Domes, Umbrellas and Tents: a Scenic Tour Guided by *Mathematica*

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Abstract

This work focuses on the mathematical aspects behind objects obtained by the intersection of general cylinders. Emphasis is given on visual representation, development of parameterization skills and computation of volume and surface areas. Extensions of Archimedian results are presented, as well as an optimization problem related to the construction of a dome tent. A possible scenario for exploiting these issues is a project-oriented Calculus course, encompassing the use of a computer algebraic system.

Introduction

Do the following objects belong together?

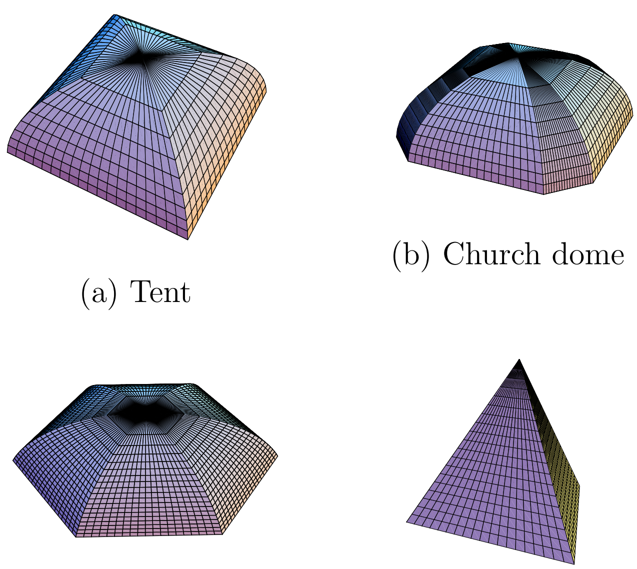
We will argue that they do bear a certain kinship, sharing the common gene of cylindrical intersection. In fact, by the end of this paper we hope the reader will be able to recognize several other members of this family in the world around him/her.

The intersection of cylinders is a construction present in everyday life that offers a wealth of interesting aspects from a mathematical point of view. The construction of visual representations of these objects presents good opportunities for exercising parameterization skills. The volumes enclosed by these surfaces suggest problems involving simple, double or triple integrals. Surface areas may be calculated by surface integrals or through single variable integral, exploiting the planar nature of cylindrical surfaces.

Lo Bello [4] looks into the shapes of famous domes that come from the upper half of the intersection of an even number of equal right circular cylinders, calculating their volumes and centroids. We follow this trend, focusing our attention on somewhat more general shapes for the cylinders. The nature of mathematical issues considered is, nevertheless, different. We explore parameterization issues, actual construction schemes, extensions of Archimedian results and an optimization problem related to the construction of a dome tent.

n Cylinder Intersections

Figure 1 shows the surface of various solids obtained via intersection of cylinders. These examples show the versatility of this simple construction: it can lead to tents, domes, pyramids, umbrellas, etc., depending on the curves that generates the cylinders, the number of cylinders and how they are positioned with respect to each other. The popular dome tent of Figure 1(a) is the surface of the intersection of two half circular (elliptical) cylinders, whose axes of symmetry are orthogonal. Various cathedral domes resemble the one in Figure 1(b), formed by the intersection of four elliptical half cylinders of different sizes. Beach umbrellas, like the one depicted in Figure 1(c), also come from the intersection of three identical cylinders. The sides of the triangular based pyramid of Figure 1(d) are embedded in the humblest kind of cylinder: it is generated by a line.



(a) Tent

(b) Church dome

(c) Beach umbrella

(d) Pyramid

Figure 1: Intersection of half cylinders.

The key to construct the pictures in Figure 1 is to identify the different types of slices that make up the surface and locate their projections (isosceles triangles) on the plane. Thus the dome tent and the umbrella are made up of identical slices. The first has four and the second, six slices. The cathedral dome has two types of slices, thin and thick ones, and the pyramid has three equal slices. Once the parameterization of a slice of a certain type is determined, the others can be obtained by the appropriate rotation. Of course there are several possible parameterizations, each producing a different visual effect. Figures 1(a) and 2 show two possibilities: on the first one it is possible to see contour curves that are equally spaced along the vertical axis, whereas on the latter the projection of the contour curves are equally spaced segments perpendicular to the x axis.

But the key to the actual construction of tents and umbrellas is the establishment of the shapes the slices assume when they are laid on the plane. Consider, for instance, the surface of the solid produced by the intersection of two elliptical half cylinders, with orthogonal axes of symmetry, the x and y axes, say. Let a and b be the semi-axes of the ellipses that generates the cylinders, see Figure 2. We will determine the region covered by the shown slice (with nonnegative x- and z-coordinates) when it is flattened (without stretching) onto the xy plane. In fact, for symmetry reasons, it is sufficient to describe half this region, e.g., its intersection with the positive xy quadrant. Points on the slice shown in Figure 2 are of the form $(a\cos t, y, b\sin t)$, and points on a neighboring slice are of the form $(x, a\cos t, b\sin t)$, for $0 \le t \le \pi/2$. The intersection of these two surfaces is the curve $(a\cos t, a\cos t, b\sin t)$, for $0 \le t \le \pi/2$, on the $x = y$ plane. Let $A = (a\cos t, 0, b\sin t)$ and $B = (a\cos t, a\cos t, b\sin t)$ be points on the same height on the slice. Then $B - A = (0, a\cos t, 0)$. When the slice is opened, they are mapped onto $P_A = (a + \ell(t), 0, 0)$ and $P_B = (a + \ell(t), a\cos t, 0)$, respectively, shown on Figure 2, where $\ell(t)$ is the arc length of the part of the ellipse from $(a, 0, 0)$ to A, given by $\ell(t) = \int_0^t \sqrt{(a\sin\theta)^2 + (b\cos\theta)^2}\, d\theta$.

The same procedure can be applied to more general objects. The formula above is easily extended to the case of a surface obtained by the intersection of n cylinders. Consider the right cylinder generated by the trace curve $(f(t), 0, g(t))$, for $0 \le t \le \underline{t}$, that is, the cylinder formed as a straight line parallel to the y axis travels along the trace curve. We assume that this trace curve lies on the half space $x \ge 0$ and is symmetric with respect to the z axis. The y axis is what we call the *spine* of the cylinder. Suppose n copies of the solid delimited by this cylinder and the xy plane are placed so that their spines are distributed at equal angles around the origin.

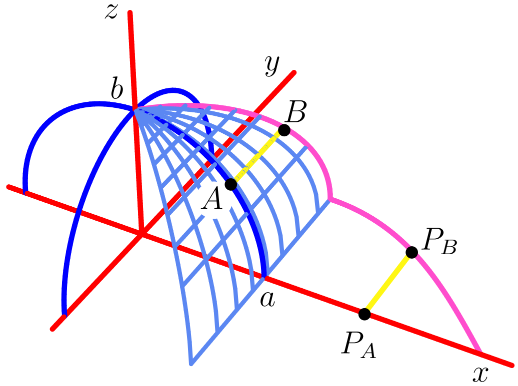


Figure 2: Flattening of a dome tent slice.

The upper surface of the intersection solid has $2n$ slices. The projections of these slices are isosceles triangles sharing a common vertex, the origin. Consider the slice whose projection is the triangle with vertices $(0, 0, 0)$, $(f(0), f(0)\tan(\pi/2n), 0)$ and $(f(0), -f(0)\tan(\pi/2n), 0)$. The intersection curve $(f(t), f(t)\tan(\pi/2n), g(t))$, border of such slice, is mapped onto the curve $(f(0) + \ell(t), f(t)\tan(\pi/2n), 0)$, where $\ell(t) = \int_0^t \sqrt{f'(\theta)^2 + g'(\theta)^2}\, d\theta$, the length of the trace curve from $(f(0), 0, 0)$ to $(f(t), 0, g(t))$. This procedure was applied to produce Figure 3 below, which depicts a possible pattern for cutting the six slices of the beach umbrella of Figure 1(c).

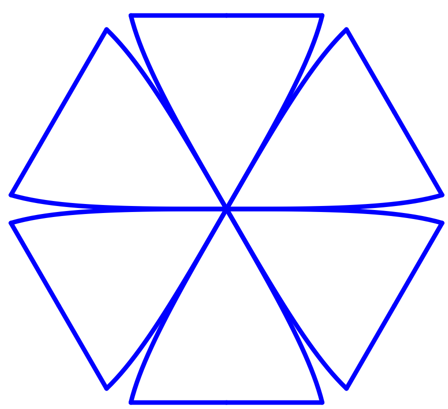


Figure 3: Beach umbrella pattern.

An Archimedian Ratio

Archimedes requested that a picture depicting a sphere circumscribed in a right circular cylinder of same height as its diameter be carved on his tombstone, cf. [1], see Figure 4. This picture spells out the relationship he established, namely that the ratio of the volume (resp. surface area) of the sphere to the volume (resp., surface area) of the cylinder is 2:3. Inspired by [3], we extend this result to a similar relationship involving objects obtained by the intersection of several cylinders and prisms that circumscribe these objects.

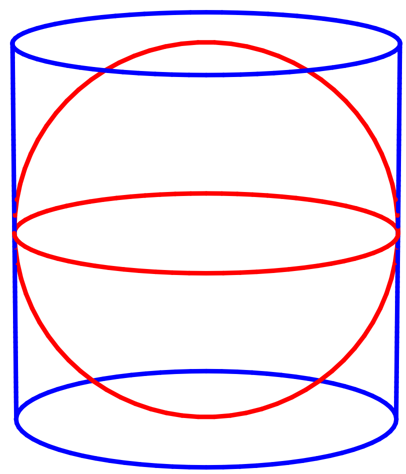


Figure 4: Picture carved on Archimedes tombstone.

Consider the object formed by the intersection of n right circular cylinders of radius r, whose axes of symmetry are distributed at equal angles around a point on a common plane. The intersection of this object with this plane is a $2n$ sided regular polygon. How could we obtain similar shaped solids such that the corresponding polygon has an odd number of sides? Instead of obtaining the solid by cylinder intersections, we'll construct it by putting blocks together. Starting with an n sided regular polygon with apothem r, put the origin O of our coordinate system on the center of the polygon. The polygon can be decomposed into n isosceles triangles, with one vertex on the origin and the other two on adjacent vertices of the polygon. The part of the object whose projection is a given isosceles triangle, say OAB, is the circular right cylinder with axis of symmetry parallel to AB and containing O, and radius r. This is one of the blocks that make up our solid, others are obtained in a similar fashion or simply by rotating the first block, around the normal to the polygon, by angles of $2\pi j/n$, for $j = 1, \ldots, n-1$. Notice that the solid just described encompasses the one obtained by cylinder intersections. When n equals four, the upper half part of the solid is a dome, like the one on Figure 1(a). We call an n sided object, the union of n blocks, an *n-double dome*.

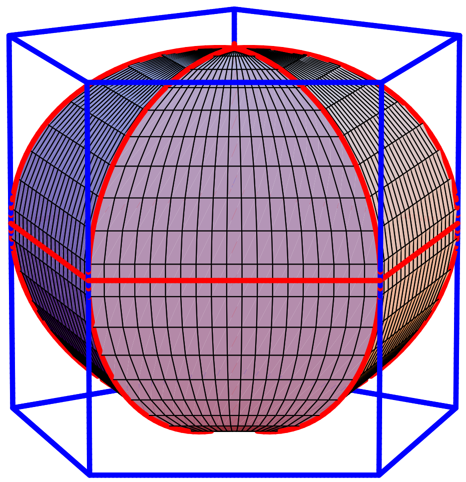


Figure 5: Five-double dome and circumscribing prism.

The n-double dome is circumscribed by an n right prism, whose basis is a translation of the n sided regular polygon with apothem r, see Figure 5. The volumes of the n double dome and the circumscribing prism are given by

$$\text{VolDDome} = \frac{4}{3} n r^3 \tan\left(\frac{\pi}{n}\right) \quad \text{and}$$

$$\text{VolPri} = 2 n r^3 \tan\left(\frac{\pi}{n}\right),$$

and the surface areas by

$$\text{AreaDDome} = 4 n r^2 \tan\left(\frac{\pi}{n}\right) \quad \text{and}$$

$$\text{AreaPri} = 6 n r^2 \tan\left(\frac{\pi}{n}\right).$$

Thus the ratio 2:3 between volumes (resp., surface areas) is also valid for these objects.

Moreover, the n double dome tends to a sphere of radius r as n tends to infinity, whereas the corresponding prism tends to this sphere's circumscribing cylinder. These limit processes furnish alternative methods for computing the sphere's volume and surface area that generalize the calculus of the area of the circle as a limit of the areas of circumscribed polygons.

Optimal Tent Problem

The Calculus textbook by Edwards and Penney [2] presents exercises on the construction of an optimal tent that provided inspiration for our model problem. In exercises 53–55 of section 3.6, p. 162, the student is asked to choose the optimal (maximum volume) design for a square base pyramidal tent, that is to be cut from a square piece of canvas.

We consider a similar optimization problem, with a dome tent instead. The determination of the pattern that should be cut from the canvas in order to construct the tent was indicated in Figure 2. Summarizing, the problem is to choose the lengths of the semi-axes a and b of the ellipses that generate the dome tent that maximize its volume, assuming that the tent pattern has to be cut from a 20 foot wide square piece of canvas. It is quite obvious that the best arrangement for such a pattern is achieved when the square that will constitute the floor of the tent has its sides aligned with the diagonals of the piece of canvas and that the pattern should extend to the boundary of the material, as illustrated in Figure 6.

The optimization problem may then be formally stated as

$$\begin{array}{lrcl} \text{maximize} & \operatorname{vol}(a,b) & & \\ \text{subject to} & 2a+\ell(a,b) & = & 20\sqrt{2} \\ & a,b & \geq & 0 \end{array}$$

where $\operatorname{vol}(a,b) = 8a^2b/3$ is the volume of the tent, $2a$ is the width of the tent's floor, b is the tent's height and $\ell(a,b)$ is half the arc length of the ellipse with semi-axes a and b.

Since an elliptical integral appears in the constraint, *Mathematica*'s `Solve` command

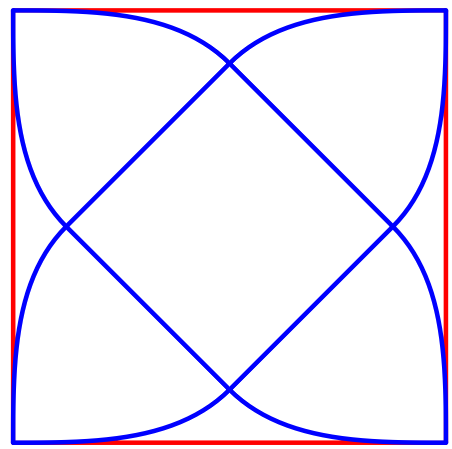


Figure 6: Tent pattern.

fails to find a solution for the Lagrangian system associated with the optimization problem. A numerical method for solving the system is needed, and a good initial estimate comes in handy. This estimate may be graphically obtained by drawing together the level curves of $\mathrm{vol}(a,b)$ and the graph of the equation $h(a,b) = 2a + \ell(a,b) = 20\sqrt{2}$. Figure 7 shows the result of a zoom around the candidate solution in the mentioned drawing. The optimal point is located on the dot, the graph of the equation is the lighter curve passing through the dot and the level curve associated with the optimal value is the darker one. This takes care of the a, b components of the system, but we also need an estimate for the Lagrange multiplier. Using the graphical estimation $(\overline{a}, \overline{b})$ of the optimal point as initial guesses for a and b, and letting, for instance,

$$\overline{\lambda} = \left. \frac{\frac{\partial h}{\partial a} + \frac{\partial h}{\partial b}}{\frac{\partial \mathrm{vol}}{\partial a} + \frac{\partial \mathrm{vol}}{\partial b}} \right|_{(a,b)=(\overline{a},\overline{b})}$$

be an initial guess for the Lagrange multiplier.

The actual figures for the given problem are $a^* = 5.337452773754006$ and $b^* = 5.866765258813976$. Investigation of second order conditions is deemed unnecessary due to the information given by the level curve map. The context may inspire several other problems such as maximizing volume subject to fixed height of tent and available canvas, maximizing volume subject to fixed surface area (supposing canvas not used for the tent may be applied to the manufacture of other products—purses?), minimize surface area subject to fixed volume, minimize total cost subject

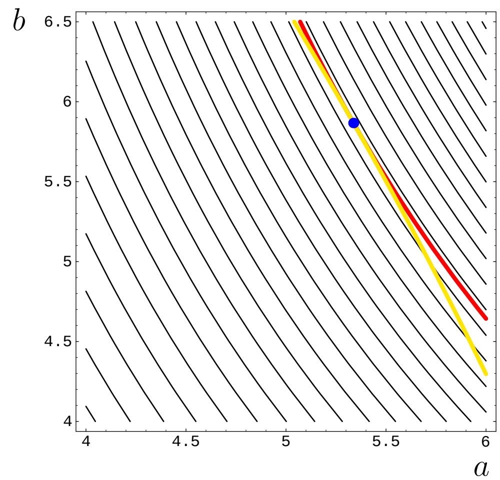


Figure 7: Level curves and equation graph.

to fixed volume (in this case we may assign different materials and thus costs for the floor and sides), etc.

Final Remarks

In this work we have focused on the mathematical aspects behind objects obtained by the intersection of general cylinders. Emphasis is given on visual representation, development of parameterization skills and computation of volume and surface areas. A possible scenario for exploiting these issues is a project-oriented Calculus course.

The problem described in the last section has all the qualities we look for when selecting optimization problems for a Calculus course from the enlarged realm of "solvable" problems made possible by the availability of a Computer Algebraic System such as *Mathematica*. It is a down to earth problem, involving an object most people have seen and can relate to, the objective is quite straightforward, it is not overly simplified, so as to render its solution useless, the mathematics involved is not trivial but also not beyond the scope of a regular course, assuming, of course, students have access to a package that may do the dirty work (calculating, plotting and solving systems involving elliptical integrals) for them. In fact, this kind of problem, as opposed to the usual pen-and-paper textbook problem, is an absolute requirement, if students are to feel that the investment in learning how to use such a sophisticated software is a sound one.

References

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[4] Lo Bello, A.J. The volumes and centroids of some famous domes. *Mathematics Magazine*, 61:3, pp. 164–170, 1988.