# Nonsmooth Continuous-Time Optimization Problems: Necessary Conditions*

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**Abstract :** First-order necessary optimality conditions of both Fritz-John and Karush-Kuhn-Tucker types are provided for Lipschitz continuous-time nonlinear optimization problems.

**Keywords :** Continuous nonlinear programming, necessary conditions, nonsmooth optimization

## 1 Introduction

Consider the primal continuous-time nonlinear programming problem below.

$$\left.\begin{array}{c} \text{Minimize } \ \phi(x) = \int_0^T f(t, x(t))dt, \\ \\ \text{subject to } \ g_i(t, x(t)) \leq 0 \text{ a.e. } \ t \in [0, T], \\ i \in I = \{1, \ldots, m\}, \quad x \in X. \end{array}\right\} \quad (CNP)$$

Here $X$ is an open, nonempty convex subset of the Banach space $L_\infty^n[0, T]$ of all n-dimensional vector valued Lebesgue mesurable functions, which are essentially bounded, defined on the compact interval $[0, T] \subset I\!R$, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \leq j \leq n} \operatorname{ess\,sup}\{|x_j(t)|, \ 0 \leq t \leq T\}$$

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where for each $t \in [0,T]$, $x_j(t)$ is $j$th component of $x(t) \in I\!\!R^n$, $\phi$ is a real valued function defined on $X$, $g(t,x(t)) = \gamma(x)(t)$ and $f(t,x(t)) = \Gamma(x)(t)$, where $\gamma$ is map from $X$ into the normed space $\Lambda_1^m[0,T]$ of all Lebesgue measurable essentially bounded $m$-dimensional vector functions defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1\leq j\leq m} \int_0^T |y_j(t)|dt,$$

and $\Gamma$ is map from $X$ into the normed space $\Lambda_1^1[0,T]$.

This class of problems was introduced in 1953 by Bellman [1] in connection with production-inventory "botleneck processes". He considered a type of optimization problems, which is now known as continuous-time linear programming, formulated its dual and provided duality relations. He also suggested some computational procedure.

Since then, a lot of authors have extended his theory to wider classes of continuous-time linear problems (e.g. [2], [3], [4], [5, 6], [7], [8], [9] and [10]).

On the other hand, optimality conditions in the spirit of Karush-Kuhn-Tucker type for continuous nonlinear problems were first investigated by Hanson and Mond [11]. They considered a class of linear constrained nonlinear programming problems. Assuming a nonlinear integrand in the cost funtion was twice differentiable, they linearized the cost function and applied Levinson's duality theory [3] to obtain the Karush-Kuhn-Tucker optimality conditions. Also applying linearization, Farr and Hanson [15] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems (both cost function and constraints were nonlinear).

Assuming some kind of constraint qualifications and using direct methods, further generalizations of the theory of optimality conditions for continuous-time nonlinear problems are to be found in Scott and Jefferson [12], Abraham and Buie [13], Reiland and Hanson [14] and Zalmai [16, 17, 18, 26, 27]. However, the development of nonsmooth necessary optimality conditions theories for problem (CNP) is not yet satisfactory.

Our aim in this paper is to provide first-order necessary optimality conditions in the form of Fritz-John and Karush-Kuhn-Tucker theorems for a general class of nonsmooth continuous-time Lipschitz programming problems. This is accomplished through generalizations of the differentiable continuous versions of Fritz-John and Karush-Kuhn-Tucker theorems in [16] to the Lipschitz case. Sufficient conditions is pursued in another paper [21].

Related results can be found in Craven [22]. However, his arguments are via approximation of smooth functions rather than alternative theorems.

This work is organized as follows. In Section 2 we recall some basic properties of Lipschitz nonsmooth analysis, support functions, Integration of multifunctions and state the generalized Gordan's Theorem. In Section 3 we establish the nonsmooth geometric optimality conditions for (CNP). The nonsmooth versions of Fritz-John and Karush-Kuhn-Tucker continuous-time optimality conditions are obtained on Sections 4 and 5, respectively.

# 2 Preliminaries

In this section we summarize basic concepts and tools from nonsmooth analysis, including supporting funtions and integration of multifunctions. Most of material included here can be found in Clarke [23]. We also state the generalized Gordan's Theorem, which has been a very usuful tool in optimization theory.

In what follows, $\mathcal{B}$ denotes a real linear space with norm $\|\cdot\|$ and $\mathcal{B}^*$ its topological dual with norm given by

$$\|\xi\|_* = \sup\{\langle \xi, v\rangle : v \in \mathcal{B},\ \|v\| \leq 1\},$$

where $\langle \cdot, \cdot \rangle$ is the canonic dual map between $\mathcal{B}^*$ and $\mathcal{B}$.

Let $f : \mathcal{B} \to I\!R$ be Locally Lipschitz, i.e., for all $x \in \mathcal{B}$ there is $\epsilon > 0$ and constant $K$ depending on $\epsilon$ such that

$$|f(x_1) - f(x_2)| \leq K\|x_1 - x_2\|,\ \forall x_1,\ x_2 \in x + \epsilon B.$$

Here $B$ denotes the open unit ball of $\mathcal{B}$. We also say that $f$ is Lipschitz of rank $K$ near $x$.

Let $v \in \mathcal{B}$. The *generalized directional derivative* of $f$ at $x$ in the direction $v$, denoted by $f^0(x; v)$, is defined as follows:

$$f^0(x; v) = \limsup_{\substack{y \to x \\ s \to 0^+}} \frac{f(y + sv) - f(y)}{s}.$$

Here $y \in \mathcal{B}, \quad s \in (0, +\infty)$.

The *generalized gradient* of $f$ at $x$, denoted by $\partial f(x)$, is the subset of $\mathcal{B}^*$ given by

$$\{\xi \in \mathcal{B}^* \ / \ f^0(x; v) \geq \langle \xi, v\rangle\ , \quad \forall v \in \mathcal{B}\}.$$

For every $v \in \mathcal{B}$, one has

$$f^0(x;v) = \max\{\langle \xi, v\rangle \ / \ \xi \in \partial f(x)\}.$$

We say that $f$ is *regular* at $x \in \mathcal{B}$ if

(i) $\forall v \in \mathcal{B}$, the directional derivative $f'(x;v)$ exists;

(ii) $\forall v \in \mathcal{B},\ f'(x;v) = f^0(x;v)$.

## 2.1 Support Functions

We recall that the *support function* of a nonempty subset $D$ of $\mathcal{B}$ is the function $\sigma_D : \mathcal{B} \to I\!R \cup \{+\infty\}$ defined by

$$\sigma_D(\xi) = \sup\{\langle \xi, x\rangle \ / \ x \in D\}.$$

We now state some basic known results of support functions which are needed in the sequel.

**Proposition 2.1** *(Hörmander [24]). Let $C, D$ be nonempty closed convex subsets of $\mathcal{B}$. And let $\Sigma,\ \Delta$ be nonempty weak* closed convex subsets of $\mathcal{B}^*$. Then,*

$$C \subseteq D \quad \textit{iff} \quad \sigma_C(\xi) \le \sigma_D(\xi), \quad \forall \xi \in \mathcal{B}^*,$$

$$\Delta \subseteq \Sigma \quad \textit{iff} \quad \sigma_\Delta(x) \le \sigma_\Sigma(x), \quad \forall x \in \mathcal{B}.$$

**Proposition 2.2** *(Hörmander [24]) Let $C, D$ be nonempty closed convex subsets of $\mathcal{B}$, and $\Sigma,\ \Delta$ be nonempty weak* closed convex subsets of $\mathcal{B}^*$. Let also $\mu,\ \lambda \ge 0$ be given scalars. Then*

$$\mu\sigma_C(\xi) + \lambda\sigma_D(\xi) = \sigma_{\{\mu C + \lambda D\}}(\xi) \quad \xi \in \mathcal{B}^*;$$

$$\mu\sigma_\Delta(x) + \lambda\sigma_\Sigma(x) = \sigma_{\{\mu\Delta + \lambda\Sigma\}}(x) \quad x \in \mathcal{B}.$$

## 2.2 Integration of Multifunctions

Given a multifunction $G : [0,T] \to I\!R^n$, denote by $S^1([0,T])$, the following set

$$S^1([0,T]) = \{f \in L_1^n[0,T],\ f(t) \in G(t) \text{ a.e. } t \in [0,T]\}.$$

We define the integral of $G$, denoted by $\int_0^T G(t)dt$, as the following subset of $I\!R^n$ :

$$\int_0^T G(t)dt := \left\{ \int_0^T f(t)dt : f \in S^1([0,T]) \right\}.$$

A multifunction $G$ is said to be integrably bounded if $G$ is measurable and there exists a integrable function $z : [0,T] \to I\!R_+$ such that

$$\|G(t)\| \leq z(t) \text{ a.e. on } [0,T].$$

**Theorem 2.3** *If $G$ is a integrably bounded multifunction taking values compact subsets of $I\!R^n$, then*

$$\sigma_{\int_0^T G(t)dt}(v) = \int_0^T \sigma_{G(t)}(v)dt, \ \forall v \in R^n.$$

The proof of this theorem can be found, for example, in [25].

## 2.3 The generalized Gordan Theorem

In this subsection, we state a transposition theorem, known as the Generalized Gordan's Theorem (Zalmai [18]). It is the key to move from the geometric optimality condition obtained above to the main results on first-order necessary optimality conditions in this work.

For the next result the domain of definition of the elements of the the spaces $L^n_\infty[0,T]$, $L^m_\infty[0,T]$, $\Lambda^m_1[0,T]$ are replaced with a nonzero Lebesgue measure set $A \subset [0,T]$.

**Theorem 2.4** *Let $A \subset [0,T]$ be a set of positive Lebesgue measure and $X$ be a nonempty convex subset of $L^n_\infty(A)$ and $p_i : V \times A \to I\!R$, $i \in I = \{1,\ldots,m\}$ be defined by $p_i(t,x(t)) = \pi_i(x)(t)$, where $V$ is an open subset of $I\!R^n$, $\pi = (\pi_1,\ldots,\pi_m)$ is a map from $X$ to $\Lambda^m_1(A)$ and suppose that $p_i$ is convex with respect to its argument on $V$ througout $A$. Then, exactly one of the following systems is consistent:*

*(i) there is $x \in X$ such that $p_i(t,x(t)) < 0$ a.e. $t \in A$, $i \in I$,*

*(ii) there is a nonzero m-vector function $u \in L^m_\infty(A)$, $u_i(t) \geq 0$ a.e. $t \in A$, $i \in I$, such that*

$$\int_0^T \sum_{i \in I} u_i(t) p_i(t,x(t)) dt \geq 0$$

*for all $x \in X$.*

*Proof.* The proof of Theorem 2.7 follows in similar fashion as that of Theorem 3.2 in [18], replacing $[0,T]$ by $A$. .

# 3 Geometric Caracterization of a Minimum

We recall problem (CNP) from the introduction:

$$\left.\begin{array}{c} \text{Minimize } \ \phi(x) = \int_0^T f(t,x(t))dt, \\ \\ \text{subject to } \ g_i(t,x(t)) \le 0 \ \text{ a.e. } \ t \in [0,T], \\ i \in I = \{1,\ldots,m\}, \quad x \in X. \end{array}\right\} \quad (CNP)$$

Let $I\!\!F$ be the set of all feasible solutions to problem (CNP) (we suppose nonempty), i.e.,

$$I\!\!F = \{x \in X : g_i(t,x(t)) \le 0 \text{ a.e. } t \in [0,T], \ i \in I\}.$$

Let $V$ be an open subset of $I\!\!R^n$ containing the set

$$\{x(t) \in I\!\!R^n : x \in I\!\!F, \ t \in [0,T]\}.$$

$f$ and $g_i$, $i \in I$, are real functions defined in $[0,T] \times V$. The function $t \to f(t,x(t))$ is assumed to be Lebesgue measurable and integrable for $x \in X$.

For all $\overline{x} \in I\!\!F$, and $i \in I$, let $A_i(\overline{x})$ denote the set

$$\{t \in [0,T] : g_i(t,\overline{x}(t)) = 0\}.$$

Hereon, we assume that, given $a \in V$, there exist an $\epsilon > 0$ and a positive number $k$ such that $\forall t \in [0,T]$, and $\forall x_1, \ x_2 \in a + \epsilon B$ ($B$ denotes the unit ball of $I\!\!R^n$) we have

$$|f(t,x_1) - f(t,x_2) \le k\|x_1 - x_2\|.$$

Similar hypothesis are assumed for $g_i$, $i \in I$. Thence, $f(t,\cdot)$ and $g_i(t,\cdot)$, $i \in I$, are Lipschitz near every $\overline{x} \in V$, throughout $[0,T]$.

We suppose, the Lipschitz constant is the same for all functions involved.

Now, assume $\overline{x} \in X$ and $h \in L^n_\infty[0,T]$ are given. So, the Clarke generalized directional derivatives

$$g_i^0(t,\overline{x}(t);h(t)) := \gamma_i^0(\overline{x};h)(t) := \limsup_{\substack{y \to x \\ \lambda \to 0^+}} \frac{\gamma_i(y+\lambda h)(t) - \gamma_i(y)(t)}{\lambda},$$

and

$$f^0(t, \overline{x}(t); h(t)) := \Gamma^0(\overline{x}; h)(t) := \limsup_{\substack{y \to x \\ \lambda \to 0^+}} \frac{\Gamma(y + \lambda h)(t) - \Gamma(y)(t)}{\lambda}$$

are finite a.e. $t \in [0, T]$ and $\phi^0(\overline{x}; h)$ is finite.

For ease of reading and presentation, in the rest of this work, we use the notations $f^0(t, \overline{x}(t); h(t))$ and $g_i^0(t, \overline{x}(t); h(t))$ rather than $\Gamma^0(\overline{x}; h)(t)$ and $\gamma_i(\overline{x}; h)(t)$. It follows easily from the assumptions that

$$t \to f^0(t, \overline{x}(t)); h(t)),$$
$$t \to g_i^0(t, \overline{x}(t)); h(t)), \ i \in I,$$

are Lebesgue mesurable and integrable for all $\overline{x} \in X,$ and $h \in L_\infty^n[0, T]$.

Consider the following cones in $L_\infty^n[0, T]$ with zero vertices:

$$\mathcal{K}(\phi; \overline{x}) = \{h \in L_\infty^n[0, T] : \phi^0(\overline{x}; h) < 0\},$$

$$\mathcal{K}(g_i; \overline{x}) = \{h \in L_\infty^n[0, T] : g_i^0(t, \overline{x}(t); h(t)) < 0 \text{ a.e. } t \in A_i(\bar{x})\}, \ i \in I.$$

We are now in position to provide a geometric caracterization of a local minimum for problem (CNP).

**Theorem 3.1** *Let $\overline{x}$ be an optimal solution of problem (CNP). Then*

$$\bigcap_{i \in I} \mathcal{K}(g_i; \overline{x}) \cap \mathcal{K}(\phi; \overline{x}) = \emptyset. \tag{1}$$

*Proof.* Suppose the intersection of cones (1) is nonempty and take $h \in L_\infty^n[0, T]$ in this intersection. It follows from limsup properties and contituity of the functions involved that there is a real number $\delta > 0$ such that, $\forall \ 0 < \lambda < \delta, \ \overline{x} + \lambda h \in X,$

$$g_i(t, \overline{x}(t) + \lambda h(t)) \leq 0, \text{ a.e. } t \in [0, T], \ i \in I,$$
$$\phi(\overline{x} + \lambda h) < \phi(\overline{x}).$$

But, that means $\overline{x} + \lambda h, \ \forall 0 < \lambda < \delta$, is a feasible solution for (CNP) with objective value better than $\overline{x}$. This contradicts the optimality of $\overline{x}$ for problem (CNP). Therefore, the intersection (1) is empty. .

# 4 The Fritz-John Otimality Conditions

In this section we derive a new continuous-time analogue of the Fritz-John necessary optimality conditions translating the geometric optimality conditions into algebraic statements. This is made possible through the use of the Generalized Gordan Theorem. We also point out that the new Fritz-John necessary conditions generalizes the smooth case treated by Zalmai ([16], Thm.3.3).

**Theorem 4.1** *Let $\overline{x} \in I\!\!F$. Let $f(t,\cdot)$ and $g(t,\cdot)$ be Lipschitz near $\overline{x}(t)$. If $\overline{x}$ is a local optimal solution of (CNP), then there exist $\overline{u}_0 \in I\!\!R$, $\overline{u}_i \in L^m_\infty[0,T]$, $i \in I$, such that*

$$0 \in \int_0^T \{\overline{u}_0 \partial_x f(t,\overline{x}(t)) + \sum_{i\in I} \overline{u}_i(t)\partial_x g_i(t,\overline{x}(t))\}dt; \tag{2}$$

$$\overline{u}_0 \geq 0,\ \overline{u}(t) \geq 0 \ \textit{a.e.}\ t \in [0,T]; \tag{3}$$

$$(\overline{u}_0, \overline{u}(t)) = (\overline{u}_0, \overline{u}_1(t), \cdots, \overline{u}_m(t)) \not\equiv 0 \ \textit{a.e}\ t \in [0,T]; \tag{4}$$

$$\overline{u}_i(t) g_i(t,\overline{x}(t)) = 0 \ \textit{a.e.}\ t \in [0,T],\ i \in I. \tag{5}$$

*Proof.* We shall proceed under the *Interim Hypothesis*: (CNP) has only one contraint

$$g(t,x(t)) \leq 0 \text{ a.e. in } [0,T].$$

The removal of this interim hypothesis will be done at the end of the proof.

We denote

$$A(\bar{x}) = \{t \in [0,T] : g(t,\bar{x}(t)) = 0\};$$
$$K(g,\bar{x}) = \{h \in L^n_\infty[0,T] : g^0(t,\bar{x}(t);h(t) < 0, \quad t \in A(\bar{x})\}.$$

**Lemma 4.2** *Let $\overline{x} \in I\!\!F$. Let $f(t,\cdot)$, $g(t,\cdot)$ be Lipschitz near $\overline{x}(t)$ throughout $[0,T]$. If $\overline{x}$ is a local optimal solution of (CNP), then there exist $\bar{u}_0 \in I\!\!R$, $\bar{u} \in L^\infty[0,T]$, such that*

$$0 \leq \int_0^T \{\bar{u}_0 f^0(t,\overline{x}(t);h(t)) + \bar{u}(t) g^0(t,\overline{x}(t);h(t))\}dt, \quad \forall h \in L^n_\infty[0,T]; \tag{6}$$

$$\bar{u}_0 \geq 0,\ \bar{u}(t) \geq 0 \ \textit{a.e.}\ t \in [0,T]; \tag{7}$$

$$(\bar{u}_0, \bar{u}(t)) \not\equiv 0 \ \textit{a.e.}\ t \in [0,1]; \tag{8}$$

$$\bar{u}(t) g(t,\overline{x}(t)) = 0 \ \textit{a.e.}\ t \in [0,T],\ i \in I. \tag{9}$$

*Proof.* If $\overline{x}$ be a local optimal solution to problem (CNP), then by Theorem 3.1

$$\mathcal{K}(g;\overline{x}) \cap \mathcal{K}(\phi;\overline{x}) = \emptyset.$$

Hence, there is no $h \in L^n_\infty[0,T]$ such that

$$\phi^0(\overline{x};h) < 0,$$
$$g^0(t,\overline{x}(t);h(t)) < 0, \text{ a.e. } t \in A(\bar{x}).$$

We can conclude, by making use of Theorem 2.4, that there are $u_0,\ u \in L^\infty[0,T]$, with $u_0(t) \geq 0,\ u(t) \geq 0$ a.e. in $[0,T]$, not all identically zero such that

$$0 \ \leq \ \int_{A(\bar{x})} \{u_0(t)\phi^0(\overline{x};h) + u(t)g^0(t,\overline{x}(t);h(t))\}dt \quad \forall h \in L^n_\infty[0,T].$$

Setting $\bar{u}_0 = \int_{A(\bar{x})} u_0(t)dt$ and $\bar{u}(t) = u(t)$ if $t \in A(\bar{x})$ and $\bar{u}(t) = 0$ otherwise, we obtain

$$\begin{aligned} 0 \ &\leq \ \bar{u}_0\phi^0(\overline{x};h) + \int_0^T \bar{u}(t)g^0(t,\overline{x}(t);h(t))dt \\ &\leq \ \int_0^T \{u_0 f^0(t,\bar{x}(t);h(t)) + \bar{u}(t)g^0(t,\overline{x}(t);h(t))\}dt \end{aligned}$$

for all $h \in L^n_\infty[0,T]$. (Fatou's lemma is used in the last inequality.) Thus (6) is proved. The remaining assertions of the Lemma 4.2 follow immediately. .

Let $\bar{x}$ be an optimal solution to (CNP). It follows from Lemma 4.2 that there exist $\overline{u}_0 \in I\!R$ and $\overline{u} \in L^\infty[0,T]$, satisfying (6)-(9).

It remains to prove assertion (2) to conclude the proof of the theorem. Statement (6) can be rewritten as follows:

$$\begin{aligned} 0 \ &\leq \ \int_0^T \{\bar{u}_0\sigma_{\partial_x f(t,\overline{x}(t))}(h(t)) + \bar{u}(t)\sigma_{\partial_x g(t,\overline{x}(t))}(h(t))\}dt, \\ &= \ \int_0^T [\sigma_{\{\bar{u}_0\partial_x f(t,\overline{x}(t))+\bar{u}(t)\partial_x g(t,\overline{x}(t))\}}(h(t))]dt, \end{aligned}$$

$\forall h \in L^n_\infty[0,T]$ (The equality above follows from Proposition 2.5). Since the above inequality holds for all $h \in L^n_\infty[0,T]$, it holds, in particular, for constant functions $h(t) = v \in I\!R^n,\ \forall t \in [0,T]$.

It can be easily verified that the multifunction

$$t \to \bar{u}_0\partial_x f(t,\overline{x}(t)) + \bar{u}(t)\partial_x g(t,\overline{x}(t))$$

is integrably bounded and takes values compact subsets of $I\!R^n$. By Theorem$\tilde{2}$.6 we have

$$\begin{aligned} 0 \ &\leq \ \int_0^T [\sigma_{\{\bar{u}_0\partial_x f(t,\overline{x}(t))+\bar{u}(t)\partial_x g(t,\overline{x}(t))\}}(v)]dt \\ &= \ \sigma_{\int_0^T [\bar{u}_0\partial_x f(t,\overline{x}(t))+\bar{u}(t)\partial_x g(t,\overline{x}(t))]dt}(v). \end{aligned}$$

But, by Proposition 2.4, this is equivalent to

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \overline{x}(t)) + \bar{u}(t) \partial_x g(t, \overline{x}(t))] dt,$$

which finishes the proof of the theorem under the interim hypothesis.

**Removal of the Interim Hypothesis.** Suppose (CNP) has $m$ constraints $g_i(t, x(t)) \leq 0$ a.e. in $[0, T]$, and $\bar{x}$ as a local optimal solution. Reduce the $m$ constraints of (CNP) to just one by defining $g(t, x(t)) = \max_{1 \leq m} g_i(t, x(t))$ a.e. in $[0, T]$. The point $\bar{x}$ is also an optimal solution of the modified problem. Let $I(t, x) := \{i \in I : g_i(t, x(t)) = g(t, x(t))$. From what has been proved under the interim hypothesis there exist $\overline{u}_0 \in I\!R,\ u \in L^\infty[0, T]$, satisfying

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \overline{x}(t)) + u(t) \partial_x g(t, \overline{x}(t))] dt \tag{10}$$

and (7)-(9). It can be deduced from (10) and the definition of integration of multi-functions that there exists a measurable function $e(t) \in \partial_x g(t, \overline{x}(t))$ a.e. such that

$$0 \in \int_0^T [\bar{u}_0 \partial_x f(t, \overline{x}(t)) + u(t) e(t)] dt. \tag{11}$$

We have the following lemma.

**Lemma 4.3** *There exists $v \in L_\infty^m[0, T],\ v \geq 0$ a.e., satisfying*

1. *$v_i(t) = 0$ whenever $g_i(t, \bar{x}(t)) \neq g(t, \bar{x}(t)),\ i = 1, \ldots, m$;*
2. *$\sum\limits_{i=1}^{m} v_i(t) = 1$ a.e.;*
3. *$e(t) \subset \sum\limits_{i=1}^{m} v_i(t) \partial_x g_i(t, \overline{x}(t))$ a.e.*

*Proof.* For each $t$ where $\partial_x g(t, \overline{x}(t))$ is well defined it follows from [23] that

$$\partial_x g(t, \overline{x}(t)) \subset \mathrm{co}\{\partial_x g_i(t, \overline{x}(t)) : i \in I(t, \bar{x})\}.$$

Since $e(t) \in \partial_x g(t, \overline{x}(t))$ a.e. we obtain

$$e(t) \in \mathrm{co}\{\partial_x g_i(t, \overline{x}(t)) : i \in I(t, \bar{x})\}.$$

Define

$$V(t) \quad := \quad \{(v_1, \dots, v_m) \in I\!\!R^m : \sum_i^m v_i = 1,\ v_i \geq 0,$$
$$v = 0 \text{ if } g_i(t, \bar{x}(t)) < g(t, \bar{x}(t)),$$
$$e(t) \in \sum_{i=1}^m v_i \partial_x g_i(t, \overline{x}(t))\}$$

The set $V(t)$ is obviously nonmepty and closed a.e., and $V$ is a mesurable set-valued function defined a.e. on $[0, T]$. It follows from standard measurable selection theorems (see e.g., [23]) that we can choose measurable functions $v_1(t), \dots, v_m(t)$ defined on $[0, T]$ such that $(v_1(t), \dots, v_m(t)) \in V(t)$ a.e. in $[0, T]$. The proof of the lemma follows immediately. .

Now defing $\bar{u}_i(t) := u(t) v_i(t)$ it follows easily from Lemma 4.3 and (11) that assertions (2)-(5) of Theorem 4.1 are valid. .

In Theorem 4.1 if $f(t, \cdot)$ and $g(t, \cdot)$ are Clarke regular, then the condition (2) can be changed by

$$0 \in \partial_x L(\overline{x}, u_0, u),$$

where,

$$L(x, u_0, u) := \int_0^T \{u_0 f(t, x(t)) + \sum_{j=1}^m u_i(t) g_i(t)\} dt.$$

# 5 Karush-Kuhn-Tucker Optimality Conditions

In the necessary conditions, proved in the previous section, there is no garantee that the Lagrange multiplier associated with the objective function will be nonzero. It is usual to assume some kind of regularity condition on the restrictions of problem to make sure that multiplier is in fact nonzero. These regularity conditions are usually refered to as constraint qualifications. We assume the following natural constraint qualification:

$$\bigcap_{i \in I} \mathcal{K}(g_i, \overline{x}) \neq \emptyset. \tag{12}$$

We now state and prove the following Karush-Kuhn-Tucker type theorem.

**Theorem 5.1** *(Karush-Kuhn-Tucker) Let $\overline{x} \in I\!\!F$ and suppose the constraint qualification (12) is satisfied for functions $g_i$, $i \in I$. If $\overline{x}$ is a local minimum of problem*

*(CNP), then there exist $\tilde{u}_i \in L^\infty[0,T],\ i \in I$, such that*

$$0 \in \textstyle\int_0^T [\partial_x f(t,\overline{x}(t)) + \sum_{i\in I} \tilde{u}_i(t)\partial_x g_i(t,\overline{x}(t))\} dt; \tag{13}$$

$$\tilde{u}(t) \geq 0 \ a.e.\ t \in [0,T],\ i \in I; \tag{14}$$

$$\tilde{u}_i(t) g_i(t,\overline{x}(t)) = 0 \ a.e.\ t \in [0,T],\ i \in I. \tag{15}$$

*Proof.* We first prove Theorem 5.1 under the *interim hypothesis*: (CNP) has only one constraint $g(t,x(t)) \leq 0$ a.e. in $[0,T]$.

If $\overline{x}$ is a local optimal solution to problem (CNP), then by Lemma 4.2, there exist $\bar{u}_0 \in I\!R,\ \bar{u}(t) \in L^\infty[0,T]$, such that (6)-(9) hold true. If $\bar{u}_0 = 0$ then (6) would reduce to

$$0 \leq \int_0^T u(t) g^0(t,\overline{x}(t);h(t))dt, \quad \forall h \in L^n_\infty[0,T].$$

Hence, by the Generalized Gordan's Lemma, there is no $h \in X$ such that

$$g^0(t,\overline{x}(t);h(t)) < 0 \text{ a.e. in } [0,T],$$

contradicting the constraint qualification (12). So, $\bar{u}_0 \neq 0$. Set

$$\tilde{u}_0 = \frac{\bar{u}_0}{\bar{u}_0}, \quad \tilde{u}(t) = \frac{\bar{u}(t)}{\bar{u}_0}.$$

and the theorem follows from inequality

$$0 \leq \int_0^T \{f^0(t,\overline{x}(t);h(t)) + \tilde{u}(t) g^0(t,\overline{x}(t);h(t))\} dt, \quad \forall h \in L^n_\infty[0,T],$$

by using similar arguments to those in the proof of condition (2) of Theorem4.1.

**Romoval of the Interim Hypothesis.** Let $g(t,x(t)) := max\{g_i(t,x(t)) :\ i \in I\}$. We need the following technical result.

**Lemma 5.2** *The constraint qualification (10) for $m$ constraints implies $K(g,\bar{x}) \neq \emptyset$.*

*Proof.* It follows from the measurable selection theorem [23], by using standard arguments, that there exists $\xi \in L^n_\infty[0,T]$ such that $\xi(t) \in \partial_x g(t,\bar{x}(t))$ and

$$g^0(t,\bar{x}(t);h(t)) = \sum_{i=1}^n \xi_i(t)h_i(t) \text{ a.e. in } [0,T], \quad \forall h \in L^n_\infty[0,T].$$

Let $h \in L^n_\infty[0,T]$ be given but arbitrary. An application of Lemma 4.3 and Proposition$\tilde{2}$.4 implies that there exists an essentially bounded function $u(t) \in V(t)$ *a.e.* ($V$ as defined in the proof of Lemma 4.3) such that

$$\xi(t) \in \sum_{i\in I} u_i(t)\partial_x g_i(t,\bar{x}(t));$$
$$g^0(t,\bar{x}(t);h(t)) \le \sigma_{\sum\limits_{i\in I} u_i(t)\partial_x g_i(t,\bar{x}(t))}(h(t))$$

a.e. in $[0,T]$. It follows from the above inequality and Proposition 2.5 that

$$\begin{aligned} g^0(t,\bar{x}(t);h(t)) &\le \sum_{i\in I} u_i(t)\sigma_{\partial_x g_i(t,\bar{x}(t))}(h(t)) \quad \text{a.e. in } [0,T]; \\ &= \sum_{i\in I} u_i(t) g_i^0(t,\bar{x}(t);h(t)) \quad \text{a.e. in } [0,T]. \end{aligned}$$

Therefore,

$$\bigcap_{i\in I} K(g_i,\bar{x}) \ne \emptyset \Rightarrow K(g,\bar{x}) \ne \emptyset.$$

Now, if $\bar{x}$ is optimal for (CNP) with $m$ constraints then it is also optimal for (CNP) with the constraint $g(t,x(t)) \le 0$ a.e. $\in [0,T]$. It follows from Lemma 5.2 and the theorem in question as proved so far, that there exists $u \in L^\infty[0,T],\ u \ge 0$ such that

(i) $u(t)g(t,\bar{x}(t)) = 0$ a.e. in $[0,T]$;

(ii) $0 \in \int_0^T \partial_x f(t,\bar{x}(t))dt + \int_0^T u(t)\partial_x g(t,\bar{x}(t))dt.$

Arguments similar to those in the proof of Theorem 4.2 yields the desired result. .

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