2.$$

Hence, it follows,

$$\alpha|\theta(t)|^2 + |\xi(t)|^2 \le (\alpha|\theta(0)|^2 + |\xi(0)|^2)\exp(-L_1 n t)$$

$(n \ge n_0)$, which implies

$$\alpha|\theta(t)|^2 + |\xi(t)|^2 = 0$$

and finally $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{h}_1 = \mathbf{h}_2$.

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