Algebras of symmetric holomorphic functions on ℓ_p

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Abstract

Spectra of algebras of holomorphic symmetric functions on ℓ_p are investigated

By a symmetric function defined on ℓ_p we mean a function which is invariant under any reordering of the sequence in ℓ_p. Symmetric polynomials in finite dimensional spaces can be studied in [9] or [11]; in the infinite dimensional Hilbert space they already appear in [10]. Throughout this note $\mathcal{P}_s(\ell_p)$ is the space of symmetric polynomials on a complex space ℓ_p, $1 \leq p < \infty$ and $\mathcal{H}_s(\ell_p)$ is the space of symmetric holomorphic functions on ℓ_p. We will use the notation $\mathcal{H}_{bs}(\ell_p)$, $A_{us}(B_{\ell_p})$ for the algebras of symmetric holomorphic functions which are bounded on bounded sets of ℓ_p and uniformly continuous on the open unit ball B_{ℓ_p} of ℓ_p respectively. The purpose of this paper is the description of such algebras and their spectra. The spectrum of algebras of holomorphic functions on Banach spaces was studied in [1],[2], [7].

1. Algebra of symmetric polynomials

Let X be a Banach space and $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. The sequence $(G_i)_i$ of polynomials is called an algebraic basis of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is $q \in \mathcal{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(G_1(x), \ldots, G_n(x))$, in other words, if G is the mapping $x \in \mathbb{C}^n \rightsquigarrow G(x) := (G_1(x), \ldots, G_n(x)) \in \mathbb{C}^n$, $P = q \circ G$.

Let $< p >$ be smallest integer number that is greater than or equal to p. In [8] is proved that polynomials $F_k(\sum a_i e_i) = \sum a_i^k$ for $k =< p >, < p > +1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$. So if $< p_1 >=< p_2 >$ then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer p. Throughout we will assume that p is an integer number, $1 \leq p < \infty$.

It is well known [9] XI §52 that for $n < \infty$ any polynomial in $\mathcal{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the *elementary symmetric polynomials* $(R_i)_{i=1}^n$, $R_i = \sum_{k_1 < \ldots < k_i} x_{k_1} \ldots x_{k_i}$

Lemma 1.1 *Let $G_1, \ldots, G_n$ be an algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$ $G_i(y) = \xi_i$, $i = 1, \ldots, n$ then $x = y$ up to a permutation.*

Proof. First we suppose that $G_i = R_i$. Then according to Vieta formulas [9] the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \ldots (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x) = \xi_i$ so $x = (x_1, \ldots, x_n)$ as required. Let now G_i be an arbitrary algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials $v_i \in \mathbb{C}^n$. Setting v as the polynomial mapping $x \in \mathbb{C}^n \rightsquigarrow v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \to \mathbb{C}^n$ such that $G = w \circ R$, hence $R = (v \circ w) \circ R$, so $v \circ w = id$. Then v and w are inverse each other since $w \circ v$ coincides with the identity on the open set $Im(w)$. In particular, v is one to one.

Now, the solutions $x_1, \ldots, x_n$ of the equation

$$x^n - v_1(\xi_1) x^{n-1} + \ldots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$

satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, hence $\xi = G(x)$. $\square$

Corollary 1.1. *Given $\xi_1, \ldots, \xi_n \in \mathbb{C}^n$ there is $x \in \ell_p^{n+p-1}$ such that*

$$F_p(x) = \xi_1, \ldots, F_{p+n-1}(x) = \xi_n.$$

Let us say that $x \sim y$, $x, y \in \ell_p$ if there is a permutation T of basis in ℓ_p such that $x = T(y)$. For any point $x \in \ell_p$ δ_x will denote the linear multiplicative functional on $\mathcal{P}_s(\ell_p)$ "evaluation" at x. It is clear that if $x \sim y$ then $\delta_x = \delta_y$.

Theorem 1.1. *Let $x, y \in \ell_p$ and $F_i(x) = F_i(y)$ for every $i > p$. Then $x \sim y$.*

Proof. Call $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$. Without loss of generality, we can assume that $1 = |x_1| = \ldots = |x_k| > |x_{k+1}| \geq \ldots$ and $1 \geq |y_1| \geq |y_2| \geq \ldots$

If $|y_1| < 1$ then for many big j, $|F_j(x)|$ will be close to k while for all big j, $F_j(y)$ will be close to 0. Thus $|y_1| = 1$. Suppose that $1 = |y_1| = \ldots = |y_m| > |y_{m+1}| \geq \ldots$ Claim: $m = k$. Suppose for a contradiction, that $m < k$. Then, for many big j, $|F_j(x)|$ is close to k, while for all big j, $|F_j(y)| < m + 1/2 < k$. This contradiction shows that $m < k$ is fals; similarly, $k < m$ is fals, and so $m = k$.

Let $\tilde{x} = (x_1, \ldots, x_k)$ and $\tilde{y} = (y_1, \ldots, y_k)$. Also, let x^j denote the point $(x_1^j, x_2^j, \ldots)$, etc. We claim that $\tilde{x} \sim \tilde{y}$. Consider the function $f : (S^1)^{2k} \to \mathbb{C}$ given by

$$f(\tilde{u}, \tilde{v}) = f(u_1, \ldots, u_k, v_1, \ldots, v_k) = [u_1 + \ldots + u_k] - [v_1 + \ldots + v_k].$$

Since $F_j(x - \tilde{x})$ and $F_j(y - \tilde{y}) \to 0$ as $j \to \infty$ and since we are assuming that $F_j(x) = F_j(y)$ for all $j \geq p$, it follows that $f(\tilde{x}^j, \tilde{y}^j) \to 0$ as $j \to \infty$. Now, f is obviously a continuous function, and so it follows that for any point $(u, v) \in (S^1)^{2k}$ which is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geq p\}$, $f(u, v) = 0$.

Next, the point $(1, \ldots, 1) \in (S^1)^{2k}$ is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geq p\}$. If $(\tilde{x}^{j_t}, \tilde{y}^{j_t})_t \to (1, \ldots, 1)$, then $(\tilde{x}^{j_t+1}, \tilde{y}^{j_t+1})_t \to (\tilde{x}, \tilde{y})$. Consequently, $f(\tilde{x}, \tilde{y}) = 0$, or in other words $F_1(\tilde{x}) = F_1(\tilde{y})$. Similarly, $F_j(\tilde{x}) = F_j(\tilde{y})$ for all j. From Lemma 1.1 it follows that $\tilde{x} \sim \tilde{y}$. So $F_j(x - \tilde{x}) = F_j(y - \tilde{y})$ for every $j \geq p$ i.e.

$$F_j(0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots) = F_j(0, \ldots, 0, y_{k+1}, y_{k+2}, \ldots)$$

for every $j \geq p$. If $|x_{k+1}| = 0$ and $|y_{k+1}| = 0$ then $x_i = 0$ and $y_i = 0$ for $i > k$. Let $|x_{k+1}| = a \neq 0$ then we can repeat the above argument for vectors $x' = (x_{k+1}/a, x_{k+2}/a, \ldots)$ and $y' = (y_{k+1}/a, y_{k+2}/a, \ldots)$ and by induction we will see that $x \sim y$. $\Box$

Corollary 1.2. *Let $x, y \in \ell_p$ and for arbitrary integer $m \geq p$ $F_i(x) = F_i(y)$ for each $i \geq m$. Then $x \sim y$.*

Proof. Since $m \geq p$ then $x, y \in \ell_m$ and from Theorem 1.1 it follows that $x \sim y$ in ℓ_m. So $x \sim y$ in ℓ_p. $\Box$

Lemma 1.2. *Let $P_1, \ldots, P_m \in \mathcal{P}_s(\ell_p)$ and $\ker P_1 \cap \ldots \cap \ker P_m = \emptyset$. Then there is $Q_1, \ldots, Q_m \in \mathcal{P}_s(\ell_p)$ such that*

$$\sum_{i=1}^{m} P_i Q_i \equiv 1.$$

Proof. Let $n = \max_i(\deg P_i)$. Then we can assumed that $P_i(x) = g_i(F_p(x), \ldots, F_n(x))$ for some $g_i \in \mathcal{P}(\mathbb{C}^{n-p+1})$. Let us suppose that at some point $\xi \in \mathbb{C}^{n-p+1}$, $\xi = (\xi_1, \ldots, \xi_{n-p+1})$ $g_i(\xi) = 0$. Then by Corollary 1.1 there is $x_0 \in \ell_p$ such that $F_i(x_0) = \xi_i$. So the common set of zero of all g_i is empty. Thus by Hilbert Nullstellensatz there is $q_1, \ldots, q_m$ such that $\sum_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(F_p(x), \ldots, F_n(x))$. $\Box$

2. Finitely generated symmetric algebras

Below it will be shown that in general, the spectrum of the algebra generated by symmetric polynomials is not exhausted by point evaluation functionals. Now we consider a special case, where they coincide.

Let us denote by $\mathcal{P}_s^n(\ell_p)$, $n \geq p$ the subalgebra of $\mathcal{P}_s(\ell_p)$, generated by $\{F_p, \ldots, F_n\}$; one easily realizes, by appealing to Corollary 1.1, that $\mathcal{P}_s^n(\ell_p) \cap \mathcal{P}(^k\ell_p)$, is a *sup-norm* closed subspace of $\mathcal{P}(^k\ell_p)$ for every $k \in \mathbb{N}$.

Let $A_{us}^n(B_{\ell_p})$ and $\mathcal{H}_{bs}^n(\ell_p)$ be the closed subalgebras of $A_{us}(B_{\ell_p})$ and $\mathcal{H}_{bs}(\ell_p)$ generated by $\{F_p, \ldots, F_n\}$, that is the closure of $\mathcal{P}_s^n(\ell_p)$ in each of the corresponding algebras. Note that for any $f \in \mathcal{H}_{bs}^n(\ell_p)$ with $f = \sum P_k$, the Taylor series expansion of f at 0, we have $P_k \in \mathcal{P}_s^n(\ell_p)$. Indeed, if $f \in \mathcal{P}_s^n(\ell_p)$, is immediate that $P_k \in \mathcal{P}_s^n(\ell_p) \cap \mathcal{P}(^k\ell_p)$ for all k. Then the same holds for any $f \in \mathcal{H}_{bs}^n(\ell_p)$ by recalling the continuity of the map which assigns to a holomorphic function its k^{th} Taylor polynomial.

By [6] III. 1.4, we may, and we do, identify the spectrum of $A_{us}^n(B_{\ell_p})$ with the joint spectrum of $\{F_p, \ldots, F_n\}$, $\sigma(F_p, \ldots, F_n)$.

Let us denote by $\mathcal{F}_p^n$ a mapping from ℓ_p to $\mathbb{C}^{n-p+1}$ such that $\mathcal{F}_p^n : x \mapsto (F_p(x), \ldots, F_n(x))$. Then $D_p^n := \mathcal{F}_p^n(B_{\ell_p})$ is a subset of the unit disk D of $\mathbb{C}^{n-p+1}$.

Let K be a bounded set in $\mathbb{C}^n$. Recall that a point x belongs to the *polynomial convex hull* of $K, [K]$, if for every polynomial f , $|f(x)| \leq \sup_{z \in K} |f(z)|$. A set is *polynomially convex* if it coincides with its polynomial convex hull. Recall that the sup norm on K of a polynomial coincides with the sup norm on $[K]$. It is well known (see f.e. [6]) that the spectrum of the uniform Banach algebra $P(K)$ generated by polynomials on the compact set K coincides with the polynomially convex hull of this set. Let $[D_p^n]$ be the polynomial convex hull of D_p^n. We denote as usual $P([D_p^n])$ for the algebra of continuous functions in $[D_p^n]$ which are uniformly approximable by polynomials.

Theorem 2.1.
(i) The composition operator $C_{\mathcal{F}_p^n} : \mathcal{H}_b(\mathbb{C}^{n+1-p}) \to \mathcal{H}_{bs}^n(\ell_p)$ *given by* $C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n$ *is a topological isomorphism.*
(i′) The composition operator $C_{\mathcal{F}_p^n} : P([D_p^n]) \to A_{us}^n(B_{\ell_p})$ *given by* $C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n$ *is a topological isomorphism.*
(ii) $\mathcal{M}(\mathcal{H}_{bs}^n(\ell_p)) = \mathbb{C}^{n+1-p}$.
(ii′) $\mathcal{M}(A_{us}^n(B_{\ell_p})) = [D_p^n]$.

Proof. Clearly the composition operators are well defined and one to one, so it remains to prove that they are onto.

In (i), let $f \in \mathcal{H}_{bs}^n(\ell_p)$ and $f = \sum P_k$ be the Taylor series expansion of f at 0. Since $P_k \in \mathcal{P}_s^n(\ell_p)$, there is a homogeneous polynomial $g_k \in \mathcal{P}(\mathbb{C}^{n+1-p})$ such that $P_k(x) = g_k(F_p(x), \ldots, F_n(x))$. Put $g(\xi_1, \ldots, \xi_{n-p+1}) = \sum_{k=1}^{\infty} g_k(\xi_1, \ldots, \xi_{n-p+1})$; since g is a convergent power series in each variable, it is separately holomorphic, hence holomorphic. Note that $f = g \circ \mathcal{F}_p^n$.

In (i'), observe that for any $g \in P([D_p^n])$, $||C_{\mathcal{F}_p^n}(g)|| = \sup_{x\in B_{\ell_p}} |g \circ \mathcal{F}_p^n(x)| = ||g||_{D_p^n} = ||g||_{[D_p^n]}$. Thus $C_{\mathcal{F}_p^n}$ is an open mapping, hence its range is a closed subspace, which moreover contains $\mathcal{P}_s^n(\ell_p)$, therefore $C_{\mathcal{F}_p^n}$ is onto $A_{us}^n(B_{\ell_p})$.

(ii) and (ii') it follow from (i), (i') and the theory of entir functions of several variables [**?**] and the theory of uniform algebras [6]. □

Lemma 2.1. *If* $(\xi_1^0, \dots, \xi_m^0) \in [D_p^m]$ *and* $n < m$ *then* $(\xi_1^0, \dots, \xi_n^0) \in [D_p^n]$.

Proof. If $(\xi_1^0, \dots, \xi_n^0) \notin [D_p^n]$, there is a polynomial of n variables such that

$$|q(\xi_1^0, \dots, \xi_n^0)| > \sup_{(\xi_1,\dots,\xi_n)\in D_p^n} |q(\xi_1, \dots, \xi_n)|.$$

Consider a polynomial $\tilde{q}$ of m variables such that $\tilde{q}(\xi_1, \dots, \xi_m) = q(\xi_1, \dots, \xi_n)$, i.e., the composition of q and the projection onto the first n variables. Then,

$$\sup_{(\xi_1,\dots,\xi_m)\in D_p^m} |\tilde{q}(\xi_1, \dots, \xi_m)| = \sup_{x\in B_{\ell_p}} |\tilde{q}(F_p(x), \dots, F_{p+m-1}(x))| =$$

$$\sup_{x\in B_{\ell_p}} |q(F_p(x), \dots, F_{p+n-1}(x))| < |q(\xi_1^0, \dots, \xi_n^0)| = |\tilde{q}(\xi_1^0, \dots, \xi_m^0)|.$$

But this means $(\xi_1^0, \dots, \xi_m^0) \notin [D_p^m]$, a contradiction. □

3. Spectrum of $A_{us}(B_{\ell_p})$

For studying the spectrum of $A_{us}(B_{\ell_p})$ the most decisive feature is that the polynomials $\{F_p^n\}_{n=p}^{\infty}$ generate a dense subalgebra. Actually for every $f \in A_{us}(B_{\ell_p})$ its Taylor polynomials are easily seen to be symmetric by a uniqueness argument.

First we will show that the spectrum of the uniform algebra of symmetric holomorphic functions on B_{ℓ_p} does not coincide with the point evaluation functionals.

Example 3.1. For every n put $v_n = \frac{1}{n^{1/p}}(e_1, \dots, e_n) \in B_{\ell_p}$. Then $\delta_{v_n}(F_p) = 1$ and $\delta_{v_n}(F_j) \to 0$ as $n \to 0$ for every $j \geq p$. By compactness of the $\mathcal{M}(A_{us}(B_{\ell_p}))$ there is an accumulation point ϕ of the sequence $\{\delta_{v_n}\}$. Then $\phi(F_p) = 1$ and $\phi(F_j) = 0$ for all $j > p$. From Corollary 1.2 it follows that there is no point z in ℓ_p such that $\delta_z = \phi$.

Let us denote by $\Sigma_p := \{(a_i)_{i=p}^{\infty} \in \ell_\infty : (a_i)_{i=p}^n \in [D_p^n]\}$. As a consequence of Lemma 2.1, Σ_p is the *limit of the inverse sequence* ([5] 2.5) of $\{[D_p^n], \pi_n^m, \mathbb{N}\}$ where π_n^m is the projection onto the first n coordinates. When Σ_p is endowed with the product topology, that is, the coordinatewise convergence, it is a non-empty compact Hausdorff space by [5] 3.2.13. Σ_p is a weak-star compact subset of the closed unit ball ℓ_∞ since the weak star topology and the pointwise convergence topology coincide on the closed unit ball of ℓ_∞.

Now we describe the spectrum of $A_{us}(B_{\ell_p})$.

Theorem 3.1. *Σ_p is homeomorphic to the spectrum of $A_{us}(B_{\ell_p})$.*

Proof. First of all observe that any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$, is completely determined by the sequence of values $\{\Psi(F_n)\}$ since Ψ is determined by its behaviour on $\mathcal{P}_s(\ell_p)$, the algebra generated by $\{F_n\}$, which in turn is dense in $A_{us}(B_{\ell_p})$.

We construct an embedding

$$j : (a_i)_{i=p}^\infty \in \Sigma_p \rightsquigarrow \Phi \in \mathcal{M}(A_{us}(B_{\ell_p})),$$

and prove that it is a homeomorphism. Given $(a_i)_{i=p}^\infty \in \Sigma_p$ a homomorphism $j[(a_i)_{i=p}^\infty] := \Phi$ on $A_{us}(B_{\ell_p})$ is defined in the following way: Every polynomial $P \in \mathcal{P}_s(\ell_p)$ may be written as $g \circ \mathcal{F}_p^n$ for some $n \in \mathbb{N}$, thus we may define, and we do, $\Phi(P) = g(a_p, \ldots, a_n)$. Certainly $\Phi(P)$ is well defined since if $P = h \circ \mathcal{F}_p^m$ for some other polynomial h, and, say, $m > n$, then by Corollary 1.1, $h = \tilde{g}$, where $\tilde{g}$ has the same meaning as in Lemma 2.1. Hence $g(a_p, \ldots, a_n) = \tilde{g}(a_p, \ldots, a_n, \ldots, a_m) = h(a_p, \ldots, a_n, \ldots, a_m)$. It is easy now to see that Φ is linear and multiplicative on the subalgebra of symmetric polynomials. Also $|\Phi(P)| = |g(a_p, \ldots, a_n)| \leq ||g||_{[D_p^n]} = ||g||_{D_p^n} \leq ||P||$, therefore Φ is uniformly continuous on $\mathcal{P}_s(\ell_p)$, hence it has a continuous linear and multiplicative extension to the closure of $\mathcal{P}_s(\ell_p)$ that is, to $A_{us}(B_{\ell_p})$. We still denote this extension by Φ.

Obviously, j is one to one. Moreover j is also an onto mapping: Indeed, for any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$, the sequence $\{\Psi(F_n)\} \in \Sigma_p$ because $\{\Psi(F_n)_{n=p}^{n=m}\}$ is an element of the joint spectrum of $\mathcal{M}(A_{us}^m(B_{\ell_p}))$ (obtained just by taking the restriction of Ψ to $A_{us}^n(B_{\ell_p})$) which we know to be $[D_p^m]$; of course, $j[\{\Psi(F_n)\}] = \Psi$ since they coincide on each F_n.

This embedding is continuous since the $\mathcal{M}(A_{us}(B_{\ell_p}))$ is an equicontinuous subset of the dual space $(A_{us}(B_{\ell_p}))^*$, so the weak-star topology coincides on it with the topology τ of pointwise convergence on the elements of the dense set of all symmetric polynomials, hence on the generating system $\{F_n\}_{n=p}^\infty$.

Finally j is a homeomorphism as a bijection between two compact Hausdorff spaces. □

We can view Σ_p as *"the joint spectrum"* of the sequence $\{F_n\}_{n=p}^\infty$, since $\Phi(F_n) = a_n$. Note that $\mathcal{F}_p(B_{\ell_p}) \subset \Sigma_p$.

Remark 3.1. *The set $D_p = \mathcal{F}_p(B_{\ell_p}) \subset B_{\ell_\infty}$ pointwise coincides with the set of point evaluation multiplicative functionals on $A_{us}(B_{\ell_p})$.*

It is clear that $\overline{D_p^n} \subset [D_p^n] \subset \overline{D}$ but we do not know whether this embeddings are proper. This is related to a corona type theorem for $A_{us}(B_{\ell_p})$ since D_p *is dense in* Σ_p *if* $\overline{D_p^n} = [D_p^n]$ $\forall n \in \mathbb{N}$.

Note that if $q > p$ (where q and p are not necessarily integer) then $D_p \subset D_q$ and the inclusion is strict. Indeed, let $x \in B_{\ell_q}$ so that $x \notin \ell_p$. If $\mathcal{F}_q(y) = \mathcal{F}_p(x)$ for some $y \in \ell_q$ then $x \sim y$ in ℓ_q so in ℓ_p.

Proposition 3.1.- *Σ_p is the polynomial convex hull of $D_p \subset (\ell_\infty, \tau_p)$.*

Proof. Let $(a_i)_{i=p}^\infty \in \ell_\infty$ such that $|P((a_i))| \leq ||P||_{\Sigma_p}$ for all polynomials $P \in \mathcal{P}(\ell_\infty)$. For any $n \geq p$ and any $g \in \mathcal{P}(\mathbb{C}^{n+1-p})$, the mapping Q given by $(x_i)_{i=p}^\infty \in \ell_\infty \rightsquigarrow g(x_p, \ldots, x_n)$ is a polynomial in ℓ_∞, hence

$$|g(a_p, \ldots, a_n)| = |Q((a_i))| \leq ||Q||_{\Sigma_p} \leq ||g||_{[D_p^n]}.$$

Therefore $(a_p, \ldots, a_n) \in [D_p^n]$, as we want and Σ_p is polynomially convex. So to finish, it is enough to check that Σ_p is contained in the polynomial convex hull of D_p. To do this, let now $(a_i)_{i=p}^\infty \in \Sigma_p$ and $P \in \mathcal{P}((\ell_\infty, \tau_p))$. As P is τ_p continuous, it depends on a finite number of variables, say $x_p, \ldots, x_n$, thus the mapping q given by $(x_p, \ldots, x_n) \rightsquigarrow P(x_p, \ldots, x_n, 0, \ldots, 0, \ldots)$ is a polynomial in $\mathbb{C}^{n+1-p}$ and as $(a_p, \ldots, a_n) \in [D_p^n]$,

$$|P((a_i))| = |P(a_p, \ldots, a_n, 0, \ldots, 0, \ldots)| = |q(a_p, \ldots, a_n)| \leq ||q||_{[D_p^n]} = ||q||_{D_p^n} \leq ||P||_{D_p},$$

it follows that $(a_i)_{i=p}^\infty$ belongs to the polynomially convex hull of D_p. □

Theorem 3.2. *There is an algebraic and topological isomorphism between $A_{us}(B_{\ell_p})$ and a uniform Banach algebra generated by $w^*(\ell_\infty, \ell_1)$ continuous functionals on Σ_p.*

Proof. For every $f \in A_{us}(B_{\ell_p})$ and $\phi \in \mathcal{M}(A_{us}(B_{\ell_p}))$ denote by $\hat{f}(\phi) = \phi(f)$ the Gelfand transform which is known to be an algebraic isometry into $C(\Sigma_p)$. Recall that the range of the Gelfand transform is a closed subalgebra which, as we are going to see, will coincide with $\mathcal{A}$, the uniform Banach subalgebra of $C(\Sigma_p)$ generated by the coordinate functionals $\{\pi_k\}_{k=p}^\infty$.

Since $\hat{F}_k(\xi) = \xi_k$ for $\xi = (\xi_i)_i \in \Sigma_p$, it follows that the Gelfand transform of $\hat{F}_k$ is the k^{th} coordinate functional on ℓ_∞. As $A_{us}(B_{\ell_p})$ is the closure of the algebra generated by $\{F_k : k \geq p\}$, it follows that $\hat{f} \in \mathcal{A}$, for every $f \in A_{us}(B_{\ell_p})$. Therefore $\mathcal{A}$ is precisely the range of the Gelfand transform. □

We do not know what is closure of Δ_p in weak-star topology. But it is easy to see that if $\mathcal{M}(A_{us}(B_{\ell_p})) \supset B_{\ell_1}$ then $\mathcal{M}(A_{us}(B_{\ell_p})) = B_{\ell_\infty}$.

Questions.

1. What is polynomially convex hull of D_p^n?
2. What is spectrum of $\mathcal{H}_{bs}(B_{\ell_p})$?

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