MAXWELL THEORY IS STILL A SOURCE OF SURPRISES

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Abstract

From time to time there are claims in the literature that Maxwell theory is inconsistent even at the classical level. In this paper we analyse a new electrodynamics proposed by Chubykalo and Smirnov-Rueda (C&SR) which has been proposed because C&SR thought that they "demonstrated" through an example that the Liénard-Wiechert potentials are inadequate and lead to inconsistencies when applied to the fields generated by a uniformly accelerated charge. We show that this particular claim is *non sequitur* by exhibiting the serious flaws in C&SR calculations. We comment also on several other misleading statements in their alternative formulation of electrodynamics. The "separated potential method" discussed by C&SR leads them to state that the complete inhomogeneous wave equation for the potentials must be separated in a Poisson equation (PE) (describing infinite spread velocity of longitudinal perturbations) plus a free wave equation. We discuss the inconsistency of these results and to show how classical electrodynamics is a complex and tricky subject we present genuine and exact non-transverse solutions of Maxwell equations that travel with speed $v \neq 1$. Superluminal solutions called $SEXWs$ are discussed since finite aperture approximations to them have been produced recently. The existence of $SEXWs$ leads us to discuss the meaning of the Poynting vector. In particular we show that in the case of steady state electromagnetic configurations the right use of the Poynting vector is compatible with existing experiments which disprove the results obtained by C&SR with their "new electrodynamics". Finally we comment on a recent exact solution of the free Maxwell equations, obtained without the Green's function method, which implies that a uniformly accelerated charge does not radiate, a result that really puts doubts on the general applicability of the Liénard-Wiechert potentials, since it contradicts their prediction, at least for this case.

1. Introduction

In [1] (and also in errata [2] and [3]) Chubykalo and Smirnov-Rueda (C&SR) introduce a new electromagnetic theory in substitution to Maxwell theory. They think to have enough motivation for the enterprise since they claim that Maxwell theory produces errors when applied in many physical situations. Among the errors they claim to have proved through an example that the Liénard-Wiechert potentials are inadequate and lead to inconsistencies of the conventional classical electrodynamics. C&SR statement is based on their analysis of the potentials and fields generated by a uniformly accelerated charge. According to them the x-component E_x of the electric field produced by a uniformly accelerated charge moving in the x direction and calculated with the Liénard-Wiechert potentials does not satisfy the wave equation $\Box E_x = 0$ for spacetime points $(t, \vec{x})$ not occupied by the charge. If their result were true, it would imply in a serious inconsistency. However, in section 1 we show that they made serious mistakes and when the correct calculations are done we get $\Box E_x = 0$, as it should. All our calculations have been checked using REDUCE 3.0.

In section 3 we comment on other claims by C&SR having to do with their reformulation of classical electrodynamics. In sections III and IV of their paper C&SR introduced their "separated potential method" for solving the inhomogeneous wave equation for the vector potential $(A^\mu) = (\varphi, \vec{A})$. They separated φ and $\vec{A}$ as $\varphi = \varphi_0 + \varphi^*$, $\vec{A} = \vec{A}_0 + \vec{A}^*$ where $(\varphi_0, \vec{A}_0)$ solves a Poisson equation and $(\varphi^*, \vec{A}^*)$ solves the homogeneous wave equation. They claim that the Poisson equation satisfied by $(\varphi_0, \vec{A}_0)$ may be "considered as a wave equation with infinite spread velocity of longitudinal perturbations". This result forgot all the classical results concerning solutions of elliptic and hyperbolic equations and is *nonsequitur*. Also from the physical point of view, C&SR's proposal is at complete variance with the QED description of *e.g.* the Coulomb field [4] and they tried to relate their "longitudinal perturbations" to the so called $B(3)$ theory of Evans [5,6,7], created to explain the so called inverse Faraday effect and where that author claims that the normal transverse solution of the free Maxwell equations is always accompanied by a longitudinal ($B(3)$) magnetic field which is phase free. Evans' theory is according to our view misleading, but it deserves a whole comment which will not be presented here. Instead, we prefer to show that the free Maxwell equations have "extraordinary" genuine and exact solutions which are not transverse waves and which propagate with speed $v \neq 1$[8,9,10]. In particular superluminal electromagnetic field configurations called *SEXWs* have been produced recently[11]. See also a comment on the experiment described in [11] in [27].

The existence of *SEXWs* lead us to examine more closely the meaning of the Poynting vector and to show explicitly that C&SR statement (sec. V) that "the flux of electromagnetic energy in the steady satate has no sense since no presence of the free electromagnetic field is supposed in this case" is simply wrong. Indeed, there exists also an experimental verification that steady state electromagnetic fields (even static ones) store angular momentum [12].

In section 4 we present our conclusions and briefly discuss a new solution for the electromagnetic field produced by a uniformly accelerated charge found by Turakulov [13] (without using the method of Green's functions) which shows no presence of radiation. This solution gives results different from the one obtained through the use of the Liénard-Wiechert potentials and, in some sense, shows that C&SR's criticism of the use of these potentials has some reason. Anyway, Turakulov's new solution reopens an old problem that has been discussed by eminent physicists, *e.g.* Born [14] and Pauli [15] (see *e.g.* [16,17] for a discussion).

2. Liénard-Wiechert Potentials Do Not Imply In Any Inconsistency

Let (M, g, D) be Minkowski spacetime, $M \simeq R^4$, g is the Lorentz metric and D is the Levi–Civita connection of g. Let $\langle x^\mu \rangle$ be a Lorentz-Einstein coordinate chart for M: $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. We use, in what follows, units such that $c = 1$. In this section we show that, contrary to C&SR statement, the x component $E_x(t, \vec{x})$ of the electric field generated by a uniformly accelerated charge, moving in the x direction, satisfies the wave equation for all $x \in M$ not occupied by the charge, *i.e.*,

$$\Box E_x(t, \vec{x}) = 0 \,. \tag{1}$$

The fundamental flaw in C&SR's calculations is that they use the *classical*, instead of the *relativistic* equation of motion for a uniformly accelerated charge. This error is clear in their eq.(5). The use of the wrong equation of motion produces errors in the calculations of the second order derivatives of E_x, which is an implicit function of t and $\vec{x}$ due to the retarded Green's function used to obtain the Liénard-

Wiechert potential.

The correct calculations are: Let $\sigma : R \supset I \mapsto M$, $\sigma : s \mapsto \sigma(s)$ be the world-line of a uniformly accelerated charge (hyperbolic motion) where s is the proper time. The equations of motion for the particle moving in the x direction along the x axis are, as it is well known,

$$x^1(\sigma(s)) = \frac{1}{a}\cosh(as)\,;\; x^0(\sigma(s)) = \frac{1}{a}\sinh(as)\,; \tag{2}$$

with $x^2(\sigma(s)) = x^3(\sigma(s)) = 0$. Then, writing

$$x(\sigma(t)) \equiv x(t) = \frac{1}{a}\sqrt{1+a^2t^2}\,, \tag{3}$$

we can proceed as follows.

We begin by writing down the complete solution of Maxell's equations for a point charge q endowed with a constant acceleration, obtained with the help of the Liénard-Wiechert potentials. The electric field $\vec{E}$ at a spacetime point $(t, \vec{x})$ is

$$\begin{aligned} \vec{E}(t,\vec{x}) &= q\frac{(\vec{R} - R\vec{v}_0/c)(1 - v_0^2/c^2)}{(R - \vec{R}.\vec{v}_0/c)^3} \\ &+ q\frac{\vec{R} \times [(\vec{R} - R\vec{v}_0/c) \times \dot{\vec{v}}_0/c^2]}{(R - \vec{R}.\vec{v}_0/c)^3} \end{aligned} \tag{4}$$

In this equation $R = |\vec{R}|$, $v = |\vec{v}|$ and $\vec{R} = \vec{x} - \vec{x}_0$, where $\vec{x}_0$ is the position of the charge at an instant of time t_0 (retarded time) given by the implicit function

$$F(\vec{x}, \vec{x}_0, t, t_0) = t - t_0 - \frac{R}{c} = 0\,; \tag{5}$$

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2} \tag{6}$$

and where $\vec{v}_0$ is the velocity of the charge at time t_0 and $\dot{\vec{v}}_0$ its acceleration. To simplify our equations we now put $c = 1$ and $q = 1$. We also suppose that the motion takes place along the x axis, so that $\vec{R} = (x - x_0, y, z,)$, $\vec{v}_0 = (v_0, 0, 0)$ and $\dot{\vec{v}}_0 = (\dot{v}_0, 0, 0)$. The electric field at an observation point $(t, \vec{x})$ is then

$$\vec{E}(t,\vec{x}) = \frac{(1 - v_0^2)(x - x_0 - Rv_0, y, z)}{[R - (x - x_0)v_0]^3}$$

$$+ \quad \frac{(-y^2 - z^2, (x - x_0)y, (x - x_0)z)\dot{v}_0}{[R - (x - x_0)v_0]^3} \, . \tag{7}$$

The x component of the electric field at an arbitrary point $(t, \vec{x})$ of spacetime (with the obvious exception of the very position of the charge) is then

$$E_x = \frac{(1 - v_0^2)[(x - x_0) - Rv_0] - r^2\dot{v}_0}{[R - (x - x_0)v_0]^3} \, , \tag{8}$$

where we make use of cylindrical coordinates for which $y^2 + z^2 = r^2$ and where $R = [r^2 + (x - x_0)^2]^{1/2}$. The final expression for the E_x is obtained by replacing x_0, v_0 and $\dot{v}_0$ by their values at time t_0, according to eq.(3). We simplify the calculations by putting $a = 1$. We have

$$x_0 = x(t_0) = \sqrt{1 + t_0^2} \, ; \tag{9}$$

$$v_0 = v(t_0) = \frac{t_0}{\sqrt{1 + t_0^2}} = \frac{t_0}{z_0} \, ; \tag{10}$$

$$\dot{v}_0 = \frac{1}{(1 + t_0^2)^{3/2}} = z_0^{-3} \, . \tag{11}$$

The final result is written

$$E_x = \frac{x_0(x - x_0) - Rt_0 - r^2}{[x_0 R - t0(x - x_0)]^3} = \frac{N}{D^3} \, , \tag{12}$$

with

$$N = x_0(x - x_0) - Rt_0 - r^2 \tag{13}$$

and

$$D = x_0 R - t_0(x - x_0) \, . \tag{14}$$

Thus E_x becomes an explicit function of N and D, which are in turn explicit functions of r, z and t_0 only. The variable t_0 is itself an implicit function of r, z and t through equation (5).

Now, if α is one of the independent variables r, z or t, the first derivative of E_x with respect to α is simply

$$\frac{\partial E_x}{\partial \alpha} = \frac{DN_\alpha - 3ND_\alpha}{D^4} \, , \tag{15}$$

where

$$D_\alpha = \frac{\partial D}{\partial \alpha} + \frac{\partial D}{\partial t_0}\frac{\partial t_0}{\partial \alpha} \tag{16}$$

and

$$N_\alpha = \frac{\partial N}{\partial \alpha} + \frac{\partial N}{\partial t_0}\frac{\partial t_0}{\partial \alpha} \tag{17}$$

are the partial derivatives of N and D with respect to the α variable. The last derivative in the r.h.s. of eqs. (16) and (17) is given by the formula

$$\frac{\partial t_0}{\partial \alpha} = -\frac{\partial F/\partial \alpha}{\partial F/\partial t_0}\,. \tag{18}$$

After doing the calculations and collecting terms we obtain for the first order derivatives of E_z:

$$\frac{\partial E_x}{\partial r} = \frac{-r[D^2 + 3N(N + r^2 + 1)]}{D^5}\,; \tag{19}$$

$$\frac{\partial E_x}{\partial x} = \frac{x[D^2 - 3N(N + r^2)]}{D^5}\,; \tag{20}$$

$$\frac{\partial E_x}{\partial t} = \frac{-(t_0 + R)[D^2 - 3N(N + r^2)]}{D^5}\,. \tag{21}$$

This procedure thus greatly simplifies the calculations. For instance, the second derivative of E_x with respect to r, according to (16), (17) and (19), is given by

$$\begin{aligned} \frac{\partial^2}{\partial r^2}(E_x) &= \frac{\partial}{\partial r}\left(\frac{\partial E_x}{\partial r}\right) + \frac{\partial}{\partial N}\left(\frac{\partial E_x}{\partial r}\right) N_r \\ &+ \frac{\partial}{\partial D}\left(\frac{\partial E_x}{\partial r}\right) D_r\,. \end{aligned} \tag{22}$$

Notice that the partial derivative $\partial/\partial r$ on the r.h.s. of (22) refers only to the explicit dependence of $\partial E_x/\partial r$ on r, as the dependence implicit in the N and D terms is dealt with by using the chain rule.

The final calculations are still not easy, so that we used Reduce to perform them. The result of replacing all derivatives in the l.h.s. of the homogeneous wave equation (*HWE*) for E_x,

$$\frac{\partial^2}{\partial r^2}E_x + \frac{1}{r}\frac{\partial}{\partial r}E_x + \frac{\partial^2}{\partial x^2}E_x - \frac{\partial^2}{\partial t^2}E_x = 0\,, \tag{23}$$

is a fraction with denominator $x_0 R D^7 \neq 0$ and whose numerator turns out to be identically zero. This result shows that the x component of the electric field calculated with the Liénard-Wiechert potentials does indeed satisfy the HWE in vacuum, for the points not occupied by the charge.

3. Non transverse solutions of the free Maxwell equations

Due to their wrong calculations using the Liénard-Wiechert potentials for the uniformly accelerated charge C&SR developed a new electrodynamic theory which according to our view is also misleading. In section III of their paper they gave "reasons and foundations of the method of separated potentials." The idea behind this method of separated potentials (MSP) is to break the potential $A^\mu = (\varphi, \vec{A})$, which satisfies $\Box A^\mu = J^\mu$, $J^\mu = (\rho, \vec{\jmath})$, into two parts $(\varphi_0, \vec{A}_0)$ and $(\varphi^*, \vec{A}^*)$ with $\varphi = \varphi_0 + \varphi^*$, $\vec{A} = \vec{A}_0 + \vec{A}^*$ such that $(\varphi_0, \vec{A}_0)$ satisfy the Poisson equations

$$\nabla^2\varphi_0 = -4\pi\rho(t,\vec{x})\,, \quad \nabla^2\vec{A}_0 = -4\pi\vec{\jmath}(t,\vec{x})\,, \tag{24}$$

and $(\varphi^*, \vec{A}^*)$ satisfy the homogeneous wave equations

$$\Box\varphi^* = 0\,, \quad \Box\vec{A}^* = 0\,. \tag{25}$$

They claim that this is necessary to solve the inconsistencies of classical electrodynamics since φ_0 and $\vec{A}_0$ are implicit functions of t !

The fact is that their approach completely forgets the well formulated theory of the inhomogeneous wave equation as presented *e.g.* by Courant and Hilbert [18] and Morse and Feshbach [19]. That the separation of $\Box A^\mu = J^\mu$ into equations (24) and (25) is incorrect can be seen directly in their eq.(27) which fails to reproduce the field of a charge moving with uniform velocity, given in several textbooks, *e.g.* [20].

Another important comment that must be done here has to do with their statement (in section V) that in their MSP the Poisson equations for φ_0 and $\vec{A}_0$ may be considered as "wave equations with infinite spread velocity of longitudinal perturbations." In this statement they mix the longitudinal excitations of QED in the Gupta-Bleuler quantization method [4] with possible solutions of free Maxwell equations that are not transverse waves.* This is indeed the case since they tried

*Whittaker [49] showed that the Coulomb potential can be associated with an interference pattern of waves with frequencies $0 < \omega < \infty$ which travel from the source with arbitrary velocity. The to relate their results with the $B(3)$ theory of Evans [5,6,7] created to explain the so called inverse Faraday effect. The main claim of Evans is that the usual plane wave solution of Maxwell equations, which as is well known is a transverse wave, is always accompanied by a longitudinal magnetic field ($B(3)$) which is phase-free. We shall leave the discussion of Evan's theory to another paper. Instead we are going to show in the rest of this section that the free Maxwell equations have non purely transverse solutions which correspond in general to electromagnetic field configurations moving with velocity $v \neq 1$ (again in units where $c = 1$) [8,9,10,23].

In order to present examples of $v \neq 1$ solutions consider a function $\Phi : M \to C$ satisfying

$$\Box \Phi = 0 \,. \tag{26}$$

Define the variables

$$\xi_< = [x^2 + y^2 + \gamma_<^2 (z - v_< t)^2]^{1/2} \,; \tag{27}$$

$$\gamma_< = \frac{1}{\sqrt{1 - v_<^2}} \;; \quad \omega_<^2 - k_<^2 = \Omega_<^2 \;; \quad v_< = \frac{d\omega_<}{dk_<} \,, \tag{28}$$

$$\xi_> = [-x^2 - y^2 + \gamma_>^2 (z - v_> t)^2]^{1/2} \,; \tag{29}$$

$$\gamma_> = \frac{1}{\sqrt{v_>^2 - 1}} \;; \quad \omega_>^2 - k_>^2 = -\Omega_>^2 \;; \quad v_> = d\omega_>/dk_> \,. \tag{30}$$

We can easily verify that the functions $\Phi_<^{\ell m}$ and $\Phi_>^{\ell m}$ below are respectively subluminal and superluminal solutions of the *HWE*. We have

$$\Phi_p^{\ell m}(t, \vec{x}) = C_\ell j_\ell(\Omega_p \xi_p) P_m^\ell(\cos\theta) e^{im\theta} e^{i(\omega_p t - k_p z)} \tag{31}$$

where the index $p =<, >$, C_ℓ are constants, j_ℓ are the spherical Bessel functions, P_m^ℓ are the Legendre functions and (r, θ, φ) are the usual spherical coordinates. $\Phi_<^{\ell m}[\Phi_>^{\ell m}]$ has phase velocity $(w_</k_<) < 1[(w_>/k_>) > 1]$ and the modulation function $j_\ell(\Omega_< \xi_<)[j_\ell(\Omega_> \xi_>)]$ moves with group velocity $v_<$ $[v_>]$, where $0 \leq v_< < 1$ $[1 < v_> < \infty]$. Both $\Phi_<^{\ell m}$ and $\Phi_>^{\ell m}$ are *undistorted progressive waves* (*UPWs*). This term has been introduced by Courant and Hilbert (p. 760 of [18]). However they didn't suspect

electric field is longitudinal for Whittaker's solution (as it should), in the sense that it is parallel to the propagation vector. We comment here that Whittaker's approach is not very rigorous, and, indeed, the validity of his conclusions are true only in a manifold with the topology of $(S^2 \times R^2)$.

of UPWs moving with speeds greater than $c = 1$. For use in what follows text we write the explicit form of $\Phi^{00}_{<}$ and $\Phi^{00}_{>}$, which we denote simply by $\Phi_<$ and $\Phi_>$:

$$\Phi_p(t,\vec{x}) = C\frac{\sin(\Omega_p \xi_p)}{\xi_p} e^{i(\omega_p t - k_p z)} \quad ; \quad p = < \text{ or } > \ . \tag{32}$$

When $v_< = 0$, we have $\Phi_< \to \Phi_0$,

$$\Phi_0(t,\vec{x}) = C\frac{\sin \Omega_< r}{r} e^{i\Omega_< t}, \ r = (x^2+y^2+z^2)^{1/2}\,, \tag{33}$$

a result first found in Bateman [21]. The solution $\Phi_<$ appeared for the first time in Barut and Chandola [22].

We observe that if our interpretation of phase and group velocities is correct, then there must be a Lorentz frame where $\Phi_<$ is at rest. It is trivial to verify that in the coordinate chart $\langle x^{'\mu}\rangle$ which is a naturally adapted coordinate system to I', where $I' = (1-v_<^2)^{-1/2}\partial/\partial t + (v_</\sqrt{1-v_<^2})\partial/\partial z$ is a Lorentz frame moving with speed $v_<$ in the z direction relative to the inertial frame $I = \partial/\partial t$, Φ_p goes in $\Phi_0(t',\vec{x}')$ given by eq.(33) with $t \mapsto t'$, $\vec{x} \mapsto \vec{x}'$. We can also verify that there is no Lorentz frame with velocity parameter $0 < V < 1$ where $\Phi_>$ is at rest.

We present still another example of a $v > 1$ solution of $\Box\Phi = 0$, called the X-wave [9,10,23]. This example is given here to show the general way to obtain $v \neq 1$ solutions of the wave equation. For more details see [8,9,10,24,25].

Let $\tilde{\Phi}(\omega,\vec{k})$ be the Fourier transform of $\Phi(t,\vec{x})$, *i.e.*

$$\tilde{\Phi}(\omega,\vec{k}) = \int_{R^3} d^3x \int_{-\infty}^{+\infty} dt\, \Phi(t,\vec{x}) e^{-i(\vec{k}\vec{x}-\omega t)}\,, \tag{34}$$

$$\Phi(t,\vec{x}) = \frac{1}{(2\pi)^4}\int_{R^3} d^3\vec{k} \int_{-\infty}^{+\infty} d\omega\, \tilde{\Phi}(\omega,\vec{k}) e^{i(\vec{k}\vec{x}-\omega t)}\,, \tag{35}$$

Inserting (34) in the HWE (26) gives

$$(\omega^2 - \vec{k}^2)\tilde{\Phi}(\omega,\vec{k}) = 0 \tag{36}$$

and we are going to look for solutions of the HWE and eq.(36) in the sense of distributions. We rewrite (36) as

$$(\omega^2 - k_z^2 - \Omega^2)\tilde{\Phi}(\omega,\vec{k}) = 0\,. \tag{37}$$

It is obvious that (37) is always satisfied if we take

$$\Phi(\omega,\vec{k}) = \Xi(\bar{k},\eta)\delta(k_z - \bar{k}\cos\eta)\delta(\omega - \bar{k})\,, \tag{38}$$

where $\Xi(\bar{k},\eta)$ is an arbitrary function and where

$$k_z = \bar{k}\cos\eta\,,\ \cos\eta = \frac{k_z}{\omega}\,,\ \omega > 0\,,\ \Omega = \bar{k}\sin\eta\,,\ \bar{k} > 0\,. \tag{39}$$

We recall that $\vec{\Omega} = (k_x, k_y)$, $\vec{\rho} = (x,y)$ and we choose $\vec{\Omega}.\vec{\rho} = \Omega\rho\cos\theta$. Now, putting eq.(38) in eq.(35) we get

$$\begin{aligned}\Phi(t,\vec{x}) &= \frac{1}{(2\pi)^4}\int_0^\infty d\bar{k}\ \bar{k}\sin^2\eta\, e^{i(\bar{k}\cos\eta\, z - \bar{k}t)} \times \\ &\times \left[\int_0^{2\pi} d\theta\ \Xi(\bar{k},\eta) e^{i\bar{k}\rho\sin\eta\cos\theta}\right]. \end{aligned}\tag{40}$$

Choosing

$$\Xi(\bar{k},\eta) = (2\pi)^3 \frac{z_0 e^{-\bar{k}z_0\sin\eta}}{\bar{k}\sin\eta}, \tag{41}$$

where $z_0 > 0$ is a constant, we obtain

$$\begin{aligned}\Phi(t,\vec{x}) &= z_0\sin\eta\int_0^\infty d\bar{k}\, e^{-\bar{k}z_0\sin\eta} \times \\ &\times \left[\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\bar{k}\rho\sin\eta\cos\theta}\right] e^{i\bar{k}(\cos\eta\, z - t)}\,. \end{aligned}\tag{42}$$

Calling $z_0\sin\eta = a_0 > 0$, the last equation becomes

$$\Phi^{>}_{X_0}(t,\vec{x}) = a_0\int_0^\infty d\bar{k} e^{-\bar{k}a_0} J_0(\bar{k}\rho\sin\eta) e^{i\bar{k}(\cos\eta\, z - t)}. \tag{43}$$

Writing $\bar{k} = \bar{k}_<$ we see that

$$J_0(\bar{k}_<\rho\sin\eta) e^{i\bar{k}_<(z\cos\eta - t)} \tag{44}$$

is a subluminal Bessel beam [9,26], a solution of the HWE moving in the positive z direction with group velocity less than one, since for this beam $\omega^2 - k_<^2 = \Omega^2$. From eq.(43) we also have

$$\Phi^{>}_{X_0}(t,\vec{x}) = \frac{a_0}{\sqrt{(\rho\sin\eta)^2 + (a_0 + i(z\cos\eta - t))^2}}\,, \tag{45}$$

so that this solution propagates with speed $v = 1/\cos\eta$. For sound waves this result has been confirmed experimentally. More precisely, a finite aperture approximation (Rayleigh-Sommerfeld method [10,26,27,28]) to $\Phi^{>}_{X_0}$ was produced by a special transducer [29] and it has been verified [17,25] that it travels with speed $c_s/\cos\eta$, where c_s is the "sound velocity" that appears in the acoustical wave equation.

From the $v \neq 1$ solutions of $\Box\Phi = 0$ we can immediately obtain solutions of the free Maxwell equations travelling with $v < 1$ or $v > 1$ by means of the so called Hertz potential method [8,9,10,30]. The results are more easily presented using the formalism of differential forms (see *e.g.* [31]). Let (M, g, D) be Minkowski spacetime and let $e_\mu \in \sec TM$ (TM is the tangent bundle) be an orthonormal basis, $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ ($\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$) and $e_\mu = \partial/\partial x^\mu$, $\langle x^\mu\rangle$ being Lorentz-Einstein coordinate functions. Let $\gamma^\mu \in \sec T^*M$ (T^*M is the cotangent bundle) be the dual basis and let $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. We have $\gamma^\mu = dx^\mu$. Now, let $\bigwedge(M) = \sum_{p=1}^{4}\bigwedge^p(M)$ be the real exterior algebra bundle, where $\bigwedge^0(M)$ is identified with the ring of real functions and $\bigwedge^1(M) \equiv T^*M$. By $C\otimes\bigwedge(M)$ we mean the complex exterior algebra bundle. As is well known the electromagnetic field is represented by a two-form $F \in \sec\bigwedge^2(M)$. We have

$$F = \frac{1}{2}F^{\mu\nu}\gamma_\mu\wedge\gamma_\nu,\;\; F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{46}$$

where (E^1, E^2, E^3) and (B^1, B^2, B^3) are respectively the Cartesian components of the electric and magnetic fields.

If $\star : \bigwedge^p(M) \to \bigwedge^{4-p}(M)$ is the Hodge star operator and $\delta : \bigwedge^p(M) \to \bigwedge^{p+1}(M)$, $\delta(\omega_p) = (-)^p \star^{-1} d \star \omega_p$, $\star^{-1}\star = 1$ and where $d : \bigwedge^p(M) \to \bigwedge^{p+1}(M)$ is the exterior derivative operator, then the free Maxwell equations read

$$dF = 0\,, \quad \delta F = 0\,. \tag{47}$$

Now, if $A \in \sec\bigwedge^1(M)$ is the vector potential, such that $F = dA$, then if $\delta A = 0$ (Lorenz gauge[†]) we have the following

[†]Not Lorentz gauge, please. See *e.g.* Penrose and Rindler [32].

Theorem: Let $\Pi \in \sec \bigwedge^2(M)$ be the so called Hertz potential. Suppose that Π satisfies the wave equation, *i.e.* $\Box\Pi = 0$. If we take $A = -\delta\Pi$ we have that $F = dA$ satisfies $dF = 0$, $\delta F = 0$ [16].

From this theorem we see that if $\Phi \in \sec C \otimes \bigwedge^0(M)$ satisfies $\Box\Phi = 0$ we can find nontrivial solutions of of eqs.(47) by choosing, *e.g.*,

$$\Pi = \Phi\gamma^1 \wedge \gamma^2 \,, \tag{48}$$

where Φ is one of the functions shown above in eqs.(32,45), and then taking $F = \mathrm{Re}\,[-d\delta\,\Pi]$.

For example, if $\Phi = \Phi_0$ given by eq.(33) we get, putting

$$\Pi = \mathrm{Re}\,\Phi_0\gamma^1 \wedge \gamma^2 - \star(\mathrm{Im}\,\Phi_0\gamma^1 \wedge \gamma^2) \,, \tag{49}$$

that

$$\begin{aligned} F_0 = \frac{C}{r^3}[&\sin\Omega t(\alpha\Omega r\sin\theta\sin\varphi - \beta\cos\theta\sin\theta\cos\varphi)\gamma_0 \wedge \gamma_1 \\ - &\sin\Omega t(\alpha\Omega r\sin\theta\cos\varphi + \beta\sin\theta\cos\theta\sin\varphi)\gamma_0 \wedge \gamma_2 \\ + &\sin\Omega t(\beta\sin^2\theta - 2\alpha)\gamma_0\gamma_3 + \cos\Omega t(\beta\sin^2\theta - 2\alpha)\gamma_1 \wedge \gamma_2 \\ + &\cos\Omega t(\beta\sin\theta\cos\theta\sin\varphi + \alpha\Omega r\sin\theta\cos\varphi)\gamma_1 \wedge \gamma_3 \\ + &\cos\Omega t(-\beta\sin\theta\cos\theta\cos\varphi + \alpha\Omega r\sin\theta\sin\varphi)\gamma_2 \wedge \gamma_3] \end{aligned} \tag{50}$$

with $\alpha = \Omega r\cos\Omega r - \sin\Omega r$, $\beta = 3\alpha + \Omega^2 r^2\sin\Omega r$. Writing explicitly $\vec{E}$ and $\vec{B}$ we have

$$\vec{E} = \vec{W}\sin\Omega t \,, \quad \vec{B} = \vec{W}\cos\Omega t \,, \tag{51}$$

with

$$\vec{W} = -C\left(\frac{\alpha\Omega y}{r^3} - \frac{\beta xz}{r^5}, -\frac{\alpha\Omega x}{r^3} - \frac{\beta yz}{r^5}, \frac{\beta(x^2+y^2)}{r^5} - \frac{2\alpha}{r^3}\right) \,. \tag{52}$$

We notice in passing that $\nabla \times \vec{W} = \Omega\vec{W}$, the so called force-free field that appears in magnetohydrodynamics problems. We see from eq.(52) that F is a non-transverse stationary solution of the free Maxwell equations in vacuum. We can also easily verify that the field invariants are non null.

If we take

$$\Pi = \Phi_{X_0}\gamma^1 \wedge \gamma^2 \,, \tag{53}$$

where Φ_{X_0} is given by eq.(45), we get the superluminal electromagnetic X-wave (*SEXW*) [10,23]. Explicitly we get an F_{X_0} corresponding to electric and magnetic fields $\vec{E}_{X_0}$ and $\vec{B}_{X_0}$, whose components in polar coordinates are

$$(\vec{E}_{X_0})_\rho = 0\,; \tag{54a}$$

$$(\vec{E}_{X_0})_\theta = \frac{1}{\rho} i \frac{M_6}{\sqrt{M} M_2} \Phi_{X_0}\,; \tag{54b}$$

$$(\vec{B}_{X_0})_\rho = \cos\eta (\vec{E}_{X_0})_\theta\,; \tag{54c}$$

$$(\vec{B}_{X_0})_\theta = 0\,; \tag{54d}$$

$$(\vec{B}_{X_0})_z = -\sin^2\eta \frac{M_7}{\sqrt{M}} \Phi_{X_0}\,. \tag{54e}$$

The functions $M_i (i = 2, \ldots, 7)$ in (54) are

$$M_2 = \tau + \sqrt{M}\,; \tag{55a}$$

$$M_3 = \frac{1}{\sqrt{M}}\tau\,; \tag{55b}$$

$$M_4 = \frac{3}{\sqrt{M}}\tau\,; \tag{55c}$$

$$M_5 = \tau\,; \tag{55d}$$

$$M_6 = \rho^2 \sin^2\eta \frac{M_4}{M} M_2\,; \tag{55e}$$

$$M_7 = -\frac{1}{\sqrt{M}} + \frac{3}{\sqrt{M^3}}\tau^2\,. \tag{55f}$$

We immediately see from eqs.(54) that the F_{X_0} is indeed a superluminal solution of the free Maxwell equations which propagates with speed $1/\cos\eta$ in the z direction.

For the Poynting vector we get

$$(\vec{P}_{X_0})_\rho = -\mathrm{Re}\,\{(\vec{E}_{X_0})_\theta\}\mathrm{Re}\,\{(\vec{B}_{X_0})_z\}\,; \tag{56a}$$

$$(\vec{P}_{X_0})_\theta = 0\,; \tag{56b}$$

$$(\vec{P}_{X_0})_z = |\mathrm{Re}\,\{(\vec{E}_{X_0})_\theta\}|^2\,; \tag{56c}$$

and for the energy density we obtain

$$u_{X_0} = (1+\cos^2\eta)\left[|\mathrm{Re}\,\{(\vec{E}_{X_0})_\theta\}|^2\right] + |\mathrm{Re}\,\{(\vec{B}_{X_0})_z\}|^2\,; \tag{57}$$

and we have that $|\vec{P}|/u < 1$, a general result. Now, the question cannot be eluded of what is the velocity of transport of energy of the $v \neq 1$ (in particular, the $v > 1$) solutions of Maxwell equations ?

We can find in many physics textbooks (*e.g.* [33]) and in scientific papers [34] the following argument. Consider an arbitrary solution of *ME* in vacuum $dF = \delta F = 0$. Then it follows that the Poynting vector and the energy density of the field are

$$\vec{P} = \vec{E}\times\vec{B}\,, \quad u = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \tag{58}$$

It is obvious that the following inequality always holds:

$$v_\varepsilon = \frac{|\vec{P}|}{u} \leq 1. \tag{59}$$

Now, the law of conservation of energy-momentum over a finite volume V with boundary $S = \partial V$ reads in integral form

$$\frac{\partial}{\partial t}\left\{\iiint_V d\mathbf{v}\frac{1}{2}(\vec{E}^2 + \vec{B}^2)\right\} = \oint_S d\vec{S}.\vec{P} \tag{60}$$

Eq.(60) is interpreted saying that $\oint_S d\vec{S}.\vec{P}$ is the field energy flux across the surface $S = \partial V$, so that $\vec{P}$ is the flux density — the amount of field energy passing through a unit area of the surface in unit time.

Now, for plane wave solutions of Maxwell equations,

$$v_\varepsilon = 1 \tag{61}$$

and this result gives origin to the "dogma" that free electromagnetic fields transport energy at speed $v_\varepsilon = c = 1$.

However, $v_\varepsilon \leq 1$ is always true, even for superluminal solutions of ME, as the ones we have just mentioned. The same is true for the superluminal modified Bessel beam found by Band [34]. He maintains there that since $v_\varepsilon \leq 1$ there is no conflict between superluminal solutions of ME and Relativity Theory since what Relativity forbids is the propagation of energy with speed greater than c.

Here we challenge this conclusion. (A discussion on this issue and of whether Relativity Theory is compatible or not with the superluminal solutions of Maxwell equations can be found *e.g.* in [9,10,28].) The fact is that as well known $\vec{P}$ is not uniquely defined. Eq.(60) continues to hold true if we make the substitution $\vec{P} \mapsto \vec{P} + \vec{P}'$ with $\nabla.\vec{P}' = 0$. But of course we can easily find for subluminal, luminal or superluminal solutions of Maxwell equations a $\vec{P}'$ such that

$$\frac{|\vec{P} + \vec{P}'|}{u} \geq 1. \tag{62}$$

We thus come to the conclusion that the question of the transport of energy in superluminal *UPWs* solutions of *ME* is an experimental question. For the acoustic superluminal X-solution of the *HWE* the energy around the peak region seems to flow together with the wave, *i.e.* with speed $c_1 = c_s/\cos\eta$ where c_s is the velocity parameter appearing in the acoustic wave equation. The same seems to be true for the *SEXW* launched in free space by Saari and Reivelt[11]. These examples (together with results obtained for the flow of energy in the tunneling of microwaves [50]) show that, at least in the extension occupied by the field configuration, energy can flow with superluminal velocities.

Before ending we give another example to illustrate that eq.(59) is devoid of physical meaning. Consider a spherical conductor in electrostatic equilibrium with uniform superficial charge density (total charge Q) and with a dipole magnetic moment. Then, we have

$$\vec{E} = Q\frac{\mathbf{r}}{r^2}\,; \quad \vec{B} = \frac{C}{r^3}(2\cos\theta\,\mathbf{r} + \sin\theta\,\boldsymbol{\theta})\,, \tag{63}$$

$$\vec{P} = \vec{E} \times \vec{B} = \frac{CQ}{r^5}\sin\theta\,\boldsymbol{\varphi}\,, \tag{64}$$

$$u = \frac{1}{2}\left(\frac{Q^2}{r^4} + \frac{C^2}{r^6}(3\cos^2\theta + 1)\right)\,. \tag{65}$$

Thus

$$v_\varepsilon = \frac{|\vec{P}|}{u} = \frac{2CQr\sin\theta}{r^2Q^2 + C^2(3\cos^2\theta + 1)} \neq 0 \quad \text{for } r \neq 0. \tag{66}$$

Since the fields are static the conservation law eq.(60) continues to hold true, as there is no motion of charges and for any closed surface containing the spherical conductor we have

$$\oint_S d\vec{S}.\vec{P} = 0. \tag{67}$$

But *nothing* is in motion !

To end this section we recall that in sec. 2.19 of his book Stratton [30] presents a discussion of the Poynting vector and energy transfer which essentially agrees with the view presented above. Indeed he finishes that section with the words [30]: "By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision."

The last example seems at first sight to endorse C&SR statement in sec. V of their paper that "the flux of electromagnetic energy in the steady state has no sense since no presence of the free electromagnetic field is supposed in this case." However, from eq.(64) it is clear that there is angular momentum stored in the static electromagnetic field. If the sphere is discharged by its south pole, it will acquire a mechanical angular momentum which (supposing there is no radiation) is equal to the electromagnetic angular momentum stored in the field, as can be seen by an elementary calculation. An experimental verification of this not so well known phenomenon has been obtained by Graham and Lahoz [12]. By charging and discharging a capacitor in a magnetic field orthogonal to the $\vec{E}$ field inside the capacitor they showed that the capacitor oscillates, thus transforming electromagnetic angular momentum in mechanical angular momentum. Graham and Lahoz's experiment shows definitively that C&SR statement concerning the meaning of the Poynting vector for the steady state is completely wrong. However, as one can see from the other examples discussed above, care must be taken with the concept of propagation of energy and other results derived from the relation $|\vec{P}|/u < 1$.

To end this long section we recall that even if Saari and Reivelt[11] agree that they have produced a superluminal electromagnetic field configuration, they think that no trouble with Relativity theory occurs because they think that *SEXWs* are not signals in the Sommerfeld-Brillouin sense[35]. We do not agree with their opinion and our view is expressed in[27,28].[‡] The main problem—object of intense discussions on the Workshop: "Superluminal (?) Velocities" held at Cologne in June 98[§]—is: what constitutes a real signal? The Sommerfel-Brillouin signal is a pure mathematical concept and cannot exist in the physical world since it must have an infinite frequency spectrum. This point has been thoughtfully discussed by Nimtz[36] (one of the organizers of the Workshop quoted above) who showed that real signals must be necessarily frequency band limited, a fact that, he argues, implies that superluminal signals of this kind can violate Einstein's causality. We discuss this claim in [50].

4. Conclusions

In this paper we showed that contrary to the statements of C&SR the Liénard-Wiechert potentials do not seem to lead to any inconsistency when applied to the fields of a charge in hyperbolic motion. Nevertheless this does not mean that the question "does a uniformly accelerated charge radiate ?" has a consensuous answer. This is well illustrated by the dozens of papers in the literature dealing with this subject. A good review with the main references is the paper by Fulton and Rohrlich [16], where the authors conclude that there *is* radiation. This conclusion, based on the Liénard-Wiechert potentials, is achieved by a *clever* definition of what radiation means and the use of a Lorentz invariant definition.

However, Turakulov [13] has recently found an *exact* solution of Maxwell equations (with the method of separation of variables) for the fields generated by a charge in hyperbolic motion that shows no radiation. The existence of this exact solution shows that the uncritica use of Liénard-Wiechert potentials for any conceivable electromagnétic situation may be *non sequitur*. If the charge is moving in the z direction and if (t, ρ, θ, z) are the usual cylindrical coordinate functions naturally adapted to the inertial frame $I = \partial/\partial t$ [37], Turakulov's solution, valid for the domains $|z| < |t|$ and $|z| > |t|$, is

$$A_t = \frac{z}{z^2 - t^2}\left(\frac{z^2 - t^2 + \rho^2 + a^2}{\sqrt{R_+ R_-}} - 1\right) ; \tag{68a}$$

$$A_z = \frac{t}{z^2 - t^2}\left(\frac{z^2 - t^2 + \rho^2 + a^2}{\sqrt{R_+ R_-}} - 1\right) ; \tag{68b}$$

where

‡A more detailed discussion about the velocity of energy propagation (including the case of superluminal tunneling of microwaves) is given in [50].

§The papers presented at the Workshop appeared in Ann. der Physik. **7** (1998).

$$R_+ = \left(\sqrt{|z^2 - t^2|} + a\right)^2 + \rho^2 \tag{69a}$$

and

$$R_- = \left(\sqrt{|z^2 - t^2|} - a\right)^2 + \rho^2 \tag{69b}$$

It must be mentioned that this solution is similar to the Born solution [14] but they indeed differ from the Liénard-Wiechert potentials and give, as can be easily verified, a null magnetic field at the moment of zero velocity. The criticisms of Fulton and Rohrlich [16] to the Born solution which, being valid for both domains $z - t > 0$ and $z - t < 0$, describes the retarded field of a uniformly accelerated charge q moving with equation of motion $z(t) = \frac{1}{a}\sqrt{1 + a^2t^2}$ together with the advanced field of a uniformly accelerated charge $-q$ moving with equation of motion $z(t) = \frac{-1}{a}\sqrt{1 + a^2t^2}$ does not apply to Turakulov's solution. It is important to emphasize here that he proved [38], using the method of separation of variables, that there is emission of radiation under infinitesimally small changes of the acceleration of a moving charge. Also, solution (68) solves the problem of the charge in a gravitational field (equivalence principle) in a trivial way.

We end this paper recalling that there are yet several claims in the literature that classical electrodynamics is an inconsistent theory. Some claims are based on experimental results like those by Pappas [39] and Graneau [40,41], where they maintain that there are several experiments concerning the action of a current in metalic conductors which are in disagreement with the Biot-Savart or the Lorentz force laws and which do agree only with the Ampère force law [42]. Recently, Pappas's experiment on the force acting on an Ampère bridge has been repeated by Cavalleri and colaborators [43]. They found that contrary to Pappas's claims the Lorentz force explains the experimental results with great accuracy.

From the theoretical point of view there are also several claims that the Ampère and Lorentz force laws predict different results for the forces that a circuit produces in a part of itself. This result has been found to be wrong by Assis [44] and has been reconfirmed by Cavalleri [43]. Another well difused statement is that a conducting wire carrying a current does not generate an electric field outside the wire. This statement is false, both theoretically [47] and experimentally [48].

The examples discussed in this paper show that classical electromagnetic theory is a complex subject, less known than we suppose and that care must be taken in studying every one of its problems.

We think that there are still many surprises in Maxwell theory. Besides the ones studied above, we quote also the Maxwell-Dirac equivalence [45,46] which reveals unsuspected connections between electrodynamics and relativistic quantum mechanics.

References

1. A. E. Chubykalo and R. Smirnov-Rueda, Phys. Rev. E **53**, 5373 (1996).
2. A. E. Chubykalo and R. Smirnov-Rueda, Phys. Rev. E **55**, 3793 (1997).
3. A. E. Chubykalo and R. Smirnov-Rueda, Phys. Rev. E **57**, 3683 (1998).
4. J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer, Berlin, 1976), second expanded edition.
5. M. W. Evans, Physica B **182**, 227 (1992); Mod. Phys. Lett. **7**, 1247 (1993); Found. Phys. Lett. **7**, 67 (1994).
6. M. W. Evans and S. Kielich, *Modern Nonlinear Optics* (Wiley, New Yor, 1993, 1994), Advances in Chemical Physicis, vols. 85(1), 85(2), 85(3).
7. M. W. Evans and J.-P. Vigier, *The Enigmatic Photon* (Kluwer, Dordrecht, 1994, 1995), vols. 1 and 2.
8. W. A. Rodrigues, Jr. and J. Vaz, Jr., Subluminal and Superluminal Solutions in Vacuum of the Maxwell Equations and the Massless Dirac Equation, in J. Keller and Z. Oziewicz (eds.), *Proc. of the Conference: The Theory of the Electron*, Mexico City 1995, Adv. in Apllied Clifford Algebras **7**(S), 453(1997).
9. W. A. Rodrigues and J. E. Maiorino, Random Oper. and Stochastic Equations **4**, 355 (1996).
10. W. A. Rodrigues, Jr. and J.Y. Lu, Found. Phys.**27**, 435 (1997).
11. P. Saari and K. Reivelt, Phys. Rev. Lett. **21**, 4135 (1997).
12. G. M. Graham and A. G. Lahoz, Nature (London) **285**, 154 (1980).
13. Z. Ya. Turakulov, Turkish J. Phys. **18**, 479 (1994); J. Geom. Phys. **14**, 305 (1994).
14. M. Born, Ann. Phys. (Leipzig) **30**, 1 (1909).
15. W. Pauli, *Theory of Relativity* (Pergamon, New York, 1958), p. 93.
16. T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) **9**, 499 (1960).
17. F. Rohrlich, Nuovo Cimento **21**, 811 (1961).
18. R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley, New York, 1966), vols. I and II.
19. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), vols. I and II.
20. J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), second edition.
21. H. Bateman, *Electrical and Optical Motion* (Cambridge Univ. Press, Cambridge, 1915).
22. A. O. Barut and H. C. Chandola, Phys. Lett. A **180**, 5 (1993).
23. J.-Y. Lu and J. F. Greenleaf, "Limited Diffraction Solutions to Maxwell and Schroedinger Equations", Biodynamics Res. Unit, May Clinic and Foundation, Rochester, 1995 (unpublished).
24. R. Donnelly, Proc. R. Soc. London Ser. A **437**, 673 (1992); **460**, 541 (1993).
25. A. M. Shaarawi, I. M. Besieries, and R. W. Ziolkowski, J. Math. Phys. **31**, 2511 (1996).
26. J.-Y. Lu and J. F. Greenleaf, Ultrasound Med. Biol. **17**, 265 (1991).
27. E. Capelas de Oliveira and W. A. Rodrigues, Jr., Superluminal Electromagnetic Waves in Free Space, Ann. der Phys. **7**, 654 (1998).
28. J. E. Maiorino and W. A. Rodrigues, Jr., What is Superluminal Wave Motion, Sci. and Tech. Mag. **2**(4), 1999, electronic journal, "http://www.cptec.br/stm".

29. J.-Y. Lu and J. F. Greenleaf, IEEE Trans. Ultrason. Ferroelec. Freq. Control **39**, 19 (1992).
30. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
31. Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982).
32. R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge Un. Press, Cambridge, 1986).
33. B. Jancewicz, *Multivectors and Clifford Algebras in Electrodynamics* (World Scientific, Singapore, 1988).
34. W. Band, Found. Phys. **18**, 549 (1988); **18**, 625 (1988).
35. L. Brillouin, *Wave Propagation and Group Velocity*, Academic Press, New York and Londom, 1960.
36. G. Nimtz, Ann. der Phys. **7**, 618 (1998).
37. W. A. Rodrigues Jr. and M. A. F. Rosa, Found. Phys. **19**, 705 (1989).
38. Z. Ya. Turakulov, Turkish J. Phys. **19**, 1567 (1995).
39. P. T. Pappas, Nuovo Cimento **76 B**, 189 (1983); P. T. Pappas and P. G. Moyssides, Phys. Lett. **A 111**, 193 (1985); P. G. Moyssides and P. T. Pappas, J. Appl. Phys. **59**, 19 (1986).
40. P. Graneau, Nature (London) **295**, 311 (1982); J. Appl. Phys. **53**, 6648 (1982); **55** 2598 (1984); **58**, 3638 (1985); Nuovo Cimento **78**, 213 (1983); Phys. Lett. **A 97**, 253 (1983); **A 120**, 77 (1987); *Ampère-Neumann Electrodynamics of Metals* (Hadronic Press, Nonantum, 1985).
41. P. Graneau and P. N. Graneau, Appl. Phys. Lett. **46**, 468 (1985); Nuovo Cimento **7**, 31 (1986); Phys. Lett. **A 161**, 1 (1992).
42. A. K. T. Assis, *Weber's Electrodynamics* (Kluwer, Dordrecht, 1994).
43. G. Cavalleri, Phys. Rev. E **58**, 2505 (1998).
44. A. K. T. Assis and M. A. Bueno, IEEE Trans. Magn. **32**, 431 (1996).
45. J. Vaz, Jr. and W. A. Rodrigues, Jr., Int. J. Theor. Phys.**32**, 945 (1993).
46. W. A. Rodrigues, Jr. and J. Vaz, Jr, Found. Phys. **28**, 789 (1998).
47. A. K. T. Assis, W. A. Rodrigues Jr. and A. J. Mania, The Electric Field Outside a Stationary Resistive Wire Carrying a Constante Current, in publ. in Found. of Physics **29** (1999).
48. O. Jefimenko, Demonstration of the Electric Fields of Current-Carrying Conductors, Am. J. Phys. **30**, 19–21 (1962).
49. E. T. Whittaker, Mathematische Annalen **57**, 333 (1903).
50. W. A. Rodrigues Jr. and J. E. Maiorino, *Superluminal Wave Motion and the Foundations of Relativity and Quantum Theory* (in preparation).