The homotopy type of Lie semigroups in semi-simple Lie groups

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Abstract

Let G be a noncompact semi-simple Lie group and $S \subset G$ a Lie semigroup with nonempty interior. We study the homotopy groups $\pi_n(S)$, $n \geq 1$, of S. Generalizing a well known fact for G, it is proved that for a certain compact and connected subgroup $K(S) \subset G$, $\pi_n(S)$ is isomorphic to $\pi_n(K(S))$. Furthermore, there exists a coset $K(S)z$ contained in $\mathrm{int}S$ which is a deformation retract of S.

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1 Introduction

Let G be a connected, noncompact semi-simple Lie group. It is well known that through a Cartan decomposition $G = KS$ or an Iwasawa decomposition $G = KAN$, the topology of G is reduced to that of K. In fact, S and AN are diffeomorphic to Euclidean spaces, implying that the projections onto K are homotopy equivalences. In particular, the homotopy groups of G are isomorphic to those of K.

The purpose of this paper is to get this kind of reduction to Lie semigroups with nonempty interior in G, when G has finite center.

For such a semigroup S there are no good decompositions available providing a natural compact space which is a deformation retract of S. Instead we get the topology of S from its action in compact homogeneous spaces of G. Precisely, let AN be the solvable Iwasawa component of G and form the coset space G/AN, which is diffeomorphic to the compact Iwasawa component K. The semigroup S acts on G/AN by the restriction of the G-action. Typically this action of S is not transitive. Actually, if $S \neq G$ there are proper compact subsets of G/AN invariant under S, the so-called invariant control sets. Let C be an invariant control set of S in G/AN. The main result of this paper states that the homotopy groups of S are isomorphic to those of C.

Although in general it is not an easy task to obtain the invariant control sets of the semigroups, their possible homotopy types can be described (at least when S is generated by one-parameter semigroups) by using a previous classification of the semigroups with nonempty interior according

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to their action on the flag manifolds of G. This classification divides the semigroups into types, each type being labelled either by a conjugate class of parabolic subgroups, that is, a flag manifold of G, or by a parabolic subgroup of the Weyl group of G (see [18], [20]).

Roughly, the type of S is given by a parabolic subgroup, say $P(S)$, or rather by the corresponding flag manifold $G/P(S)$, in such a way that the geometry of the S-action on an arbitrary flag manifold is embodied in the action on $G/P(S)$. This is expressed, for instance, by the fact that the invariant control set of S in the maximal flag manifold G/MAN is given by $\pi^{-1}\left(C_{P(S)}\right)$ where π is the canonical projection onto $G/P(S)$, and $C_{P(S)}$ is the invariant control set in $G/P(S)$. In the special case when S is generated by one-parameter semigroups, $C_{P(S)}$ is contractible. So that by lifting $\pi^{-1}\left(C_{P(S)}\right)$ once again through the canonical fibration $G/AN \rightarrow G/MAN$, it follows that any invariant control set $C \subset G/AN$ is contractible to the connected component of $P(S)/AN$. This connected component is diffeomorphic to a compact subgroup $K(S)$ of K, namely the maximal compact subgroup of the semi-simple Levi component of $P(S)$.

In short, C can be continuously deformed into $K(S)$ implying that the homotopy type of S is the compact group $K(S) \subset K$. Therefore, the classification of Lie semigroups by the flag manifolds turns out to be the classification by their homotopy types.

Our proofs are not restricted to Lie semigroups, but they work for a semigroup S which contains a large Lie semigroup, in the sense of Definition 4.7 below. This slight generalization permits to work out the homotopy groups of some classical semigroups which are not infinitesimally generated, like e.g. the semigroup of positive matrices. Also, if we consider the interior of S instead of S, the conclusion of the results still hold, with the further advantage that in this case it is clear int$S$ is homotopic equivalent to $K(S)$, since these spaces are CW-complexes.

Concerning the structure of the paper, in the next section we establish notations and prove some basic facts about semigroups, their action on flag manifolds, homotopy groups, etc., which are used in the proof of the main results. In this section we also prove that the orbits of a Lie semigroup in the Riemannian symmetric space associated to the semi-simple group are contractible. This fact although not used afterwards is interesting in itself. In Section 3 the topology of the invariant control sets in G/AN is discussed. It is shown that it reduces to the topology of a compact subgroup. The statements and proofs of the main results are contained in Section 4. Direct consequences of them and further comments were included in Section 5. Finally, some examples are discussed in Section 6. First we consider the semigroup matrices in $\mathrm{Sl}(n, \mathbb{R})$ with positive entries, which is part of the class of compression semigroups of cones in $\mathbb{R}^n$. The conclusion is that the homotopy type of these semigroups is the compact group $\mathrm{SO}(n-1)$. Another class of semigroups whose homotopy groups can be computed are those contained in real rank one groups. In this case there is just one type of homotopy which is the connected component of the group M in the decomposition $P = MAN$ of a parabolic subgroup. The other examples are illustrative of our results, in the sense that what we get is already known from their structural properties. These are the semigroup of totally positive matrices that is contractible and the Ol'shanskiĭ semigroups, which have polar decompositions from where the homotopy type can be read off.

2 Preliminaries

The purpose of this section is to establish notations and background results to be used throughout the paper.

2.1 Flag manifolds and parabolic subgroups

Let G be any connected noncompact Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. The flag manifolds of G are labelled by subsets of the set of simple (restricted) roots of $\mathfrak{g}$. Precisely, choose an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let Π be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and Π^+ [respectively Σ] be the set of positive [respectively simple] roots giving rise to the nilpotent component $\mathfrak{n}$, that is,

$$\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ stands for the α-root space. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and put $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for the corresponding minimal parabolic subalgebra. By definition, the maximal flag manifold $\mathbb{B}$ of G is the set of subalgebras $\mathrm{Ad}\,(G)\,\mathfrak{p}$, where Ad stands for the adjoint representation of G in $\mathfrak{g}$. There is an identification of $\mathbb{B}$ with G/P where P is the normalizer of $\mathfrak{p}$ in P. Furthermore, $P = MAN$, $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$ and M is the centralizer of A in $K = \exp \mathfrak{k}$ ($M = \{u \in K : uh = hu$ for all $h \in A\}$).

Given a subset $\Theta \subset \Sigma$, denote by $\mathfrak{p}_\Theta$ the corresponding parabolic subalgebra, namely,

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p},$$

where $\mathfrak{n}^-(\Theta)$ is the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle\Theta\rangle$. Here $\langle\Theta\rangle$ is the set of positive roots generated by Θ. The set of parabolic subalgebras conjugate to $\mathfrak{p}_\Theta$ identifies with the homogenous space G/P_Θ, where P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \{g \in G : \mathrm{Ad}\,(g)\,\mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

This construction yields the flag manifold $\mathbb{B}_\Theta = G/P_\Theta$, $\Theta \subset \Sigma$.

Let

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha\,(H) > 0 \text{ for all } \alpha \in \Sigma\}$$

be the Weyl chamber associated to Σ. We say that $X \in \mathfrak{g}$ is split-regular in case $X = \mathrm{Ad}\,(g)\,(H)$ for some $g \in G$, $H \in \mathfrak{a}^+$. Analogously, $x \in G$ is said to be split-regular in case $x = ghg^{-1}$ with $h \in A^+ = \exp \mathfrak{a}^+$, that is, $x = \exp X$, with X split-regular in $\mathfrak{g}$.

Let $\mathfrak{n}^- = \sum_{\alpha \in \Pi} \mathfrak{g}_{-\alpha}$ be the nilpotent subalgebra opposed to $\mathfrak{n}$. Put $N^- = \exp \mathfrak{n}^-$. Then in any flag manifold $\mathbb{B}_\Theta$, the orbit (called open Bruhat cell) $\mathrm{Ad}\,(N^-)\,\mathfrak{p}_\Theta$ is open and dense. Furthermore, if $h \in A^+$ then $\lim h^k y = \mathfrak{p}_\Theta$ for any $y \in \mathrm{Ad}\,(N^-)\,\mathfrak{p}_\Theta$. In other words, $\mathfrak{p}_\Theta$ is an attractor in $\mathbb{B}_\Theta$ for any $h \in A^+$, having $\mathrm{Ad}\,(N^-)\,\mathfrak{p}_\Theta$ as stable manifold. The same way, $\mathrm{Ad}\,(g)\,\mathfrak{p}_\Theta$ is the attractor fixed point in $\mathbb{B}_\Theta$ of the split-regular $g = xhx^{-1}$ such that $\mathrm{Ad}\left(xN^-x^{-1}\right)$ is the open and dense stable manifold. In the sequel we denote the attractor fixed point of g by $\mathrm{at}\,(g)$, while the corresponding open cell, which is the stable manifold, is denoted by $\mathrm{st}\,(h)$. Although this notation does not specify the flag manifold under consideration, this will become clear from the context.

Given two subsets $\Theta_1 \subset \Theta_2 \subset \Sigma$, the corresponding parabolic subgroups satisfy $P_{\Theta_1} \subset P_{\Theta_2}$, so that there is a canonical fibration $G/P_{\Theta_1} \to G/P_{\Theta_2}$, $gP_{\Theta_1} \mapsto gP_{\Theta_2}$. Alternatively, the fibration assigns to the parabolic subalgebra $\mathfrak{q} \in \mathbb{B}_{\Theta_1}$ the unique parabolic subalgebra in $\mathbb{B}_{\Theta_2}$ containing $\mathfrak{q}$. In particular, $\mathbb{B} = \mathbb{B}_\emptyset$ projects onto every flag manifold $\mathbb{B}_\Theta$.

From the structure of the parabolic subgroup P_Θ the fiber P_Θ/P of $\mathbb{B} \to \mathbb{B}_\Theta$ is obtained. We follow closely the notation of Warner [21], Section 1.2. Denote by $\mathfrak{a}_\Theta$ the annihilator of Θ in $\mathfrak{a}$:

$$\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha\,(H) = 0 \text{ for all } \alpha \in \Theta\}.$$

Let L_Θ stand for the centralizer of $\mathfrak{a}_\Theta$ in G and put $M_\Theta(K) = L_\Theta \cap K$ for the centralizer of $\mathfrak{a}_\Theta$ in K. The Lie algebra $\mathfrak{l}_\Theta$ of L_Θ is reductive and decomposes as $\mathfrak{l}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta$ with $\mathfrak{m}_\Theta$ semi-simple. Let M_Θ^0 for the connected subgroup whose Lie algebra is $\mathfrak{m}_\Theta$ and put $M_\Theta = M_\Theta(K)\, M_\Theta^0$. It follows that the identity component of M_Θ is M_Θ^0. The Bruhat-Moore Theorem (see [21], Theorem 1.2.4.8), provides the following decompositions:

1. $P_\Theta = M_\Theta A_\Theta N_\Theta$, where $A_\Theta = \exp \mathfrak{a}_\Theta$ and N_Θ is the unipotent radical of P_Θ, that is, $N_\Theta = \exp \mathfrak{n}_\Theta$, with $\mathfrak{n}_\Theta$ the nilradical of $\mathfrak{p}_\Theta$.

2. $P_\Theta = M_\Theta(K)\, AN$.

This second decomposition ensures that the fiber $P_\Theta/P = M_\Theta(K)/M$. It turns out that $M_\Theta(K)/M = M_\Theta/(M_\Theta \cap P)$, which is the maximal flag manifold of M_Θ, since $M_\Theta \cap P$ is a minimal parabolic subgroup of M_Θ.

In the sequel we put $K(\Theta)$ for the identity component of $M_\Theta(K)$. Since P_Θ/P is connected we get further that $P_\Theta/P = K(\Theta)/(K(\Theta) \cap M)$. The subgroups $K(\Theta)$, $\Theta \subset \Sigma$, play a decisive role in the determination of the homotopy groups of the semigroups.

Finally we recall that an open Bruhat cell β in $\mathbb{B}_\Theta$ is diffeomorphic to an Euclidean space. This implies that the fiber bundle $\pi : \mathbb{B} \to \mathbb{B}_\Theta$ is trivial over β, that is, $\pi^{-1}(\beta) \approx \beta \times (P_\Theta/P)$.

2.2 Semigroups

We recall some basic facts from the general theory of semigroups as well as from the action of semigroups in flag manifolds. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We say a semigroup $S \subset G$ is exp-generated provided there exists a subset $U \subset \mathfrak{g}$ such that S is generated by the one-parameter semigroups $\exp(tX)$, $X \in U$, $t \geq 0$, that is,

$$S = \langle \exp\left(\mathbb{R}^+ U\right) \rangle.$$

In this case S is said to be generated by U.

A semigroup is said to be a Lie semigroup (or infinitesimally generated semigroup) provided it is the closure of a exp-generated semigroup (see e.g. Hilgert and Neeb [4] and Neeb [12]). Here we changed the terminology to emphasize that we do not ask for S to be closed. This requirement is not relevant for our purposes. In the sequel our typical hypothesis about a semigroup is that it contains a large enough semigroup generated by a subset in the Lie algebra. These kind of condition is fulfilled by the Lie semigroups.

Actually we work with semigroups having nonempty interior. It is well known that if S is generated by U, this condition holds if and only if U is generating, which means that the Lie algebra generated by U is $\mathfrak{g}$. Furthermore, in case U is generating $\mathrm{int}S$ is dense in S.

Now, we turn to semigroups in semi-simple groups. The following facts can be proved for any semigroup S with $\mathrm{int}S \neq \emptyset$, provided G has finite center. Consider the action of S in the flag manifolds of G. It was proved in [20], Theorem 6.2, that S is not transitive in $\mathbb{B}_\Theta$ unless $S = G$. Moreover, there exists just one closed invariant subset $C_\Theta \subset \mathbb{B}_\Theta$ such that Sx is dense in C_Θ for all $x \in C_\Theta$. This subset is called the invariant control set of S in C_Θ (abbreviated S-i.c.s.) . Since S is not transitive, $C_\Theta \neq B_\Theta$.

The fact that Sx is dense in C_Θ for all $x \in C_\Theta$ implies the existence of an open subset $C_\Theta^0 \subset C_\Theta$ such that for all $x, y \in C_\Theta^0$ there exists $g \in S$ with $gx = y$. Furthermore, C_Θ^0 is dense in C_Θ. This subset C_Θ^0 is called the set of transitivity of C_Θ, and is given by $C_\Theta^0 = (\mathrm{int}S)\, x \cap (\mathrm{int}S)^{-1}\, x$, for all $x \in C_\Theta$. In case S is exp-generated, it follows that $C_\Theta^0 = \mathrm{int}C_\Theta$ (see [20], Section 2).

The semigroups in G are distinguished according to the geometry of their invariant control sets. There exists $\Theta \subset \Sigma$ such that $\pi_{\Theta}^{-1}(C_{\Theta}) \subset \mathbb{B}$ is the invariant control set in the maximal flag manifold. Among the subsets Θ satisfying this property there is one which is maximal, in the sense that it contains the others. We denote this subset by $\Theta(S)$ and say that it is the type of S. Alternatively, we denote the type of S by the corresponding flag manifold $\mathbb{B}(S) = \mathbb{B}_{\Theta(S)}$ (see [20], [18] for further discussions about the of a semigroup).

When $\Theta = \Theta(S)$, the invariant control set $C_{\Theta(S)}$ has the following nice property: The set $R(S)$ of split-regular elements in int$S$ is not empty. Let $h \in R(S)$ and let as before at(h) and st(h) stand for its attractor and stable open cell in $\mathbb{B}_{\Theta(S)}$. Then $C_{\Theta(S)} \subset \mathrm{st}(h)$ and $\mathrm{at}(h) \in C^0_{\Theta(S)}$. As will be seen afterwards, this property completely describes the topology of the invariant control set in $\mathbb{B}$.

2.3 Reversibility

Recall that a subsemigroup T of a group L is right [left] reversible if one of the following equivalent conditions are satisfied

i) For all $h_1, h_2 \in H$, $Th_1 \cap Th_2 \neq \emptyset$ $[h_1T \cap h_2T \neq \emptyset]$.

ii) $T^{-1}T$ $[TT^{-1}]$ is a subgroup.

Clearly, T is right reversible if and only if T^{-1} is left reversible. In case L is connected and int$T \neq \emptyset$ then T is right [left] reversible if and only if $T^{-1}T$ $[TT^{-1}]$ coincides with L.

The following lemma encodes basic known properties of reversibility to be used later.

Lemma 2.1 *The subsemigroup $T \subset L$ is right reversible if and only if for any finite subset $\{h_1, \ldots, h_k\} \subset L$, $k \geq 1$, one of the following conditions holds*

1. $(Th_1) \cap \cdots \cap (Th_k) \neq \emptyset$.
2. *There exists $h \in L$ such that $h_i \subset hT$, $i = 1, \ldots, k$ (and hence $h_iT \subset hT$).*

Symmetric conditions are true for left reversibility.

Proof: The first condition is clearly sufficient. To see that it is necessary take $h_1, \ldots, h_k \in L$, $k \geq 3$, and proceed by induction. Assume that

$$(Th_1) \cap \cdots \cap (Th_{k-1}) \neq \emptyset$$

and take h in this intersection. Then $Th \subset (Th_1) \cap \cdots \cap (Th_{k-1})$. By reversibility $Th \cap Th_k \neq \emptyset$, so the lemma follows.

As to the second condition, right reversibility is equivalent to $\left(h_1T^{-1}\right) \cap \cdots \cap \left(h_kT^{-1}\right) \neq \emptyset$ for any finite subset. Now $h \in L$ belongs to this intersection if and only if $h_i \in hT$, $i = 1, \ldots, k$. Hence the equivalence of the condition follows. □

Another known fact about reversibility is that compact sets can be translated inside the reversible semigroups (c.f. [4], Lemma 3.37). We reproduce the proof here in order to emphasize that the translation is done by an element of the semigroup and not of the group as stated in [4].

Lemma 2.2 *Let $T \subset L$ be an open semigroup which is right [left] reversible. If $K \subset L$ is compact then there exist $g \in T$ such that gK $[Kg]$ is contained in T.*

Proof: Suppose that T is right reversible. Then $L = T^{-1}T$ so that $K \subset \bigcup_{g \in T} g^{-1}T$. By compactness there are finite $g_1, \ldots, g_k \in T$ such that

$$K \subset g_1^{-1}T \cup \cdots \cup g_k^{-1}T.$$

By the second condition in the previous lemma there exists $g \in L$ with $g_i^{-1} \in g^{-1}T$, $i = 1, \ldots, k$, so that $K \subset g^{-1}T$, that is, $gK \subset T$. Since $g_i \in T$, it follows that $g \in T$. □

2.4 Homotopy

We regard the n-th homotopy group $\pi_n(X, x_0)$ of a space based at $x_0 \in X$ as the set of homotopy classes of pointed maps $\gamma : (\mathbb{S}^n, s_0) \to (X, x_0)$ where $\mathbb{S}^n$ stands for the n sphere and s_0 is a base point in $\mathbb{S}^n$, e.g., $s_0 = (1, 0, \ldots, 0)$. The following facts about the homotopy of semigroups will be used extensively in the sequel.

Proposition 2.3 *Suppose that S is connected and contains an* exp*-generated semigroup T with* $\mathrm{int}T \neq \emptyset$. *Let $i : \mathrm{int}S \to S$ be the inclusion map. Then the induced homomorphism $i_* : \pi_n(\mathrm{int}S) \to \pi_n(S)$ is an isomorphism.*

Proof: Since S is path connected we can fix a base point $g_0 \in \mathrm{int}(S)$. Let $\gamma : (\mathbb{S}^n, s_0) \to (\mathrm{int}S, g_0)$ be such that $i_*[\gamma] = [i \circ \gamma] = 1$ in $\pi_n(S)$. Then there exists a continuous map $\Phi : [0,1] \times \mathbb{S}^n \to S$ such that

$$\Phi(0, \tau) = \gamma(\tau),\ \Phi(1, \tau) = g_0$$

for all $\tau \in \mathbb{S}^n$. By the assumption on T, there exists a continuous curve $\beta : [0,1] \to S$ such that $\beta(0) = 1$ and $\beta(]0,1]) \subset \mathrm{int}T \subset \mathrm{int}S$ (see [4], Theorem 3.8). Define $\Psi : I \times \mathbb{S}^n \to S$ by

$$\Psi(t, \tau) = \Phi(t, \tau)\beta(t(1-t)).$$

Then Ψ is a deformation of γ to the constant path in g_0. Furthermore, $\mathrm{im}\Psi \subset \mathrm{int}S$ because $\mathrm{int}S$ is a dense semigroup ideal of S and $\beta(t(1-t)) \in \mathrm{int}S$ if $t \neq 0, 1$. Hence $[\gamma] = [g_0]$ in $\mathrm{int}S$, showing that i_* is injective. On the other hand, pick $[\gamma] \in \pi_n(S)$ with $\gamma : (\mathbb{S}^n, s_0) \to (S, g_0)$. Then

$$\Phi : I \times \mathbb{S}^n \to S, (t, \tau) \mapsto \beta(t)\gamma(\tau)$$

deforms the path γ into a path, based at g_0, which lies entirely in $\mathrm{int}S$. Hence $[\gamma] \in \mathrm{im}(i_*)$, implying that i_* is surjective. □

Lemma 2.4 *Suppose that $T \subset S$ is* exp*-generated. Let $\gamma : X \to S$ be a continuous map, with X a topological space, and take $h \in T$. Then $h\gamma$ and γh are homotopic to γ in S.*

Proof: Since T is exp-generated, there exists a continuous path $h_t \in T \subset S$, $t \in [0, u]$, such that $h_0 = 1$ and $h_u = h$. Define $\Phi : [0, u] \times X \to S$ by

$$\Phi(t, x) = h_t \gamma(x).$$

Clearly, Φ is continuous because of the continuity of the product map in G. Moreover, $\Phi(0, x) = \gamma(x)$ and $\Phi(u, x) = h\gamma(x)$ for all $x \in X$. Since $\Phi(t, x) \in T \subset S$ for all $(t, x) \in [0, u] \times X$, it follows that Φ is a homotopy between γ and $h\gamma$ in S. To see that γh is homotopic to γ just take the homotopy $\gamma(x) h_t$. □

Lemma 2.5 *With the same assumptions and notations as in the previous lemma, the maps* $\pi_n(S,g) \to \pi_n(S,hg)$, $[\gamma] \mapsto [h\gamma]$ *and* $\pi_n(S,g) \to \pi_n(S,gh)$, $[\gamma] \mapsto [\gamma h]$ *are isomorphisms.*

Proof: The left or right translation isomorphism coincide with the isomorphism given by the path $h_t g$ or gh_t, where as before h_t is a path joining 1 to h in T (c.f. the proof of Theorem 7.2.3 in Maunder [10]). □

For the rest of this subsection we digress from the main stream of the paper with the purpose of discussing homotopy and reversibility in homogeneous spaces. It will emerge from this discussion that the orbits of S in the (Riemannian) symmetric space associated to G are contractible. This fact, although not used in the proof of the main result, reinforces it by showing that as happens with G the topology of S is embodied in the compact part of G.

Let H be a group, $L \subset H$ a subgroup and $T \subset H$ a subsemigroup. We say that T is reversible modulo L in case $Tx \cap Ty \neq \emptyset$ for all $x, y \in H/L$. Of course, a right reversible semigroup is in particular reversible modulo $L = \{1\}$. Also, T is reversible modulo L if and only if $T^{-1}Tx = H/L$ for all $x \in H/L$. Using this property and proceeding like in Lemma 2.2 it can proved – in the topological setting – an analogous of that lemma. Namely if we assume that T is open then for a given compact $Q \subset H/L$ and $x \in H/L$, there exists $g \in T$ such that $gQ \subset Tx$. Using this fact we can prove the following generalization of Theorem 3.38 in [4]. Although the proof is almost the same we reproduce it for the sake of completeness.

Proposition 2.6 *Let* H *be a Lie group and* $T \subset H$ *an* exp*-generated semigroup with nonempty interior. Suppose that* T *is reversible modulo* L*, with* L *a closed subgroup. Take* $x \in H/L$ *and consider the inclusion* $i : Tx \to H/L$*. Then the induced homomorphism* $i_* : \pi_n(Tx) \to \pi_n(H/L)$ *is injective.*

Proof: Fix $x_0 \in Tx$ and let $\gamma_0 : (\mathbb{S}^n, s_0) \to (Tx, x_0)$ be a cycle such that $i_*[\gamma_0] = 1$ in $\pi_n(H/L)$. Then there exists a homotopy based at x_0, $\Phi : I \times \mathbb{S}^n \to H/L$ such that $\Phi(0,\cdot) = \gamma_0$ and $\Phi(1,\cdot) = x_0$. The image Q if Φ is compact in H/L, hence there exists $g \in T$ such that $gQ \subset Tx$. The map $\Psi(t,s) = g\Phi(t,s)$ is a homotopy based at gx_0 carrying the cycle $g\gamma_0$ into gx_0. Hence $[g_0\gamma_0] = 1$ in $\pi_n(Tx, gx_0)$.

Now, we use the fact that T is exp-generated and proceed as in Lemma 2.5 to check that $\gamma \mapsto g\gamma$ defines an isomorphism between $\pi_n(Tx, x_0)$ and $\pi_n(Tx, gx_0)$. Implying that $[\gamma_0] = 1$ in $\pi_n(Tx, x_0)$. □

After the following lemma proved by Furstenberg [3], Lemma 3.2 (and its corollary), the above proposition implies immediately that the orbits of Lie semigroups in Riemannian symmetric spaces are contractible.

Lemma 2.7 *Suppose that* G *is a noncompact semi-simple Lie group and let* G/K *be the corresponding symmetric space, where* K *is a maximal compact subgroup of* G*. Let* $T \subset G$ *be a semigroup with* $\mathrm{int}T \neq \emptyset$*. Then* T *is reversible modulo* K*.*

Corollary 2.8 *With the same notations as above let* $S \subset G$ *be an* exp*-generated semigroup with* $\mathrm{int}S \neq \emptyset$*. Then* $(\mathrm{int}S)x$ *is contractible for any* $x \in G/K$*.*

Proof: It follows from the previous lemma that $\mathrm{int}S$ is reversible modulo K. Then Proposition 2.6 implies that the inclusion $i : (\mathrm{int}S)x \to G/K$ is injective. Now, G/K is diffeomorphic to an Euclidean space. Hence the homotopy groups $(\mathrm{int}S)x$ of are trivial. Being open in G/K, $(\mathrm{int}S)x$

is a manifold and hence a CW-complex. Therefore by the Theorem of Whitehead it follows that $(\mathrm{int}S)\,x$ is contractible. □

3 Invariant control sets in G/AN

The homotopy groups of the semigroup $S \subset G$ will be proved to be isomorphic to the homotopy groups of the invariant control sets of S in G/AN. Before going into the homotopy groups, we discuss in this section the geometry of these invariant control sets.

We assume here that S is connected. Also we put $\Theta = \Theta(S)$, and let C_Θ stand for the unique invariant control set of S in $\mathbb{B}_\Theta = G/P_\Theta$. The set of transitivity of C_Θ is denoted by C_Θ^0. Put $C(\mathbb{B})$ for the unique invariant control set in the maximal flag manifold G/MAN. It is given by $C(\mathbb{B}) = \pi_\Theta^{-1}(C_\Theta)$, where $\pi_\Theta : \mathbb{B} \to \mathbb{B}_\Theta$ is the canonical fibration. Also, its set of transitivity is $C(\mathbb{B})_0 = \pi_\Theta^{-1}\left(C_\Theta^0\right)$. These inverse images are actually diffeomorphic to Cartesian products. In fact, recall that C_Θ is contained in some open Bruhat cell, say σ, of $\mathbb{B}_\Theta$. Since the bundle $\pi_\Theta : \mathbb{B} \to \mathbb{B}_\Theta$ is trivial over σ, it follows that $\pi_\Theta^{-1}(\sigma) \approx \sigma \times F_\Theta$, where F_Θ is the fiber P_Θ/P. Therefore $C(\mathbb{B}) \approx C_\Theta \times F_\Theta$ and $C(\mathbb{B})_0 \approx C_\Theta^0 \times F_\Theta$.

Now, we lift the invariant control set $C(\mathbb{B})$ to G/AN. Consider the fibration

$$\pi_1 : G/AN \longrightarrow G/MAN.$$

This is a principal bundle whose fiber is the compact group $M \approx MAN/AN$. The projection π_1 is equivariant with respect to the actions of G on the homogenous spaces G/AN and G/MAN, i.e., $g \circ \pi_1 = \pi_1 \circ g$ for all $g \in G$. Also, M has a natural right action on G/AN, which commutes with the left action of G. The equivariance of π_1 implies that $\pi_1(C)$ is an S-i.c.s. in $\mathbb{B}$ if $C \subset G/AN$ is an invariant control set. In other words the invariant control sets of S in G/AN are contained in $\pi_1^{-1}(C(\mathbb{B}))$. Analogously, the set of transitivity C_0 of an invariant control set is contained in $\pi_1^{-1}(C(\mathbb{B})_0)$.

The assumption that S is connected implies that its invariant control sets are also connected. In particular if $C \subset G/AN$ is an S-i.c.s. then it is contained in a connected component of $\pi_1^{-1}(C(\mathbb{B}))$. Its set of transitivity C_0 is also connected and hence contained in a component of $\pi^{-1}(C(\mathbb{B})_0)$. Actually we have

Lemma 3.1 *S acts transitively on any connected component of $\pi_1^{-1}(C(\mathbb{B})_0)$.*

Proof: Consider the restriction of the principal bundle $\pi_1 : G/AN \to G/MAN$ to the open set $C(\mathbb{B})_0$. Its structural group is compact, and S acts on it as a semigroup of automorphisms of the bundle. Also, S acts transitively on the basis $C(\mathbb{B})_0$. Hence the lemma follows from Proposition 3.9 in [2], which asserts that a semigroup acting on a connected principal bundle with compact fiber is transitive provided it is transitive on the base space. □

From this lemma we get the following characterization of the invariant control sets of S in G/MAN.

Proposition 3.2 *Keep assuming that S is connected. Denote as before by C_Θ^0 the set of transitivity of the S-i.c.s. in G/P_Θ, where $\Theta = \Theta(S)$. Put $\pi : G/AN \to G/P_\Theta$ for the canonical fibration. Let $C \subset G/AN$ be an S-i.c.s. Then C is a connected component of $\pi^{-1}(C_\Theta)$ and C_0 is a connected component of $\pi^{-1}\left(C_\Theta^0\right)$. Conversely, the closure in G/AN of a connected component of $\pi^{-1}\left(C_\Theta^0\right)$ is an S-i.c.s. in G/AN.*

Proof: By the choice of G/P_Θ it follows that $C(\mathbb{B})_0 = \pi_\Theta^{-1}(C_\Theta^0)$ where π_Θ is the projection $G/P \to G/P_\Theta$. Also there exists an open Bruhat cell $\sigma \subset G/P_\Theta$ such that $C_\Theta \subset \sigma$. The restriction of π to σ defines a trivial bundle. Since C_Θ is connected and contained in σ, it follows that the connected components of $\pi^{-1}(C_\Theta)$ are contained in the connected components of $\pi^{-1}(\sigma)$. Analogously, the connected components of $\pi^{-1}(C_\Theta^0)$ are contained in the components of $\pi^{-1}(\sigma)$. This together with the fact that C_Θ^0 is dense in C_Θ implies that the closure of a component of $\pi^{-1}(C_0)$ is a component of $\pi^{-1}(C_\Theta)$, so that the closures of two different components of $\pi^{-1}(C_\Theta^0)$ are disjoint.

Now, π decomposes as

$$G/AN \to G/MAN \to G/P_\Theta.$$

Since the fiber of G/MAN is connected, it follows that the connected components of $\pi^{-1}(C_\Theta^0)$ are the components over $C(\mathbb{B})_0$ in the fibration $G/AN \to G/MAN$. Hence, the previous lemma ensures that a connected component of $\pi^{-1}(C_\Theta)$ is an invariant control set of S. Furthermore, this implies that any invariant control set is such a component, because their union is $\pi^{-1}(C_\Theta)$. □

Note that the triviality of the bundle $G/AN \to G/P_\Theta$ over C_Θ means that $\pi^{-1}(C_\Theta)$ is diffeomorphic to $C_\Theta \times F$ where $F = P_\Theta/AN$ is the fiber. Clearly, a connected component of the fiber is P_Θ^0/AN where P_Θ^0 is the identity component of P_Θ. The other components are obtained similarly from the components of P_Θ. These components are diffeomorphic to each other, since they are interchanged by the right action of M as follows by Lemma 1.2.4.5 in [21], which ensures that $P_\Theta = MP_\Theta^0$. Therefore the previous proposition together with the fact that C_Θ is connected implies the

Corollary 3.3 *With the assumption that S is connected, any invariant control set $C \subset G/AN$ is diffeomorphic to $C_\Theta \times P_\Theta^0/AN$. Moreover, $C_0 \approx C_\Theta^0 \times P_\Theta^0/AN$.*

In order to complete the picture we recall that $P_\Theta = M_\Theta(K)AN$, where $M_\Theta(K)$ is the centralizer of $\mathfrak{a}_\Theta$ in K (see [21], Theorem 1.2.4.8). We denote by $K(\Theta)$ the identity component of $M_\Theta(K)$. Then $P_\Theta^0 = K(\Theta)AN$, so that the fiber P_Θ^0/AN is diffeomorphic to $K(\Theta)$. Therefore there is following version of the above corollary which we state for later reference.

Corollary 3.4 *With the assumption that S is connected, any invariant control set $C \subset G/AN$ is diffeomorphic to $C_\Theta \times K(\Theta)$. Moreover, $C_0 \approx C_\Theta^0 \times K(\Theta)$. Here $\Theta = \Theta(S)$.*

4 Isomorphism

As before S stands for a semigroup with nonempty interior in G. Let C be an invariant control set of S in G/AN and C_0 its set of transitivity. The purpose of this section is to prove that the homotopy groups of S and C_0 are isomorphic. For the full proof it will be required that S is an exp-generated semigroup (actually the weaker assumption that S admits a large exp-generated semigroup is enough, see theorems 4.8 and 4.13 below). However, many proofs work in more generality, so that we do not specify in advance the conditions required for S, apart from having nonempty interior.

The isomorphism will be realized by the evaluation map. Fix $x \in C_0$ and denote by e_x (or simply by e) the map $e : S \to C_0$ given by $e(g) = gx$. It will be proved that the induced homomorphisms e_* between the homotopy groups are isomorphisms.

4.1 Surjectivity

In view of Corollary 3.3, C_0 is diffeomorphic to $C_\Theta^0 \times F_0$ where $F_0 = P_\Theta^0/AN$. For the semigroups considered here C_Θ^0 is shown to be contractible, so that any cycle in C_0 is homotopic to one in F_0. The surjectivity e_* will be proved by showing the existence of a cross section $\sigma : F \to \mathrm{int}S$ for the evaluation map.

In order to get such a cross section some preliminary results are required. We start with the following general statement.

Proposition 4.1 *Let $R \subset L$ be a closed normal subgroup of the Lie group L such that L is the semi-direct product $L = B \times_s R$. Take an open subsemigroup $T \subset L$ and suppose that $T/R = L/R$ and $T \cap R$ is left reversible. Then it is possible to lift any compact subset $Q \subset H/R$ inside T, that is, there exists $z \in T \cap R$ such that $Q \times \{z\} \subset T$.*

Proof: Since $T/R = L/R$ and T is open, there exists, for every $x \in Q$, a neighborhood U_x of x in L/R and $z_x \in R$ such that $U_x \times \{z_x\} \subset T$. There is a freedom in the choice of z_x. In fact, if $w \in T \cap R$ then $(U_x \times \{z_x\})\, w \subset T$ and

$$(U_x \times \{z_x\})\, w = U_x \times \{z_x w\},$$

so that we can choose any element in $z_x\,(T \cap R)$ instead of z_x.

Choose $x_1, \ldots, x_n \in Q$ such that $\{U_{x\,i}\}_{i=1,\ldots,n}$ is a covering of Q. Since $U_x \times \{z_x\,(T \cap R)\}$ is contained in T, in order to conclude the proof it is enough to check that there are $w_1, \ldots, w_n \in T \cap R$ such that $z_{x_1} w_1 = \cdots = z_{x_n} w_n$. Note that this is equivalent to

$$z_{x_1}\,(T \cap R) \cap \cdots \cap z_{x_n}\,(T \cap R) \neq \emptyset,$$

which by Lemma 2.1 is equivalent to left reversibility of $T \cap R$. □

Now we apply this proposition to lift the fiber F inside $\mathrm{int}S$. Fix $x \in C_0$ and put $y = \pi\,(x)$. We can assume without loss of generality that AN is the isotropy subgroup at x and P_Θ is the isotropy of y. These assumptions imply that $\mathrm{int}S$ meets AN and P_Θ. Take a decomposition $P_\Theta = M_\Theta A_\Theta N_\Theta$ of P_Θ (see [21], Theorem 1.2.4.8). The subgroup $A_\Theta N_\Theta$ is closed and normal and P_Θ becomes the semi-direct product of M_Θ by $A_\Theta N_\Theta$.

Let P_Θ^0 be the connected component of the identity of P_Θ. Then $P_\Theta^0 = M_\Theta^0 A_\Theta N_\Theta$ where M_Θ^0 is the identity component of M_Θ. Since P_Θ has a finite number of components and $\mathrm{int}S \cap P_\Theta \neq \emptyset$, it follows that $\mathrm{int}S$ meets P_Θ^0 as well.

Now, P_Θ^0 is the semi-direct product of M_Θ^0 and $A_\Theta N_\Theta$, hence the previous proposition can be applied to $L = P_\Theta^0$, $R = A_\Theta N_\Theta$ and $T = (\mathrm{int}S) \cap P_\Theta^0$, as soon as we check that

1. $T/A_\Theta N_\Theta = P_\Theta^0/A_\Theta N_\Theta$, and
2. $T \cap A_\Theta N_\Theta = (\mathrm{int}S) \cap A_\Theta N_\Theta$ is left reversible.

The first of these equalities were proved in [20]. Here is a sketch of the proof: Since $\pi_\Theta^{-1}\,(C_\Theta)$ is the invariant control set in G/MAN, it follows that T is transitive on the fiber $\pi_\Theta^{-1}\,(y)$. However, $A_\Theta N_\Theta$ fixes every point of this fiber, so that the projection $\mathrm{pr}\,(T)$ of T into M_Θ^0 is transitive on $\pi_\Theta^{-1}\,(y)$. However this fiber is the maximal flag manifold of the connected semi-simple group M_Θ^0. So that the transitivity of $\mathrm{pr}\,(T)$ on the fiber implies that $\mathrm{pr}\,(T) = M_\Theta^0$. This means that $T/A_\Theta N_\Theta = P_\Theta/A_\Theta^0 N_\Theta$, showing that (1) holds.

For the second point observe that $T \cap A_\Theta N_\Theta$ is an open subsemigroup of $A_\Theta N_\Theta$, which is not empty because T projects onto $P_\Theta^0 / A_\Theta N_\Theta$. Now we use the following lemma proved by Ruppert [15], Lemma 4.6:

Lemma 4.2 *Suppose that R is a solvable Lie group and that $U \subset R$ is an open semigroup that contains an element $\exp X$ such that $\mathrm{Re}\,\lambda \geq 0$ for all eigenvalues λ of $\mathrm{ad}\,(X)$. Then U is left reversible.*

The semigroup $T \cap A_\Theta N_\Theta$ fulfills the requirements of this lemma. To check this recall that through the exponential map $A_\Theta N_\Theta$ is diffeomorphic to $\mathfrak{a}_\Theta + \mathfrak{n}_\Theta$. Take $h \in T \cap A_\Theta N_\Theta$ and write $h = \exp(H + Y)$, $H \in \mathfrak{a}_\Theta$, $Y \in \mathfrak{n}_\Theta$. Since $T \cap A_\Theta N_\Theta$ is open, we can assume that H is regular in $\mathfrak{a}_\Theta$, in the sense that the no root in $\Sigma \setminus \langle \Theta \rangle$ annihilates in H. Then it is well known (see e.g. [21, Prop. 1.2.4.10]) that the map

$$X \mapsto \exp(\mathrm{ad}\,(X))\, H - H$$

is a diffeomorphism of $\mathfrak{n}_\Theta$ into itself. This implies that there exists $n \in N_\Theta$ such that $\mathrm{Ad}\,(n)\, H = H + Y$. Fixing this n let us change our base point x by nx, which is still in C_0, because C_0 contains the fiber $\pi_\Theta^{-1}(y)$. Then we get a new decomposition of P_Θ such that the corresponding vector group contains $h = \exp(H + Y)$. In other words, we can assume without loss of generality that there exists $h \in T \cap A_\Theta$, $h = \exp H$ with H regular in $\mathfrak{a}_\Theta$. Now we can check that the eigenvalues of $\mathrm{ad}\,(H)$ in $\mathfrak{n}_\Theta$ are ≥ 0. These eigenvalues are 0 and $\alpha(H)$ with $\alpha \notin \langle \Theta \rangle$. In case $\alpha(H) < 0$ for some α, there exists a small enough $Y \in \mathfrak{n}_\Theta^-$ such that $\exp(\mathrm{ad}\,(tH))\, Y \to \infty$ when $t \to +\infty$. This implies that $h^k \exp(Y)\, y$ accumulates outside $N_\Theta^- y$. However $y \in C_\Theta^0$ so that $\exp(Y)\, y \in C_\Theta^0$ if Y is small enough, contradicting the fact that $\Theta = \Theta(S)$, i.e., that C_Θ is a compact subset of the open Bruhat cell $N_\Theta^- y$.

Therefore we have proved that $S \cap P_\Theta$ satisfies the conditions of Proposition 4.1, proving the following statement which holds for arbitrary S with $\mathrm{int} S \neq \emptyset$ and $\Theta = \Theta(S)$.

Proposition 4.3 *Any compact subset $Q \subset M_\Theta$ can be lifted to $(\mathrm{int} S) \cap P_\Theta$, that is, there exists $z \in S \cap A_\Theta N_\Theta$ such that $Qz \subset (\mathrm{int} S) \cap P_\Theta$.*

We have also the decomposition $P_\Theta = M_\Theta AN$, so that P_Θ / AN identifies with M_Θ. From this fact and the previous proposition it becomes easy to get a cross section of the evaluation map.

Theorem 4.4 *Suppose that $\mathrm{int} S \neq \emptyset$ and put $\Theta = \Theta(S)$. Let C be one of its invariant control sets in G/AN, fix $x \in C_0$ and consider the evaluation map $e : S \to C_0$, $e(g) = gx$. Denote by F the fiber $\pi^{-1}(\pi(x))$, where $\pi : G/AN \to G/P_\Theta$ is the canonical projection. Then there exists a continuous cross section $\sigma : F \to \mathrm{int} S \cap P_\Theta$, satisfying $e\sigma = 1_F$.*

Proof: Assume without loss of generality that AN is the isotropy subgroup at x. For any $x_1 \in F$ there exists $k \in M_\Theta(K)$ such that $kx = x_1$. The assignment $\xi : x_1 \mapsto k$ settles a diffeomorphism between F and $M_\Theta(K)$, because $F = P_\Theta / AN \approx M_\Theta(K)$. Now, $M_\Theta(K)$ is a compact subset of M_Θ. Hence by the above proposition there exists $z \in AN$ such that $M_\Theta(K)\, z \subset (\mathrm{int} S) \cap P_\Theta$. Then $\sigma(x_1) = \xi(x_1)\, z$ is a well defined map $F \to (\mathrm{int} S) \cap P_\Theta$. It is a section of e. In fact

$$e(\sigma(x_1)) = \xi(x_1)\, zx = x_1$$

because $zx = x$ and $\xi(x_1)\, x = x_1$, by definition of ξ. □

In particular, this theorem ensures the lifting of any connected component F_0 of F. In case S is connected $C_0 \approx C_\Theta^0 \times F_0$, so that we get the following consequence.

Corollary 4.5 *Suppose that C_Θ^0 is contractible. Then the homomorphisms e_* between the homotopy groups induced by $e: S \to C_0$ and $e: \mathrm{int}S \to C_0$ are onto.*

Proof: Since $C_0 \approx C_\Theta^0 \times F$, any cycle in C_0 is homotopic to a cycle in F. By using the section σ we see that any cycle in F is in the image of e, showing that e_* is onto. □

For an exp-generated semigroup its invariant control set on the right flag manifold is contractible to a point as shows the following lemma proved by Mittenhuber [11], Lemma 2.11. Since the statement of [11] is restricted to rank one groups we outline the proof here.

Lemma 4.6 *Suppose that T is an* exp*-generated semigroup with* $\mathrm{int}T \neq \emptyset$. *Put $\Theta = \Theta(T)$ and let C_Θ be its invariant control set in G/P_Θ and C_Θ^0 its interior. Then C_Θ and C_Θ^0 are contractible.*

Proof: There exists a split-regular $h \in \mathrm{int}T$ such that its attractor, say $x \in G/P_\Theta$, belongs to C_0, and C is contained in the stable manifold $\mathrm{st}(h)$ of h. This implies that $h^k y \to x$ for all $y \in C_\Theta$. By compactness of C_Θ, for any neighborhood U of x there exists k_0 such that $h^k C_\Theta \subset U$ if $k \geq k_0$. In particular, we can choose U contractible to x, that is, there exists a continuous map $\Phi : [0,1] \times U \to U$ such that $\Phi(0,\cdot) = 1_U$, $\Phi(1,\cdot) = x$. On the other hand since T is exp-generated, there exists a continuous curve $g_t \in T$, $t \in [0,u]$, with $g_0 = 1$ and $g_u = h^k$. Then the map $\Phi_1 : [0,u] \times C_\Theta \to C_\Theta$, $\Phi_1(t,y) = g_t y$ contracts C_Θ into U. Joining together these maps, we get a contraction of C_Θ to x. □

Therefore for exp-generated semigroups the evaluation map induces surjective maps between the homotopy groups. Finally we note that for many examples of compression semigroups S itself may not be generated by a subset of the Lie algebra, but its invariant control set C_Θ is also the invariant control set of an exp-generated semigroup so Corollary 4.5 applies to S, ensuring that e_* is onto. Having these examples in mind we introduce the following

Definition 4.7 *Let $T_1 \subset T_2$ be semigroups with nonempty interior. Given a flag manifold $\mathbb{B}_\Theta = G/P_\Theta$, we say that T_1 is Θ-large (or $\mathbb{B}_\Theta$-large) in T_2 provided the invariant control set for both T_1 and T_2 on $\mathbb{B}_\Theta$ coincide. Also, T_1 is large in T_2 in case T_1 is Θ-large for every Θ.*

Since the invariant control sets of a semigroup are obtained by projecting the invariant control set on the maximal flag manifold $\mathbb{B}$ (see [20]), it follows that T_1 is large in T_2 if and only if it is $\mathbb{B}$-large in T_2. Also T_1 is large in T_2 in case $\Theta(T_1) = \Theta(T_2) = \Theta$ and T_1 is Θ-large in T_2. In fact in this case the invariant control set on $\mathbb{B}$ for both semigroups is the inverse image, under the projection $\mathbb{B} \to \mathbb{B}_\Theta$, of the common invariant control set on $\mathbb{B}_\Theta$.

Now, suppose that $S_1 \subset S$ are connected semigroups with $\mathrm{int}S_1 \neq \emptyset$ and S_1 large in S. Then the invariant control sets in G/AN for both semigroups are the connected components of $\pi^{-1}(C_\Theta)$ where C_Θ is the i.c.s. on G/P_Θ. So in G/AN the invariant control sets also coincide. Therefore taking into account the existence of the section σ in Theorem 4.4, we get

Theorem 4.8 *Suppose that S is connected and contains an* exp*-generated semigroup T which is large in S. Let C be an invariant control set in G/AN. Then the homomorphism $e_* : \pi_n(S) \to \pi_n(C_0)$ induced by the evaluation map $e : S \to C_0$, $e(g) = gx$, $x \in C_0$, is surjective. The same statement holds with the map $e : \mathrm{int}S \to C_0$.*

4.2 Injectivity

For the proof of the injectivity of the evaluation map we assume that S is connected and contains a Θ-large exp-generated semigroup T, where $\Theta = \Theta(S)$.

The proof goes as follows: Fix basic points $x \in C_0$ and $g_0 \in \mathrm{int} S$ such that $g_0 x = x$. If $\gamma : (\mathbb{S}^n, s_0) \to (S, g_0)$ satisfies $e_*[\gamma] = [e \circ \gamma] = 1$, we wish to prove that $[\gamma] = 1$, that is, there is a homotopy, based at g_0, carrying γ into g_0. We do not construct such homotopy directly. Instead, we build homotopies inside S carrying γ successively into smaller groups until we reach $A_\Theta N_\Theta$. Using reversibility properties of $S \cap A_\Theta N_\Theta$, it follows then that there exists a (unbased) homotopy $\Phi : [0,1] \times \mathbb{S}^n \to S$ between γ and a constant cycle g_1. Then a standard argument shows that $[\gamma] = 1$. In fact, from Φ we have the path, say α, given by the restriction of Φ to $[0,1] \times \{s_0\}$. This path settles an isomorphism $\alpha_* : \pi_n(S, g_0) \to \pi_n(S, g_1)$, where $g_1 = \Phi(1, s_0)$, such that $\alpha_*[\gamma] = [g_1] = 1$ (see e.g. [10], Theorem 7.2.3). This shows the triviality of $\ker e_*$.

We start with the following lemma, which is used in the final step of the proof. It is patterned after Proposition 2.6. Here, however, we can not assume that the semigroup is exp-generated, because the lemma will be applied to $S \cap A_\Theta N_\Theta$. Hence we do not get injectivity of the homomorphism induced by inclusion but only that any trivial cycle in the group can be translated into a cycle which is trivial inside the semigroup.

Lemma 4.9 *Let L be a connected group and $T \subset L$ an open and connected subsemigroup. Let $\gamma : (\mathbb{S}^n, s_0) \to (T, g_0)$ be a cycle in T such that $i_*[\gamma] = 1$, where $i : T \hookrightarrow L$ is the inclusion. Suppose that T is right [respectively left] reversible. Then there exists $g \in T$ such that the cycle $g\gamma$ [respectively. γg] is contractible in T.*

Proof: Consider the case where T is right reversible. By assumption there exists a homotopy $\Phi : [0,1] \times \mathbb{S}^n \to L$ such that

$$\Phi(0,\tau) = \gamma(\tau), \qquad \Phi(1,\tau) = g_0$$

for all $\tau \in \mathbb{S}^n$. Clearly, $\Phi([0,1] \times \mathbb{S}^n)$ is a compact subset of L. Hence, the right reversibility of T implies that there exists $g \in T$ such that $g\Phi([0,1] \times \mathbb{S}^n) \subset T$. Therefore $g\Phi$ is a homotopy carrying $g\gamma$ into gg_0 as claimed. The proof in the left reversible case follows by taking the inverse semigroup. □

We prove next in different steps that any cycle γ in S is homotopic within S to a cycle in $S \cap A_\Theta N_\Theta$.

Lemma 4.10 *Suppose that S contains a Θ-large* exp*-generated semigroup T with* $\mathrm{int} T \neq \emptyset$*, where $\Theta = \Theta(S)$. Fix $y \in C_\Theta^0$ and assume without loss of generality that P_Θ is the isotropy at y. Then any cycle $\gamma : (\mathbb{S}^n, s_0) \to (S, g_0)$ with $g_0 y = y$ is (unbased) homotopic to a cycle $\beta : \mathbb{S}^n \to P_\Theta$.*

Proof: By assumption there exists a split regular $h \in \mathrm{int} T$ which fixes y, so that $h^k C_\Theta$ contracts to y as $k \to +\infty$. This implies that for any neighborhood U of y there exists $k > 0$ such that $h^k C_\Theta \subset U$. Since $\gamma(\mathbb{S}^n) y \subset C_\Theta$ and $h^k \gamma$ is homotopic to γ (by Lemma 2.4), we can assume without loss of generality that $\gamma(\mathbb{S}^n) y \subset U$.

Now, take $g \in \mathrm{int} T^{-1}$ such that $gy = y$. We can choose U to be diffeomorphic to an open ball in Euclidean space and such that if we write the bundle $G \to G/P_\Theta$ locally as $U \times P_\Theta$ then $g \in U \times W \subset \mathrm{int} T^{-1}$, for some open $W \subset P_\Theta$. Under this identification $U' = \{g\} \times U$ is an Euclidean ball contained in $\mathrm{int} T^{-1}$. Then there exits a continuous contraction $\phi : I \times U' \to U'$ such that $\phi(0,l) = (g,l)$ and $\phi(1,l) = (g,g)$, for all $l \in U'$. Define in U' the cycle $\delta(z) = (g, \gamma(z) y)$.

For $(t,z) \in I \times \mathbb{S}^n$ put

$$\Phi(t,z) = \phi(t,\delta(z))^{-1}\gamma(z).$$

Note that Φ is a continuous map defined in $[0,1] \times \mathbb{S}^n$ assuming its values in S because $U' \subset S^{-1}$ and $\gamma(z) \in S$. Furthermore $\Phi(0,z) = \delta(z)^{-1}\gamma(z)$ and $\Phi(1,z) = g^{-1}\gamma(z)$, so that the cycles $\delta(z)^{-1}\gamma(z)$ and $g^{-1}\gamma(z)$ are homotopic in S. Since $g^{-1} \in T$ and T is exp-generated, it follows by Lemma 2.4, that $g^{-1}\gamma(z)$ and $\gamma(z)$ are homotopic in S. Now, $\delta(z)^{-1}\gamma(z)y = y$ from the definition of $\delta(z)$. Therefore $\delta(z)^{-1}\gamma(z)$ is contained in P_Θ concluding the proof of the lemma. □

Corollary 4.11 *Let γ and β be as in the above lemma and suppose that $e_*[\gamma] = 1$. Then $e(\beta)$ is homotopic to a point in C_0.*

Proof: In fact, applying e to the homotopy between γ and β we get a homotopy between $e(\gamma)$ and $e(\beta)$. Since $e(\gamma)$ is homotopic to a point, this holds with $e(\beta)$ as well. □

Lemma 4.12 *Let $\gamma : \mathbb{S}^n \to S \cap P_\Theta^0$ be a cycle with $\gamma(s_0) = g_1 \in A_\Theta N_\Theta$. Assume that $e_*[\gamma] = 1$. Then γ is homotopic in S to a cycle β contained in $A_\Theta N_\Theta$.*

Proof: Recall that $P_\Theta^0 = M_\Theta^0 A_\Theta N_\Theta$ and the action of P_Θ^0 on the fiber $\pi_\Theta^{-1}(y)$ is equivalent to the action on $P_\Theta^0/AN = K(\Theta) = M_\Theta^0/A(\Theta)N(\Theta)$. Therefore to say that $e_*[\gamma] = 1$ means that the projection of γ in $K(\Theta)$ is homotopic to a point in $K(\Theta)$. Now, $M_\Theta^0 = K(\Theta)A(\Theta)N(\Theta)$ is an Iwasawa decomposition of the semi-simple group M_Θ^0. Since $A(\Theta)N(\Theta)$ is diffeomorphic to an Euclidean space, it follows that the projection of γ into M_Θ^0, through the decomposition $P_\Theta^0 = M_\Theta^0 A_\Theta N_\Theta$ is homotopic to a point. Denote by δ the projected cycle. Then $\delta(s_0) = 1$, and $\delta(\tau)^{-1}$ is also homotopic to 1. Let $\Phi : I \times \mathbb{S}^n \to M_\Theta^0$ be a homotopy such that $\Phi(0,\tau) = \delta(\tau)^{-1}$ and $\Phi(1,\tau) = 1$, for all $\tau \in \mathbb{S}^n$. The subset $\Phi(I \times \mathbb{S}^n)$ is compact in M_Θ^0. Hence by Proposition 4.3, there exists $z \in S \cap A_\Theta N_\Theta$ such that $\Phi(I \times \mathbb{S}^n)z \subset S \cap P_\Theta^0$. Put

$$\Gamma(t,\tau) = \gamma(\tau)\Phi(t,\tau)z.$$

Then $\Gamma(t,\tau) \in S \cap P_\Theta^0$, and is a homotopy between $\gamma(\tau)\delta^{-1}(\tau)z$ and $\gamma(\tau)z$. Since $\gamma(\tau)z$ is homotopic to $\gamma(\tau)$ and $\gamma(\tau)\delta^{-1}(\tau)z \in A_\Theta \cap S$, for all τ, the lemma follows. □

We can now prove the injectivity of the evaluation map.

Theorem 4.13 *Assume that S is connected and contains a large* exp*-generated semigroup T with nonempty interior. As before let C be an S-i.c.s. in G/AN and C_0 its interior. Then the homomorphism $e_* : \pi_n(S) \to \pi_n(C_0)$ induced by a evaluation map $e : S \to C_0$, $e(g) = gx$, $x \in C_0$, is injective. The same statement is true with e defined in* int$S$ *instead of S.*

Proof: Observe first that since T is large in S, we can assume without loss of generality that $(\mathrm{int}T) \cap A_\Theta N_\Theta \neq \emptyset$. Now, by the Lemma 4.12 any cycle in S projecting to a contractible cycle in C_0 is homotopic within S to a cycle γ in $(\mathrm{int}S) \cap A_\Theta N_\Theta$. By Lemma 4.2 (and the discussion preceding it), $(\mathrm{int}T) \cap A_\Theta N_\Theta$ is left reversible in $A_\Theta N_\Theta$. Hence by Lemma 4.9 above, it follows that there exists $g \in \mathrm{int}T$ such that γg is homotopic to a point within T and hence inside S. Using Lemma 2.4, we conclude that γ and γg are homotopic in S.

We have showed that any cycle β in S such that $e(\beta)$ is contractible in C_0 is (unbased) homotopic to a point in S. Therefore, if we reproduce the standard argument mentioned at the beginning of this subsection we conclude that $[\beta] = 1$, showing that e_* is injective. □

4.3 Homotopy equivalence

So far we have proved that the evaluation maps $e : S \to C_0$ and $e : \mathrm{int}S \to C_0$ induce isomorphisms between the homotopy groups in case S contains a $\Theta(S)$-large exp-generated semigroup. This implies that the homotopy groups of S and $\mathrm{int}S$ are isomorphic to the homotopy groups of the compact group $K(\Theta(S)) \approx F_0$. In other words, e is a weak homotopy equivalence.

Now, $\mathrm{int}S$ is an open submanifold of G, and hence a CW-complex. Analogously, C_0 is a CW-complex, so that e is in fact a homotopy equivalence, which means that there exists $f : C_0 \to \mathrm{int}S$ such that both ef and fe are homotopic to the identity maps, that is f is a homotopy inverse of e.

We continue to assume that S admits a Θ-large, $\Theta = \Theta(S)$, exp-generated semigroup. Under this assumption we show next that a homotopy inverse of $e : \mathrm{int}S \to C_0$ is furnished by the cross section $\sigma : F_0 \to \mathrm{int}S$ constructed in Theorem 4.4. Recall that $C_0 = C_\Theta^0 \times F_0$ and that C_Θ^0 is contractible to a point. Denote by p the projection C_0 onto F_0.

Lemma 4.14 *$\sigma p : C_0 \to \mathrm{int}S$ is a homotopy inverse of e.*

Proof: Let f be a homotopy inverse of $e : \mathrm{int}S \to C_0$. We claim that f is homotopic to σp. Note that since C_Θ^0 is contractible to a point, there exists a homotopy $\Phi : I \times C_0 \to C_0$ such that $\Phi(0, x) = x$ and $\Phi(1, x) = p(x)$, that is, p is homotopic to the identity map i of C_0. Denote by $[X, Y]$ the set of homotopy classes of maps $X \to Y$. Then the induced map

$$e_* : [C_0, \mathrm{int}S] \longrightarrow [C_0, C_0]$$

is injective (see [10], Corollary 7.5.3). Now, $e_*[f] = [ef] = [i]$ and $e_*[\sigma p] = [e\sigma p] = [p]$. Since $[i] = [p]$, it follows that $[\sigma p] = [f]$, that is, $f \simeq \sigma p$, as claimed. Therefore σp is a homotopy inverse of e as well. □

From this homotopy inverse of e, it becomes easy to get $K(\Theta)$ (or rather a coset of it) as a deformation retract of $\mathrm{int}S$. In fact, recall the construction of the cross section σ in Theorem 4.4: Let $\xi : F_0 \to K(\Theta)$ be the diffeomorphism given by the requirement $\xi(x) = kx_0$, where x_0 is the base point. Then σ is given by $\sigma(x) = \xi(x)z$ where $z \in P_\Theta$ is such that the coset $K(\Theta)z \subset \mathrm{int}S$. In particular, we have that $\sigma(gx_0) = g$ for any $g \in K(\Theta)z$. This means that $\sigma pe : \mathrm{int}S \to K(\Theta)z$ satisfies $\sigma pe(g) = g$ for all $g \in K(\Theta)z$, that is, σpe is a retract of $\mathrm{int}S$. Since σpe is homotopic to the identity map, we actually get that $K(\Theta)z$ is a deformation retract of $\mathrm{int}S$. Note that in this construction we can take the section σ taking values in any coset $K(\Theta)z$ contained in $\mathrm{int}S$. Therefore we have

Theorem 4.15 *Under the assumption that S admits a large* exp*-generated semigroup with nonempty interior, there exists $z \in \mathrm{int}S \cap P_\Theta$ such that $K(\Theta)z \subset \mathrm{int}S$. Furthermore for any z satisfying this condition, $K(\Theta)z$ is a deformation retract of* $\mathrm{int}S$.

5 Remarks

5.1 Deformation retract of S

The discussion leading to Theorem 4.15 was restricted to $\mathrm{int}S$ in order to use the fact that it is an open submanifold and hence a CW-complex. Despite that the deformation retract property for S still holds. To see this let $T \subset S$ be a large exp-generated semigroup with nonempty interior. Take $g \in \mathrm{int}T$ and let $g_t \in T$ be a curve such that $g_0 = 1$ and $g_1 = g$. Then $Sg \subset \mathrm{int}S$ and S is deformed into Sg by g_t. By Theorem 4.15, there is a deformation retract $r = \sigma pe$ of $\mathrm{int}S$ into the coset $K(\Theta)\,zg$, where z is such that $K(\Theta)\,z \subset \mathrm{int}S$. Therefore if we compose r with the left translation L_g we get a map rL_g of S into $K(\Theta)\,zg$, which is not a retract. However it maps $y \in K(\Theta)\,z$ into yg, because $r(yg) = yg$. It follows that $r' = L_{g^{-1}} r L_g$ maps S into $K(\Theta)\,z$ and satisfies $r'(y) = y$ for all $y \in K(\Theta)\,z$, that is, r' is a retract of S into $K(\Theta)\,z$. Clearly $L_{g_t^{-1}}$ deforms $K(\Theta)\,zg$ into $K(\Theta)\,z$ within $\mathrm{int}S$, so that r' is indeed a deformation retract of S into $K(\Theta)\,z$.

5.2 Image under the inclusion map

As showed in [12] (see also [4], Chapter 3), the knowledge of the image of the fundamental group of S under the inclusion map $S \hookrightarrow G$ tells when S can be embedded in a given covering group of G. As a consequence of Theorem 4.15 this image is described by the inclusion of the subgroup $K(\Theta)$ in G. In fact the following statement holds for all the homotopy groups.

Proposition 5.1 *Let the assumptions and notations be as in Theorem 4.15. Then the image $i_*\pi_n(S)$ in $\pi_n(G)$ coincides with the image $j_*\pi_n(K(\Theta))$ where $j : K(\Theta) \to G$ is the inclusion.*

Proof: By Proposition 2.3, the homomorphism induced by the inclusion $\mathrm{int}S \hookrightarrow S$ is an isomorphism. Hence it is enough to prove the claim with $\mathrm{int}S$ in place of S. Since $K(\Theta)\,z$ is a deformation retract of $\mathrm{int}S$, it follows that the inclusion $K(\Theta)\,z \hookrightarrow \mathrm{int}S$ induces an isomorphism. Hence $i_*\pi_n(S)$ is the image of the homomorphism induced by $K(\Theta)\,z \hookrightarrow G$. By right translation this image coincides with $j_*\pi_n(K(\Theta))$. $\square$

5.3 Relative homotopy

From the identification of the homotopy groups of S with those of $K(\Theta)$ it follows also an identification of the relative homotopy groups. To see this, consider the following diagram

$$\begin{array}{ccccccccc}
\cdots \to & \pi_{n+1}(Kz, K(\Theta)z) & \xrightarrow{\partial_*} & \pi_n(K(\Theta)z) & \to & \pi_n(Kz) & \to & \pi_n(Kz, K(\Theta)z) & \to \cdots \\
& \downarrow \delta_* & & \downarrow \delta_* & & \downarrow \delta_* & & \downarrow & \\
\cdots \to & \pi_{n+1}(G, S) & \xrightarrow{\partial_*} & \pi_n(S) & \to & \pi_n(G) & \to & \pi_n(G, S) & \to \cdots
\end{array}$$

involving the two exact homotopy sequences of the pairs $(Kz, K(\Theta)z)$ and (G, S). Here δ stand for the inclusion maps and $z \in \mathrm{int}S$ is such that $K(\Theta)\,z \subset \mathrm{int}S$. This ladder is a commutative diagram (see e.g. [10], Theorem 7.2.18). Joining this with the fact that $\pi_n(K(\Theta)z) \approx \pi_n(S)$ for all n, it follows by a standard diagram chasing that $\delta_* : \pi_n(Kz, K(\Theta)\,z) \to \pi_n(G, S)$ is an isomorphism. Clearly $\pi_n(Kz, K(\Theta)\,z)$ is isomorphic to $\pi_n(K, K(\Theta))$ under right translation. Hence we have

Proposition 5.2 *The relative homotopy groups $\pi_n(G, S)$ are isomorphic to $\pi_n(K, K(\Theta))$.*

5.4 Deforming two semigroups

Two semigroups S_1 and S_2 of the same type $\Theta(S_1) = \Theta(S_2) = \Theta$ have the same homotopy groups. Actually they can be deformed one into another in G. In fact, there exists $z_1 \in \mathrm{int}S_1 \cap P_\Theta$ such that $K(\Theta)z_1$ is a deformation retract of $\mathrm{int}S_1$. Analogous $K(\Theta)z_2$ is deformation retract of $\mathrm{int}S_2$

for some $z_2 \in \mathrm{int} S_2 \cap P_\Theta$. Clearly $K(\Theta) z_1$ can be deformed into $K(\Theta) z_2$ in G. Composing these deformations one get a deformation of $\mathrm{int} S_1$ into $\mathrm{int} S_2$.

6 Examples

6.1 Positive matrices

Let $S = \mathrm{Sl}^+(n, \mathbb{R})$ be the semigroup of determinant one matrices having nonnegative entries. This is the compression semigroup of the positive orthant $\mathbb{R}^n_+$ in $\mathbb{R}^n$:

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_i \geq 0\}.$$

It turns out that the type of $\mathrm{Sl}^+(n, \mathbb{R})$ is the projective space $\mathbb{P}^{n-1}$, and the invariant control in $\mathbb{P}^{n-1}$ is the set $[\mathbb{R}^n_+]$ of lines contained in $\mathbb{R}^n_+$. In our previous notation, $C_\Theta = [\mathbb{R}^n_+]$.

The semigroup $\mathrm{Sl}^+(n, \mathbb{R})$ is closed but not a Lie semigroup. This can be easily seen by the fact that its unit group $H(S) = S \cap S^{-1}$ is not connected. In fact, $H(S)$ is the semi-direct product $P \times D$ where P [respectively D] is the group of determinant one permutation matrices [respectively diagonal matrices with positive entries]. However, it is well known that the unit group of a Lie semigroup is connected (see Neeb [13], Proposition III.2). On the other hand, put

$$\mathcal{L}(S) = \{X \in \mathfrak{sl}(n, \mathbb{R}) : \exp(tX) \in \mathrm{Sl}^+(n, .\mathbb{R}) \text{ for all } t \geq 0\}$$

for the Lie wedge of $\mathrm{Sl}^+(n, \mathbb{R})$. One checks easily that $\mathcal{L}(S) = \{X = (x_{ij}) : x_{ij} \geq 0, i \neq j\}$. Put $S_{\mathrm{inf}} = \langle \exp \mathcal{L}(S) \rangle$ for the corresponding exp-generated semigroup. Since $\mathcal{L}(S)$ generates $\mathfrak{sl}(n, \mathbb{R})$, S_{inf} has nonempty interior in $\mathrm{Sl}(n, \mathbb{R})$. Furthermore, the invariant control set of S_{inf} in $\mathbb{P}^{n-1}$ is also C_Θ. This can be seen by considering matrices of the form

$$H = \mathrm{diag}\{n-1, -1, \ldots, -1\}$$

with respect to a basis $\{f_1, \ldots, f_n\}$ such that $f_1 \in \mathbb{R}^n_+$ and $\mathrm{span}\{f_2, \ldots, f_n\} \cap \mathbb{R}^n_+ = 0$. It can be shown that $H \in \mathcal{L}(S)$ (see [16]), so that any $x \in C_\Theta$ is the attractor of some element of S_{inf}, implying that C_Θ is the invariant control set of S_{inf}, as well. In case $f_1 \in \mathrm{int}\left(\mathbb{R}^n_+\right)$, $H \in \mathrm{int}(\mathcal{L}(S))$, so that $C^0_\Theta = \mathrm{int}(C_\Theta)$.

Therefore, $S = \mathrm{Sl}^+(n, \mathbb{R})$ contains a $\Theta(S)$-large exp-generated semigroup. Now, it was proved in [16] that S is connected. Hence the isomorphism theorem holds for $\mathrm{Sl}^+(n, \mathbb{R})$. The subgroup P_Θ can be taken here to be the group of matrices of the form

$$\begin{pmatrix} \lambda & * \\ 0 & Q \end{pmatrix}$$

with $\lambda \in \mathbb{R}$, Q an $(n-1) \times (n-1)$ matrix and $\lambda \det Q = 1$. For the identity component P^0_Θ one must take $\lambda, \det Q > 0$. The corresponding choice of AN is the group of upper triangular matrices with positive entries on the diagonal. Hence, P^0_Θ / AN is diffeomorphic to $\mathrm{SO}(n-1)$. It follows that the homotopy groups of $\mathrm{Sl}^+(n, \mathbb{R})$ are isomorphic to the homotopy groups of $\mathrm{SO}(n-1)$.

These facts extend to the compression semigroup of a cone in $\mathbb{R}^n$. Let $W \subset \mathbb{R}$ be a pointed and generating cone and form the semigroup

$$S_W = \{g \in \mathrm{Sl}(n, \mathbb{R}) : gW \subset W\}.$$

It was proved in [16] that S_W is connected. Again the type of S_W is the projective space, and similar to the proof for $\mathrm{Sl}^+(n, \mathbb{R})$, the semigroup generated by $\mathcal{L}(S_W)$ is large in S_W. Hence the homotopy type of S_W is also $\mathrm{SO}(n-1)$.

6.2 Totally positive matrices

Let $\mathcal{T} \subset \mathrm{Sl}(n, \mathbb{R})$ be the semigroup of totally positive matrices, that is, of matrices such that all its minors are nonnegative. It is well known that $\mathcal{T}$ is a Lie semigroup (see Ando [1], Theorem 3.5 and Corollary 3.6). The type of $\mathcal{T}$ is the maximal flag manifold. This can be seen by different ways. First by [1], Theorem 6.2, any $g \in \mathrm{int}\mathcal{T}$ has real and different eigenvalues. Hence by [20], Corollary 4.4, the type of $\mathcal{T}$ is the empty set, that is the maximal flag manifold. Alternatively one can consider the semigroups $\mathcal{T}_k$ of matrices having nonnegative k-minors. The type of $\mathcal{T}_k$ is the Grassmannian $\mathrm{Gr}_k(n)$ of k-dimensional subspaces (see [18]). Clearly $\mathcal{T} = \mathcal{T}_1 \cap \cdots \cap \mathcal{T}_{n-1}$. Arguing with the parabolic subgroups the Weyl group associated to the type of the semigroups we get easily the type of $\mathcal{T}$.

It follows that the homotopy type of $\mathcal{T}$ is the connected component of MAN/AN. Since M is discrete, the homotopy groups of $\mathcal{T}$ are trivial, that is, $\mathcal{T}$ is contractible.

There exists a generalization of the semigroup of totally positive matrices to semi-simple groups whose Lie algebras are normal real forms of the complex simple Lie algebras (see Lusztig [9], and references therein). Again the type of any such a semigroup is the maximal flag manifold, implying the triviality of their homotopy groups, a fact already proved by Lusztig by another means (see [9], Section 3).

6.3 Rank one groups

In case G has real rank one there exists just one class of parabolic subgroups and hence just one flag manifold G/MAN. The proper semigroups with nonempty interior in G have all the same type, namely $\Theta = \emptyset$. The subgroup $K(\Theta)$ is the identity component of MAN/AN, that is, $K(\Theta) = M_0$ so that every semigroup S in G admitting a large exp-generated semigroup have the same homotopy groups, and they are isomorphic to the homotopy groups of M_0. Moreover, $\mathrm{int}S$ can be continuously deformed into M_0.

The subgroup M_0 is well known for each of the rank one groups. For instance if $G = \mathrm{Sl}(2, \mathbb{R})$ then $M_0 = \{1\}$ hence the homotopy groups of S are trivial, and $\mathrm{int}S$ is contractible. On the other hand let $G = \mathrm{SO}(1,p)_0$, the identity component of a real hyperbolic group. In this case $M_0 = \mathrm{SO}(p)$, so that this is the homotopy type of the Lie semigroups in $\mathrm{SO}(1,p)$.

6.4 Ol'shanskiĭ semigroups

The isomorphism theorem holds trivially for the group G itself. In this case $\Theta(G) = \Sigma$ and the flag manifold $G/P_{\Theta(G)}$ degenerates to a point as $P_{\Theta(G)} = G$. The subgroup $K(\Theta)$ is the whole K, which by means of the Iwasawa decomposition $G = KAN$, is a deformation retract of G.

In this example we discuss the class of Ol'shanskiĭ semigroups where such polar decomposition is also available, so that the homotopy groups can be read off directly from the decomposition, illustrating the isomorphism theorem. We refer to Lawson [8] for detailed discussions about these semigroups and their decompositions.

Let $(\mathfrak{g}, \tau)$ be a simple symmetric Lie algebra, where τ is an involutive automorphism of $\mathfrak{g}$. put $\mathfrak{g}_+$ [respectively $\mathfrak{g}_-$] for the subalgebra [respectively subspace] of fixed points of τ [respectively -1 eigenvectors]. There is the direct sum $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and the bracket relations $[\mathfrak{g}_\varepsilon, \mathfrak{g}_\delta] = \mathfrak{g}_{\varepsilon\delta}$, $\varepsilon, \delta = \pm$. Assume that there is wedge $W \subset \mathfrak{g}_-$

1. $\mathfrak{g}_- = W - W$,

2. W is invariant under the adjoint representation of $\mathfrak{g}_+$ and

3. $\mathrm{ad}(X)$ has real spectrum for all $X \in W$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and denote by H the connected subgroup with Lie algebra $\mathfrak{g}_+$. Then H is closed and the subset $S = (\exp W) H$ is a closed semigroup in G having nonempty interior. Furthermore, the map $W \times H \to S$, $(X, h) \mapsto (\exp X) h$ is a diffeomorphism and S is a Lie semigroup (see [8], Theorem 3.4).

Clearly, W is contractible, so that H becomes a deformation retract of S.

In order to recognize the topology of H in terms of parabolic subgroups of G choose a τ-invariant Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ so that $\mathfrak{g}_\pm = (\mathfrak{g}_\pm \cap \mathfrak{k}) \oplus (\mathfrak{g}_\pm \cap \mathfrak{s}) = \mathfrak{k}_\pm \oplus \mathfrak{s}_\pm$. The algebra $\mathfrak{g}_+$ is reductive and $\mathfrak{g}_+ = \mathfrak{k}_+ \oplus \mathfrak{s}_+$ is a Cartan decomposition. The Lie algebra of $K(H) = H \cap K$ is $\mathfrak{k}_+$ and $H = K(H) \exp(\mathfrak{s}_+)$ is a global Cartan decomposition of H. Since H is assumed to be connected, $K(H)$ is connected. Also, K and $K(H)$ is compact if G has finite center. From the global Cartan decomposition of H it follows that H, and hence S, is homotopy equivalent to $K(H)$.

Now, the existence of the invariant cone $W \subset \mathfrak{g}_-$ implies that the symmetric Lie algebra is of the regular type. This means that the centralizer $\mathfrak{z}(\mathfrak{l})$ of the subalgebra $\mathfrak{l} = \mathfrak{k}_+ \oplus \mathfrak{s}_-$ meets $W \cap \mathfrak{s}_-$ nontrivially. The assumption that $\mathfrak{g}$ is simple implies furthermore that $\mathfrak{c} = \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{s}_-$ is one-dimensional and $\mathfrak{l}$ turns out to be the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$ (see Hilgert and Neeb [5] Theorem V.I and Neeb [14], Theorem I.20(3) and Proposition IV.1). Since $\dim \mathfrak{c} = 1$ there exists a maximal abelian $\mathfrak{a} \subset \mathfrak{s}$ such that $\mathfrak{c} \subset \mathfrak{a}$. We can choose a Weyl chamber $\mathfrak{a}^+$ containing $W \cap \mathfrak{c}$ in its closure. Denote by Σ the simple system of roots defined by $\mathfrak{a}^+$ and let $\Theta(\mathfrak{c}) \subset \Sigma$ be the set of simple roots annihilating on $\mathfrak{c}$. Using again that $\dim \mathfrak{c} = 1$, it follows that $\Theta(\mathfrak{c})$ is maximal in Σ, that is, its complementary is a singleton. Hence $P_{\Theta(\mathfrak{c})}$ is a maximal parabolic subgroup, i.e., $\mathbb{B}_{\Theta(\mathfrak{c})}$ is minimal.

It was proved in [19] that $\Theta(S) = \Theta(\mathfrak{c})$. We outline the proof with an approach slightly different from [19]: Note that $\mathfrak{l}$ is the reductive component of the parabolic subalgebra $\mathfrak{p}_{\Theta(\mathfrak{c})}$. Since the Lie algebra of both $K(H)$ and $K(\Theta(\mathfrak{c}))$ are $\mathfrak{k}_+$, it follows that these subgroups coincide, so that $K(\Theta(\mathfrak{c}))$ is contained in S. Now, the orbit $O = HP_{\Theta(\mathfrak{c})}/P_{\Theta(\mathfrak{c})}$ of $P_{\Theta(\mathfrak{c})}$ in $\mathbb{B}_{\Theta(\mathfrak{c})}$ is open and its closure is S-invariant. Hence $C_{\Theta(\mathfrak{c})} = \mathrm{cl} O$. It is known that $\mathrm{cl} O$ is contained in the open Bruhat cell defined by $\mathfrak{a}^+$. Furthermore, S is transitive on the fiber over the base point $P_{\Theta(\mathfrak{c})} \in \mathbb{B}_{\Theta(\mathfrak{c})}$ because $K(\Theta(\mathfrak{c})) \subset S$. Hence $\pi^{-1}\left(C_{\Theta(\mathfrak{c})}\right)$ is the invariant control set in the maximal flag manifold $\mathbb{B}$, ensuring that the type of S is $\Theta(\mathfrak{c})$.

In particular let $S \subset \mathrm{Sl}(n, \mathbb{R})$ be the connected component of the identity semigroup of expansions of a nondegenerate quadratic form in $\mathbb{R}^n$ (see [8] for a detailed discussion of this semigroup). If the matrix of the quadratic form is

$$J = \begin{pmatrix} 1_{k\times k} & 0 \\ 0 & -1_{(n-k)\times(n-k)} \end{pmatrix}$$

then $S = \{g \in \mathrm{Sl}(n, \mathbb{R}) : g^t J g - J \geq 0\}$, where $X \geq 0$ means that the matrix X is positive semi-definite. It follows that if W is the cone composed of the symmetric matrices $X \geq 0$ then $S = \mathrm{SO}(k, n-k)_0 \exp W$. Here $H = \mathrm{SO}(k, n-k)_0$ is the identity component of the subgroup of isometries of J. Its maximal compact subgroup is $K(H) = \mathrm{SO}(k) \times \mathrm{SO}(n-k)$. On the other hand, the type of S is the Grassmannian $\mathrm{Gr}_k(n)$ of k-dimensional subspaces in $\mathbb{R}^n$. This is not hard to check after the remark that the invariant control set of S in $\mathrm{Gr}_k(n)$ is the set of $V \in \mathrm{Gr}_k(n)$ such that the restriction of J to V is positive semi-definite. Of course, the subgroup $K(\Theta)$ associated to $\mathrm{Gr}_k(n)$ is $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$, which is homotopy equivalent to S.

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