On partitions with difference conditions.
(Part II)

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Abstract

In this paper we present two general theorems having interesting special cases. From one of them we give a new proof for Theorems of Gordon using a bijection and from another we have a new combinatorial interpretation associated to a Theorem of Göllnitz.

1. Introduction

Many of the identities given by Slater [6] have been used in the proofs of several combinatorial results in partitions. In this work we use 34, 36, 48, 53 and 57 of Slater [6], listed, in this order, below

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{8n-1})(1-q^{8n-7})(1-q^{8n})}{(1-q^{2n})} \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{8n-3})(1-q^{8n-5})(1-q^{8n})}{(1-q^{2n})} \tag{2}$$

$$\prod_{n=1}^{\infty}(1-q^{12n-5})(1-q^{12n-7})(1-q^{12n}) - q\prod_{n=1}^{\infty}(1-q^{12n-1})(1-q^{12n-11})(1-q^{12n}) \tag{3}$$

$$= \prod_{n=1}^{\infty}(1-(-1)^n q^{3n-1})(1+(-1)^n q^{3n-2})(1-(-1)^n q^{3n})$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{4n^2}}{(q^4;q^4)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{12n-5})(1-q^{12n-7})(1-q^{12n})}{(1-q^{4n})} \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n+1} q^{4n(n+1)}}{(q^4;q^4)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n})}{(1-q^{4n})} \tag{5}$$

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to prove some results in partitions where we use the standard notation

$$\begin{aligned}(a;q)_0 &= 1\\ (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1})\end{aligned}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

Here we proceed in a fashion similar to the one employed by Andrews [1], in the proofs of some theorems of Göllnitz.

2. The first general theorem

Theorem 1. Let $C_k(n)$ be the number of partitions of n in distinct parts of the form $n = a_1 + \cdots + a_s$ such that $a_s \equiv (k+1)$ or $(k+2)(mod\ 4)$ with $a_s \geq k+1$, $a_j \equiv (k+1)$ or $(k+2)(mod\ 4)$ if $a_{j+1} \equiv (k+2)$ or $(k+3)(mod\ 4)$, and $a_j \equiv k$ or $(k+3)(mod\ 4)$ if $a_{j+1} \equiv k$ or $(k+1)(mod\ 4)$. Then, for $k \geq 0$,

$$\sum_{n=0}^{\infty} C_k(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)q^{n^2+kn}}{(q^4;q^4)_n}.$$

Proof. We define $f(s,n)$ as the number of partitions of the type enumerated by $C_k(n)$ with the added restriction that the number of parts is exactly s. The following identity is true for $f(s,n)$:

$$f(s,n) = f(s-1, n-2s-k+1) + f(s-1, n-4s-k+2) + f(s, n-4s) \quad (6)$$

To prove this we split the partitions enumerated by $f(s,n)$ into three classes: (a) those in which $k+1$ is a part, (b) those in which $k+2$ is a part and (c) those with all parts $> k+2$.

If in those in class (a) we drop the part $k+1$ and subtract 2 from each of the remaining parts we are left with a partition of $n-(k+1)-2(s-1) = n-2s+1$ in exactly $s-1$ parts each $\geq k+1$ and these are the ones enumerated by $f(s-1, n-2s-k+1)$. From those in class (b) we drop the part $k+2$ and subtract 4 from each of the remaining parts obtaining partitions that are enumerated by $f(s-1, n-4s-k+2)$ and for the ones in class (c) we subtract 4 from each part obtaining the partitions enumerated by $f(s, n-4s)$.

Defining

$$F(z,q) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n)z^s q^n$$

and using (6) we obtain:

$$\begin{aligned}
F(z,q) &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(f(s-1,n-2s-k+1)+f(s-1,n-4s-k+2) \\
&\quad +f(s,n-4s)z^s q^n \\
&= zq^{k+1}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s-1,n-2s-k+1)(zq^2)^{s-1}q^{n-2s-k+1} + \\
&\quad zq^{k+2}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s-1,n-4s-k+2)(zq^4)^{s-1}q^{n-4s-k+2} + \\
&\quad \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n-4s)(zq^4)^s q^{n-4s} \\
&\quad = zq^{k+1}\ F(zq^2,q) + zq^{k+2}F(zq^4,q) + F(zq^4,q).
\end{aligned} \tag{7}$$

If $F(z,q)=\sum_{n=0}^{\infty}\gamma_n z^n$ we may compare coefficients of z^n in (7) obtaining

$$\gamma_n = \gamma_{n-1}q^{2n+k-1} + \gamma_{n-1}q^{4n+k-2} + \gamma_n q^{4n}.$$

Therefore

$$\gamma_n = q^{2n+k-1}\frac{(1+q^{2n-1})}{(1-q^{4n})}\gamma_{n-1} \tag{8}$$

and observing that $\gamma_0 = 1$ we may iterate (8) to get

$$\gamma_n = \frac{(-q;q^2)_n q^{n^2+kn}}{(q^4;q^4)_n}.$$

From this

$$F(z,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+kn} z^n}{(q^4;q^4)_n}$$

and therefore

$$\begin{aligned}
\sum_{n=0}^{\infty} C_k(n) q^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n)q^n \\
&= F(1,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+kn}}{(q^4;q^4)_n}
\end{aligned}$$

If we consider $k=0$ in this theorem and denote $C_0(n)$ by $C(n)$ we have that $C(n)$ is the number of partitions of n in distinct parts of the form $n=a_1+a_2+...+a_s$ where $a_s \equiv 1$ or $2 (\text{mod } 4)$, $a_j \equiv 3$ or $4 (\text{mod } 4)$ if $a_{j+1} \equiv 1$ or $4 (\text{mod } 4)$ and $a_j \equiv 1$ or $2 (\text{mod } 4)$ if $a_{j+1} \equiv 2$ or $3 (\text{mod } 4)$.

If we let $D(n)$ denote the number of partitions of n in parts that are distinct odd $\equiv \pm 5 \pmod{12}$ or even $\equiv \pm 4 \pmod{12}$ and let $E(n)$ denote the number of partitions of n in distinct odd $\equiv \pm 1 \pmod{12}$ or even $\equiv \pm 4 \pmod{12}$ we have the following theorem:

Theorem 2. $C(n) = D(n) + E(n-1)$ for every positive integer n.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} C(n) q^n &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n} q^{4n^2}}{(q^4;q^4)_{2n}} + q \sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n+1} q^{4n^2+4n}}{(q^4;q^4)_{2n+1}} \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{12n-5})(1+q^{12n-7})(1-q^{12n})}{(1-q^{4n})} + \\
&\quad q \prod_{n=1}^{\infty} \frac{(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n})}{(1-q^{4n})} \\
&= \sum_{n=0}^{\infty} D(n) q^n + q \sum_{n=0}^{\infty} E(n) q^n.
\end{aligned} \tag{9}$$

where we have used identities (4) and (5) after replacing in (4) "q" by "-q" which completes the proof.

□

We list below the partitions of 21 enumerated by $C(21)$ and $D(21)$ and the ones of 20 enumerated by $E(20)$.

$C(21)=12$	$D(21)=5$	$E(20)=7$
21	$17+4$	20
$20+1$	$16+5$	$16+4$
$12+8+1$	$8+8+5$	$8+8+4$
$16+5$	$8+5+4+4$	$8+4+4+4$
$12+9$	$5+4+4+4+4$	$4+4+4+4+4$
$17+3+1$		$11+8+1$
$16+4+1$		$11+4+4+1$
$13+7+1$		
$13+6+2$		
$9+7+5$		
$12+5+3+1$		
$9+7+4+1$		

It is interesting to observe that

$$C(n) = F_e(n) - F_0(n) \tag{10}$$

where $F_e(n)$ (resp. $F_0(n)$) is the number of partitions of n into parts $\not\equiv 0, \pm 2 (mod\ 12)$ with no repeated multiples of 3 and with an even (resp. odd) number of parts divisible by 3.

If fact by replacing "q" by "$-q$" in (3) we have

$$\prod_{n=1}^{\infty}(1+q^{12n-5})(1+q^{12n-7})(1-q^{12n}) + q\prod_{n=1}^{\infty}(1+q^{12n-1})(1+q^{12n-11})(1-q^{12n})$$
$$= \prod_{n=1}^{\infty}(1+q^{3n-1})(1+q^{3n-2})(1-q^{3n})$$

and observing that

$$\sum_{n=0}^{\infty} C(n)q^n \overset{(9)}{=} \prod_{n=1}^{\infty} \frac{(1+q^{3n-1})(1+q^{3q-2})(1-q^{3n})}{(1-q^{4n})}$$
$$= \frac{\displaystyle\prod_{\substack{n=1 \\ n\not\equiv 0 (mod\ \ 4)}}^{\infty} (1-q^{3n})}{\displaystyle\prod_{\substack{n=1 \\ m\equiv \pm 1, \pm 4, \pm 5 (mod\ \ 12)}}^{\infty} (1-q^n)}$$
$$= \sum_{n=0}^{\infty}(F_e(n)-F_0(n))q^n.$$

we have (10).

We state and prove, next, our second general result.

3. The second general theorem

Theorem 3. For $\ell \geq 0$ let $A_\ell(n)$ be the number of partitions of n of the form $n = a_1 + a_2 + \cdots + a_{2s-1} + a_{2s}$ such that $a_{2s} \geq a$ where $\ell = 2a + \varepsilon$ ($\varepsilon = 0$ or 1), $a_{2i-3} - a_{2i} \geq 3$ and $a_{2i-1} - a_{2i} = 1$ (when $a_{2i} \geq a + \varepsilon$) or 2. Then

$$\sum_{n=0}^{\infty} A_\ell(n) q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+\ell n}}{(q^2;q^2)_n}$$

Proof. We define, for $\lambda = 1$ or 2, $g_\lambda(2s,n)$ as the number of partitions of the type enumerated by $A_\ell(n)$ with the added restriction that the number of parts is exactly 2s and such that $a_{2s-1} + a_{2s} \geq \ell + 2\lambda - 1$. For $n = s = 0$ we define $g_\lambda(2s,n) = 1$ and $g_\lambda(2s,n) = 0$ if $n < 0$ or $s < 0$ or $n = 0$ and $s > 0$.

In what follows we prove two identities for $g_\lambda(2s,n)$.

(i) $g_1(2s,n) = g_2(2s-2, n-2s-\ell) + g_1(2s-2, n-2s-\ell+1) + g_1(2s, n-2s)$

(ii) $g_2(2s,n) = g_1(2s, n-2s)$.

To prove the first one we split the partitions enumerated by $g_1(2s,n)$ into two classes: (a) those partitions in which $\left\lfloor \frac{\ell}{2} \right\rfloor$ or $\left\lfloor \frac{\ell+1}{2} \right\rfloor$ is a part; (b) those in which neither $\left\lfloor \frac{\ell}{2} \right\rfloor$ nor $\left\lfloor \frac{\ell+1}{2} \right\rfloor$ is a part.

The ones in class (a) can have as the two smallest parts either "$\left\lfloor \frac{\ell+4}{2} \right\rfloor + \left\lfloor \frac{\ell+1}{2} \right\rfloor$" or "$\left\lfloor \frac{\ell+3}{2} \right\rfloor + \left\lfloor \frac{\ell}{2} \right\rfloor$". In the first case if we remove the two smallest parts and subtract 1 from each of the remaining parts we are left with a partition of $n-(\ell+2)-(2s-2) = n-2s-\ell$ in exactly $2s-2$ parts where $a_{2s-3} - a_{2s-2} \geq \ell+3$. These are the partitions enumerated by $g_2(2s-2, n-2s-\ell)$.

From those in class (a) having "$\left\lfloor \frac{\ell+3}{2} \right\rfloor + \left\lfloor \frac{\ell}{2} \right\rfloor$" as the two smallest parts we remove these two and subtract 1 from each of the remaining parts. We are, in this case, left with a partition of $n-(\ell+1)-(2s-2) = n-2s-\ell+1$ in exactly $2s-2$ parts where $a_{2s-3} - a_{2s-2} \geq \ell+1$ which are enumerated by $g_1(2s-2, n-2s-\ell+1)$.

Is is important of observe that after doing these operations the restrictions on difference between parts is not changed.

Now we consider the partitions in class (b). Considering that neither $\left\lfloor\frac{\ell}{2}\right\rfloor$ nor $\left\lfloor\frac{\ell+1}{2}\right\rfloor$ is a part we can subtract 1 from each part obtaining partitions of $n-2s$ in $2s$ parts which are the ones enumerated by $g_1(2s, n-2s)$.

The prove of (ii) follows by the fact that if we subtract 1 from each part of a partition such that $a_{2s-1}+a_{2s} \geq \ell+3$ the resulting one is such that $a_{2s-1}+a_{2s} \geq \ell+1$.

We define, now,

$$G_\lambda(z,q) = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_\lambda(2s,n) z^{2s} q^n.$$

Using (i) we have

$$\begin{aligned}
G_1(z,q) &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s,n) g_1(2s,n) z^{2s} q^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} (g_2(2s-2, n-2s-\ell) \\
&\quad + g_1(2s-2, n-2s-\ell+1) + g_1(2s, n-2s)) z^{2s} q^n \\
&= z^2 q^{\ell+2} \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s-2, n-4s-\ell)(zq^2)^{2s-2} q^{n-4s-\ell} + \\
&\quad z^2 q^{\ell+1} \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s-2, n-2s-\ell+1)(zq)^{2s-2} q^{n-2s-\ell+1} + \\
&\quad \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s, n-2s)(zq)^{2s} q^{n-2s} \\
&= z^2 q^{\ell+2} G_1(zq^2, q) + z^2 q^{\ell+1} G_1(zq,q) + G_1(zq,q) \qquad (11)
\end{aligned}$$

where, on the third sum, we used (ii).

Now, comparing the coefficients of z^{2n} in (11) after making the substitution

$$G_1(z,q) = \sum_{n=0}^{\infty} \gamma_n z^{2n}$$

we have

$$\gamma_n = q^{4n+\ell-2}\gamma_{n-1} + q^{2n+\ell-1}\gamma_{n-1} + q^{2n}\gamma_n.$$

Therefore

$$\gamma_n = q^{2n+\ell-1}\frac{(1+q^{2n-1})}{(1-q^{2n})}.\gamma_{n-1} \tag{12}$$

and, observing that $\gamma_0 = 1$, we may iterate this $n-1$ times to get:

$$\gamma_n = \frac{(-q;q^2)_n q^{n^2+\ell n}}{(q^2;q^2)_n}.$$

Then

$$G_1(z,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+\ell n}}{(q^2;q^2)_n}z^{2n}$$

and the theorem follows since

$$\begin{aligned}\sum_{n=0}^{\infty} A_\ell(n)q^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} g_1(2s,n)q^n = G_1(1,q)\\ &= \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+\ell n}}{(q^2;q^2)_n}\end{aligned}$$

□

Particular cases of Theorem 3.

If we take $\ell = 0$ in Theorem 3 we have the following result:

Theorem 4. The number of partitions of n in an even number of parts, $2s$, such that $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 0$ is equal to the number of partitions of n in parts $\equiv \pm 1, 4 (mod\ 8)$.

Proof. By Theorem 3 we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_0(n)q^n &= \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}\\ &\overset{(2)}{=} \prod_{n=1}^{\infty}\frac{1}{(1-q^{8n-1})(1-q^{8n-4})(1-q^{8n-7})}\end{aligned}$$

□

For $\ell = 1$ we have:

Theorem 5. $A_1(n)$ is equal to the number of partitions of n in parts $\equiv 1, 5, 6 (mod\, 8)$.

Proof. By Theorem 3 and corollary 2.7, p. 21 of Andrews [2] with q replaced by q^2 and $a = -q$ we have:

$$\begin{aligned}\sum_{n=0}^{\infty} A_1(n)q^n &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)}}{(q^2;q^2)_n} \\ &= (-q^3;q^4)_\infty(-q^2;q^2)_\infty \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-2})(1-q^{8n-3})(1-q^{8n-7})}\end{aligned}$$

□

In [4] Santos and Mondek gave a family of partitions including as special case the following theorem of Göllnitz.

"Let $G_1(n)$ denote the number of partitions of n into parts, where each part is congruent to one of 1, 5 or 6 (mod 8). Let $H_1(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_j$, where $b_i \geq b_{i+1} + 2$ and strict inequality holds if b_i is odd. Then for each $n, G_1(n) = H_1(n)$."

It is clear that by our Theorem 5 we have a new combinatorial interpretation for partitions enumerated by $H_1(n)$.

For $\ell = 2$ we get the following result:

Theorem 6. The number of partitions of n in an even number of parts, $2s$, such that $a_{2j-1} - a_{2j} = 1$ or 2, and $a_{2s} \geq 1$, is equal to the number of partitions of n in parts $\equiv \pm 3, 4 (mod\ 8)$.

Proof. By Theorem 3 and equation (1) we have:

$$\begin{aligned}\sum_{n=0}^{\infty} A_2(n)q^n &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-3})(1-q^{8n-4})(1-q^{8n-5})}\end{aligned}$$

□

4. A Bijection

We now describe a transformation between the partitions defined by $A_2(n)$ and partitions of n in s parts.

The partitions of n with an even number of parts $2s$ where $a_{2j-1} - a_{2j} = 1$ or $2, a_{2s} \geq 1$ can be transformed into partitions of n in s parts just by adding the parts $a_{2j-1} + a_{2j}$, i.e., the partition

$$a_1 + a_2 + \cdots + a_{2j-1} + a_{2j} + a_{2j+1} + \cdots + a_{2s}$$

is transformed in

$$b_1 + b_2 + \cdots + b_j + \cdots + b_s$$

where $b_j = a_{2j-1} + a_{2j}$, $b_j - b_{j+1} \geq 2$ and $b_j - b_{j+1} \geq 3$ if b_{j+1} is even with the restriction $b_s \geq 3$.

This operation can be easily reversed in the following way:

if b_j is even we write it as $a_{2j-1} + a_{2j}$ where $a_{2j-1} = \dfrac{b_j}{2} + 1$ and $a_{2j} = \dfrac{b_j}{2} - 1$,
if b_j is odd we write it as $a_{2j-1} + a_{2j}$ where $a_{2j-1} = \dfrac{b_j + 1}{2}$ and $a_{2j} = \dfrac{b_j - 1}{2}$.

With this transformation we get the original one

$$a_1 + a_2 + \cdots + a_{2s}$$

with exactly the same restrictions, i.e., $a_{2j-1} - a_{2j} = 1$ or 2 and $a_{2s} \geq 1$.

To illustrate this we list the partitions of 16 as described in Theorem 6 and the ones obtained by the transformation given above.

16	$\longleftrightarrow$	$9+7$
$13+3$	$\longleftrightarrow$	$7+6+2+1$
$12+4$	$\longleftrightarrow$	$7+5+3+1$
$11+5$	$\longleftrightarrow$	$6+5+3+2$
$10+6$	$\longleftrightarrow$	$6+4+4+2$
$9+7$	$\longleftrightarrow$	$5+4+4+3$
$8+5+3$	$\longleftrightarrow$	$5+3+3+2+2+1$

Theorem 7 below follows from this transformation.

Theorem 7. The number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$, where $b_j - b_{j+1} \geq 2, b_s \geq 3$ and $b_j - b_{j+1} \geq 3$ if b_{j+1} is even is equal to the number of partitions of n into parts $\equiv \pm 3, 4 (\text{mod} 8)$.

This Theorem was proved by Gordon in [3] (Theorem 3, page 741).

Also by Theorem 4 and the bijection described we have the following theorem:

Theorem 8. The number of partitions of any positive integer n into parts $\equiv$ 1,4 or 7 (mod 8) is equal to the number of partitions of the form $n = n_1 + n_2 + ... + n_k$, where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$ if n_i is even $(1 \leq i \leq k-1)$.

This result has been proved by Gordon in [3].

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