EQUIHARMONIC TORI IN FLAG MANIFOLDS

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Abstract. Exploring the geometry of invariant $f-$structures and $f-$holomorphic curves on flag manifolds, we construct equiharmonic tori on full complex flag manifolds which are not $f-$holomorphic for any invariant $f-$structure.

§0. Introduction

Theory of harmonic maps from surfaces into (non)-symmetric flag manifolds turns out to be more difficult and subtle than the corresponding theory for symmetric spaces, in part because of the profusion of invariant metrics available. However, a distinguished class, the equiharmonic maps (harmonic with respect to all invariant metrics) have been noticed by a number of authors[Bl,Bu2,M,BPW,Ud,Uh]. Let $\phi : M \to CP^n$ be a full holomorphic curve from Riemannian surface and $\phi_0, \cdots, \phi_{n-1}$ is its harmonic sequence. The second author has in [N] showed that $\Phi := (\phi_0, \cdots, \phi_{n-1}) : M \to F(n)$ is an equiharmonic map into full flag manifold.

The equiharmonic maps that have arisen so far in the literature are, in fact, all solutions to a first order Cauchy-Riemann type system–they are "f-holomorphic with repect to a horizontal f-structure" in the sense of Black[Bl].

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Conversely, an unexpected result due to Black asserts that any equiminimal (i.e. minimal for each invariant metric on the co-domain) map of a Riemannian surface into a full flag manifold is f-holomorphic with respect to a horizontal f-structure.

Notice that equiminimality is equivalent to equiharmonicity and equiconformality. This raised the question of whether or not hypothesis of equiconformality in Black's result is essential. In §4 of this manuscript, we show that this is tha case by constructing concrete examples and show in fact

There exist two families of equiharmonic maps from the tori to the full complex flag manifold $F(n)$ which are not $f-$holomorphic for any invariant $f-$structure (Theorem 4.3 and Theorem 4.4).

We begin by show how the geometry of invariant $f-$structures on non-symmetric flag manifolds is encoded in the ϵ-matrices. As a bonus, we show how the ϵ-matrices can be used to derive the second author's result mentioned above.

§1. f-STRUCTURES ON COMPLEX FLAG MANIFOLDS

In this section, we will establish one-one correspondence between invariant $f-$structures and skew-symmetric matrices valued in $\{-1,\, 0,\, 1\}$(Theorem 1.4).

Consider the complex flag manifold

$$F(r_1,\, \cdots,\, r_n;\, N) = \frac{U(N)}{U(r_1)\times\cdots\times U(r_n)}$$

where $r_1+\cdots+r_n = N$. $F(r_1,\, \cdots,\, r_n;\, N)$ is a completely reductive homogeneous space with reductive splitting[Bl]

$$u(N) = [u(r_1)+\cdots+u(r_n)] \oplus [\oplus_{i<j} m_{ij}] \tag{1.1}$$

where $m_{ij} = \{A = (A_{kl}) \in u(N)|,\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j) \text{ and } (j,i)\}$ is the isotropy representation and $A_{kl} \in gl(r_k \times r_l;\, C)$. It will be necessary to consider the complexified version of (1.1). We have

$$m_{ij}^{\mathbb{C}} = \{A = (A_{kl}) \in gl(N;\,\mathbb{C})|\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j) \text{ and } (j,i)\} = E_{ij} \oplus E_{ji}$$

and

$$E_{ij} := \{A = (A_{kl}) \in gl(N,\,\mathbb{C})|\, A_{kl} = 0 \text{ if } (k,l) \neq (i,j)\}$$

is $U(r_1)\times\cdots\times U(r_n)$ invariant and irreducible. For arbitrary $A=(A_{kl})\in E_{ij}$ we have $A=(A^1_{kl})+\sqrt{-1}(A^2_{kl})$ where

$$A^1_{kl}=\begin{cases}\frac{A_{ij}}{2} & (k,l)=(i,j)\\ -\frac{\bar{A}^t_{ij}}{2} & (k,l)=(j,i)\\ 0 & \text{otherwise}\end{cases},\qquad A^2_{kl}=\begin{cases}\frac{A_{ij}}{2\sqrt{-1}} & (k,l)=(i,j)\\ -\frac{\bar{A}^t_{ij}}{2\sqrt{-1}} & (k,l)=(j,i)\\ 0 & \text{otherwise}\end{cases}$$

so it is easy to see that $\bar{A}=(A^1_{kl})-\sqrt{-1}(A^2_{kl})\in E_{ji}$ and vice versa, we get $\overline{E}_{ij}=E_{ji}$, and $[\oplus_{i<j}m_{ij}]^{\mathbb{C}}=\oplus_{i\neq j}E_{ij}$

Definition 1.1[Y]. An f-structure on

$$F=F(r_1,\cdots,r_n;\,N)$$

is a section $\mathcal{F}$ of $End(TF(r_1,\cdots,r_n;\,N))$ such that $\mathcal{F}^3+\mathcal{F}=0$.

Definition 1.2. An ϵ matrix, denoted by (ϵ_{ij}), is an $n\times n$ skew-symmetric matrix with values in $\{1,\,0,\,-1\}$.

An $U(N)$ invariant f-structure on $F(r_1,\,\cdots,\,r_n;\,N)$ may be identified with an H equivariant endomorphism,$\mathcal{F}$, of $\oplus_{i<j}m_{ij}$ such that $\mathcal{F}^3+\mathcal{F}=0$ where $H=U(r_1)\times\cdots\times U(r_n)$. Using Schur's Lemma [Bl, Page 15] all the $U(N)$-invariant f-structures may be constructed as following: let (ϵ_{ij}) be an ϵ-matrix and define

$$\begin{aligned}\sqrt{-1}\quad&\text{eigenspace of}\quad \mathcal{F}=\oplus_{\epsilon_{ij}=1}E_{ij}\\ -\sqrt{-1}\quad&\text{eigenspace of}\quad \mathcal{F}=\oplus_{\epsilon_{ij}=1}\overline{E}_{ij}=\oplus_{\epsilon_{ij}=-1}E_{ij}\\ 0\quad&\text{eigenspace of}\quad \mathcal{F}=\oplus_{\epsilon_{ij}=0}E_{ij}\end{aligned}\tag{1.2}$$

Determining the eigenspaces in this way defines an H equivariant endomorphism $\mathcal{F}$ of $\oplus_{i\neq j}E_{ij}$, which is seen to be the C-linear extension of an H equivariant endomorphism of $\oplus_{i<j}m_{ij}$ since $\oplus_{\epsilon_{ij}=1}E_{ij}$ and $\oplus_{\epsilon_{ij}=-1}E_{ij}$ are conjugate and $[\oplus_{\epsilon_{ij}=1}E_{ij}]\cap[\oplus_{\epsilon_{ij}=-1}E_{ij}]=\{0\}$.

Definition 1.3[Bu1]. The complex dimension of $\sqrt{-1}$ eigenspace of $\mathcal{F}$ is the rank of $\mathcal{F}$.

Suppose that f-structure $\mathcal{F}$ is defined by $\epsilon(\mathcal{F})=(\mathcal{F}_{ij})$ then

$$\begin{aligned}\text{rank}\mathcal{F}:&=\dim_{\mathbb{C}}[\sqrt{-1}\quad\text{eigenspace of}\quad\mathcal{F}]\\ &=\dim_{\mathbb{C}}\oplus_{\mathcal{F}_{ij}=1}E_{ij}\\ &=\Sigma_{\mathcal{F}_{ij}=1}\dim_{\mathbb{C}}E_{ij}=\Sigma_{\mathcal{F}_{ij}=1}r_ir_j\end{aligned}$$

In particular, if 2 rank$\mathcal{F} = \dim F(r_1, \cdots, r_n;\, N)$, i.e. $2\Sigma_{\mathcal{F}_{ij}=1} r_i r_j = N^2 - r_1^2 - \cdots - r_n^2 = \Sigma_{i \neq j} r_i r_j$. It follows that 0 eigenspace $= \{0\}$, so $\mathcal{F}$ is an almost complex structure. And vice versa. This leads to

Theorem 1.4. *There is a* $1:1$ *correspondence between* $U(N)$ *invariant* f*-structure* $\mathcal{F}$ *on* $F(r_1, \cdots, r_n;\, N)$ *and* ϵ*-matrices* $\epsilon(\mathcal{F}) = (\mathcal{F}_{ij})$ *such that*

$$\textit{rank } \mathcal{F} = \Sigma_{\mathcal{F}_{ij}=1} r_i r_j = \Sigma_{\mathcal{F}_{ij}=-1} r_i r_j = \frac{1}{2} \Sigma_{\mathcal{F}_{ij} \neq 0} r_i r_j$$

and $\mathcal{F}$ *is almost complex structure if and only if* $\mathcal{F}_{ij} \neq 0$ *for any* $i \neq j$.

We can now generalize to our case the result in[BH] concerning invariant almost complex structures.

Corollary 1.5. *There are* $3^{\binom{n}{2}} U(N)$*-invariant* f*-structures on a complex flag manifold* $F(r_1, \cdots, r_n; N)$ *obtained by choosing* ϵ_{ij} *for each* $(i < j)$.

§2. ϵ-MATRICES AND f-HOLOMORPHIC CURVES

In this section we will characterize f-holomorphicity of a smooth map from a Riemannian surface to a complex flag manifold in terms of ϵ-matrices.

Let M be a Riemannian surface. We shall consider the trivial vector bundle $\underline{C}^N$ over M whose fibre C^N continues to be endowed with the standard Hermition inner product. Consider a collection $(\mathcal{E}_i)_{1 \leq i \leq n}$ of mutually orthogonal subbundles of $\underline{C}^N$ with fibres C^{r_i} such that $\Sigma_{i=1}^n r_i = N$. Hence $\underline{C}^N = \oplus_{i=1}^n \mathcal{E}_i$ and we have simply defined a map $\phi : M \to F(r_1, \cdots, r_n; N)$ into a flag manifold. We shall call the collection $(\mathcal{E}_i)$ of subbundles describing ϕ a moving flag[BS]. Set $A'_{ij} = \pi_j \circ \frac{\partial}{\partial z} \circ \pi_i$ where π_i denotes orthogonal projection onto $\mathcal{E}_i$. For $i \neq j$ these quantities constitute the second fundamental forms of the $\mathcal{E}_i$.

Remark. The notations of the second fundamental forms are adopted here differ from those of Burstall and Salamon[BS], where A'_{ij} defines a homomorphism from $\mathcal{E}_j$ to $\mathcal{E}_i$.

Definition 2.1. A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is said to be subordinate to an ϵ-matrix (ϵ_{ij}) if $\epsilon_{ij} \neq 1$,and $i \neq j \Rightarrow A'_{ij} = 0$.

The following result generalize slightly Proposition 2 in [BS].

Proposition 2.2. *A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is f-holomorphic relative to an invariant f-structure $\mathcal{F}$ on F if and only if it is subordinate to $\epsilon(\mathcal{F})$.*

Proof. A map $\phi : M \to F(r_1, \cdots, r_n; N)$ is *f-holomorphic* if and only if $d\phi$ interwines the f-structures, i,e,

$$d\phi \circ J = F \circ d\phi \tag{2.1}$$

where J is the standard complex structure on Riemann surface. It is easy to see that (2.1) holds if and only if $d\phi(\frac{\partial}{\partial z}) \in \sqrt{-1}-$eigenspace of $\mathcal{F}$[R,p.90]. Notice that the Maurer-Cartan form gives the familiar isomorphism

$$\phi^{-1}TF(r_1, \cdots, r_n; N)^C = \oplus_{i \neq j} \bar{\mathcal{E}}_i \mathcal{E}_j = \oplus_{i \neq j} \mathrm{Hom}(\mathcal{E}_i,\, \mathcal{E}_j)$$

Furthermore under this isomorphism the component of $d\phi(\frac{\partial}{\partial z})$ in $\mathrm{Hom}(\mathcal{E}_i,\, \mathcal{E}_j)$ is A'_{ij}[BS]. By the conjugation it is clear to see that the subspace E_{ij} (see §1) corresponds to $\bar{\mathcal{E}}_i \mathcal{E}_j = \mathrm{Hom}(\mathcal{E}_i,\, \mathcal{E}_j)$, which combine with (1.2) we have $\sqrt{-1}$ eigenspace of $\mathcal{F} = \oplus_{\mathcal{F}_{ij}=1} \mathrm{Hom}(\mathcal{E}_i,\, \mathcal{E}_j)$ where $(\mathcal{F}_{ij}) = \epsilon(\mathcal{F})$. It follows that ϕ is $f-$ holomorphic related to $\mathcal{F}$ if and only if $\mathcal{F}_{ij} \neq 1, i \neq j \Rightarrow A'_{ij} = 0$

We say that an invariant f-structure $\mathcal{F}$ is *horizontal* if it satisfies $[\mathcal{F}_+,\, \mathcal{F}_-] \subset h$ where $\mathcal{F}_\pm = \oplus_{\epsilon_{ij}=\pm 1} E_{ij}$ and $h = u(r_1) + \cdots + u(r_n)$. Notice that maps which are f-holomorphic with respect to a horizontal f-structure are equiharmonic (i,e, harmonic for all invariant metrics) and equiharmonicity is preserved by homogeneous projections[Bl][Bu2]. Combine with Proposition 2.2 we have

Corollary 2.3. *Suppose that $\phi : M^2 \to F(r_1, \cdots, r_n; N)$ is subordinate to an $\epsilon-$matrix associated to a horizontal f-structure. Then $\phi = (\phi_1, \cdots, \phi_n)$ is an equiharmonic map and each $\phi_j : M^2 \to G_{r_j,N}$ is a harmonic map into complex Grassmannian for $j = 1, 2, \cdots, n$.*

Let $f : M^2 \to CP^{n-1}$ be a full holomorphic curve. A famous theorem due to Eells-Wood tells us that $\phi = \phi_r =: f_{r-1}^{\perp} \cap f_r$ is a full harmonic map into CP^{n-1} for any $r \in \{0,\, 1,\, \cdots,\, n-1\}$ where $f_\alpha : M^2 \to G_{\alpha+1,\,n}$ is α-th associate curve of f and z is the local complex coordinate on M^2. ϕ, ϕ_0, ϕ_1, $\cdots$, ϕ_{n-1} and $\Phi := (\phi_0,\, \phi_1,\, \cdots,\, \phi_{n-1})$ are called the *isotropic* map, the *harmonic sequence* and the *Eells-Wood* map respectively.

Corollary 2.4[N]. *The Eells-Wood maps: $\Phi : M^2 \to F(n)$ are equiharmonic.*

Proof. Let $\Phi := (\phi_0,\, \phi_1,\, \cdots,\, \phi_{n-1}) M^2 \to F(n)$ be an Eells-Wood map. Then we have diagram [BW]:

$$\underset{\phi_0}{\cdot} \longrightarrow \underset{\phi_1}{\cdot} \longrightarrow \cdots \longrightarrow \underset{\phi_{n-1}}{\cdot} \longrightarrow 0$$

Hence Φ is subordinate to the $\epsilon-$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & & & \cdots & 0 \\ -1 & 0 & 1 & 0 & & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & & 1 \\ 0 & & \cdots & 0 & -1 & & 0 \end{pmatrix} \tag{2.2}$$

On the other hand using a direct calculation ones get

$$[E_{ij},\, E_{kl}] = \begin{cases} 0 & \text{if} \quad i,\, j,\, k,\, l \quad \text{are distinct or} \quad j \neq l \\ E_{il} & \text{if} \quad j = k,\, i \neq l \\ E_{ii} - E_{jj} & \text{if} \quad j = k,\, i = l \end{cases}$$

It follows that (2.2) is associated to a horizontal f-structure, and Φ is equiharmonic from Corollary 2.3.

§3. Harmonic equations of closed surfaces on full complex flag manifolds

¿From this section, we restrict ourselves to full complex flag manifolds i.e. $F(n) := F(\underbrace{1, \cdots, 1}_{n}; n)$. Let $\phi : M \to F(n)$ be a smooth map from a Riemannian surface M with moving flag $(\mathcal{E}_i)$ and $\tilde{\phi} : M \to U(n)$ its lift map, i.e. $\phi = \pi \circ \tilde{\phi}$ where $\pi : U(n) \to F(n)$ is the natural projection. Let $e_1, \cdots, e_n$ be the standard basis in $\mathbb{C}^n$, i.e. $e_j = (0, \cdots, 0, 1, 0 \cdots, 0)^t$ We denote $\Pi_j : M \to gl(n, C)$ the matrix of the orthogonal projection onto E_j with respect to $e_1, \cdots, e_n$ (ref. §2). Then $\Pi_i \frac{\partial \Pi_j}{\partial z}$, denoted by A_z^{ij}, are the matrices of second fundamental forms A'_{ji}, i.e.

$$A'_{ji}(e_1, \cdots, e_n) = (e_1, \cdots, e_n) A_z^{ij} \tag{3.1}$$

For $V \in \Gamma(\phi^* TF(n))$, we set $q = \phi^* \beta(V)$ where $\phi^* \beta : \phi^* F(n) \to M \times u(n)$ is the pull-back of Maurer-Cartan form. Define the variation of ϕ by

$$\phi_t(x) := \pi \left(exp(-tq) \tilde{\phi} \right) \tag{3.2}$$

Denote associate objects by $\Pi_j(t)$, $A_z^{ij}(t)$ etc. Then we have

Lemma 3.1. *1).*$\frac{\partial}{\partial t}|_{t=0}\Pi_j(t) = [\Pi_j, q]$; *2).*$\frac{\partial}{\partial t}|_{t=0}A_z^{ij}(t) = [A_z^{ij},\, q] - \Pi_i \frac{\partial q}{\partial z}\Pi_j$

Proof. 1). From (3.1) we have

$$\Pi_j = \tilde{\phi} E_j \tilde{\phi}^* \tag{3.3}$$

where E_j will denote the matrix which has 1 in the (j,j)-position and zero elsewhere. Together with (3.2) one gets $\Pi_j(t) = e^{-tq}\Pi_j e^{tq}$. Hence $\frac{\partial}{\partial t}|_{t=0}\Pi_j(t) = -q\Pi_j + \Pi_j q = [\Pi_j, q]$.

2).It is easy to show that $\frac{\partial}{\partial z}[\Pi_j,\, q] = [\frac{\partial \Pi_j}{\partial z},\, q] + [\Pi_j,\, \frac{\partial q}{\partial z}]$. Combine with 1) we have

$$\begin{aligned}
\frac{\partial}{\partial t}|_{t=0}A_z^{ij}(t) &= \frac{\partial}{\partial t}|_{t=0}[\Pi_i(t)\frac{\partial \Pi_j(t)}{\partial z}] \\
&= \left(\frac{\partial}{\partial t}|_{t=0}\Pi_i(t)\right)\frac{\partial \Pi_j}{\partial z} + \Pi_i \frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}|_{t=0}\Pi_j(t)\right) \\
&= [\Pi_i,\, q]\frac{\partial \Pi_j}{\partial z} + \Pi_i\frac{\partial}{\partial z}[\Pi_j,\, q] = [A_z^{ij},\, q] - \Pi_i\frac{\partial q}{\partial z}\Pi_j
\end{aligned}$$

where notice that $\Pi_i\Pi_j = 0$ whenever $i \neq j$. Q.E.D.

The inner product on $gl(n,\, C)$ is defined by

$$< A,\, B > := tr(AB^*) \qquad \forall A, B \in gl(n,\, C) \tag{3.4}$$

It is easy to check that

$$< A,\, B > = \overline{< B,\, A >}, \quad < A,\, [B,\, C] > = < [B^*,\, A],\, C > \tag{3.5}$$

In particular we have

$$< A,\, B > + < B,\, A > = 2Re < A,\, B > \tag{3.6}$$

Furthermore, the inner products are preserved under correspondence (3.1). Let $ds^2_\Lambda := \sum \lambda_{ij}\omega_{i\bar{j}}\omega_{\bar{i}j}$ be an invariant metric on $F(n)$ where $\omega = (\omega_{i\bar{j}})$ is the Maurer-Cartan form on $U(n)$ and

$$\lambda_{ij} = \lambda_{ji}\begin{cases} > 0 & \text{if} \quad i \neq j \\ = 0 & \text{if} \quad i = j \end{cases}$$

¿From now we suppose that M is a closed Riemannian surface and g its Riemannian metric. Then with respect to ds^2_Λ, the energy of ϕ_t is defined by

$$E(\phi_t) := \int_M \sum \lambda_{ij} |A^{ij}_z(t)|^2 v_g \tag{3.7}$$

where $v_g = \sqrt{-1} dz \wedge d\bar{z}$. ¿From (3.6)(3.7) and Lemma 3.1 we have

$$\begin{aligned}
\frac{d}{dt}|_{t=0} E(\phi_t) &= \int_M \sum \lambda_{ij} \frac{\partial}{\partial t}|_{t=0} |A^{ij}_z(t)|^2 v_g \\
&= 2Re \int_M \sum \lambda_{ij} < A^{ij}_z, \frac{\partial}{\partial t}|_{t=0} A^{ij}_z(t) > v_g \\
&= 2Re \int_M \sum \lambda_{ij} < A^{ij}_z, [A^{ij}_z, q] - \Pi_i \frac{\partial q}{\partial z} \Pi_j > v_g
\end{aligned}$$

so we get $\frac{1}{2}\frac{d}{dt}|_{t=0} E(\phi_t) = I + II$ where

$$I = Re \int_M \Sigma \lambda_{ij} < A^{ij}_z, [A^{ij}_z, q] > v_g \tag{3.8}$$

and $II = -Re \int_M \sum \lambda_{ij} < A^{ij}_z, \Pi_i \frac{\partial q}{\partial z} \Pi_j > v_g$.

Lemma 3.2. *1).Re* $< [A^{ji}_{\bar{z}}, A^{ij}_z], q >= 0$, *2).* $< A^{ij}_z, \pi_i B \pi_j >=< A^{ij}_z, B >$ *where* $A^{ji}_{\bar{z}} := \Pi_j \circ \frac{\partial \Pi_i}{\partial \bar{z}}$.

Proof. 1).It is easy to see that

$$(A^{ji}_{\bar{z}})^* = -A^{ij}_z \tag{3.9}$$

so we have

$$[A^{ji}_{\bar{z}}, A^{ij}_z]^* = [A^{ji}_{\bar{z}}, A^{ij}_z] \tag{3.10}$$

By using (3.4),(3.6) and (3.9) one gets

$$\begin{aligned}
2Re < [A^{ji}_{\bar{z}}, A^{ij}_z], q > &=< [A^{ji}_{\bar{z}}, A^{ij}_z], q > + < q, [A^{ji}_{\bar{z}}, A^{ij}_z] > \\
&= tr([A^{ji}_{\bar{z}}, A^{ij}_z] q^*) + tr(q [A^{ji}_{\bar{z}}, A^{ij}_z]^*) \\
&= -tr([A^{ji}_{\bar{z}}, A^{ij}_z] q) + tr(q [A^{ji}_{\bar{z}}, A^{ij}_z]) = 0
\end{aligned}$$

2). Notice that $\Pi_i\Pi_j = 0$, $i \neq j$ and $\Pi_i^2 = \Pi_i$ we have

$$\begin{aligned}
< A_z^{ij}, \Pi_i B \Pi_j > &= tr(A_z^{ij}\Pi_j^* B^* \Pi_i^*) \\
&= tr(\Pi_i \frac{\partial \Pi_j}{\partial z} \Pi_j B^* \Pi_i) \\
&= tr(\Pi_i \frac{\partial \Pi_j}{\partial z} \Pi_j B^*) \\
&= -tr(\frac{\partial \Pi_i}{\partial z} \Pi_j B^*) \\
&= tr(\Pi_i \frac{\partial \Pi_j}{\partial z} B^*) =< A_z^{ij}, B >
\end{aligned}$$

It is clear to see that, from (3.5)(3.8)(3.9) and Lemma 3.2

$$I = -2Re \int_M \Sigma\lambda_{ij} < [A_{\bar{z}}^{ji}, A_z^{ij}], q > v_g = 0$$

Using Lemma 3.2 and the Stokes' theorem it is easy to see that

$$\begin{aligned}
II &= -Re \int_M \Sigma\lambda_{ij} < A_z^{ij}, \frac{\partial q}{\partial z} > v_g \\
&= Re \int_M \Sigma\lambda_{ij} < \frac{\partial A_z^{ij}}{\partial \bar{z}}, q > v_g - Re \int_M \Sigma\lambda_{ij} \frac{\partial}{\partial \bar{z}} < A_z, q > v_g \\
&= Re \int_M < \frac{\partial A_z^{\Lambda}}{\partial \bar{z}}, q > v_g
\end{aligned}$$

where $A_z^{\Lambda} = \sum_{i,j} \lambda_{ij} A_z^{ij}$ and $\frac{\partial A_z^{\Lambda}}{\partial \bar{z}} : M \to u(n)$. We have

Proposition 3.3. *$\phi : (M, g) \to (F(n), ds_{\Lambda}^2)$ is harmonic if and only if $\frac{\partial A_x^{\Lambda}}{\partial x} + \frac{\partial A_y^{\Lambda}}{\partial y} = 0$ where $A_x^{\Lambda} := \sum \lambda_{ij}\Pi_i \frac{\partial \Pi_j}{\partial x}$, $A_y^{\Lambda} := \sum \lambda_{ij}\Pi_i \frac{\partial \Pi_j}{\partial y}$.*

Proof. In fact $4Re\frac{\partial A_z^{\Lambda}}{\partial \bar{z}} = \sum_{i,j} \lambda_{ij} Re \left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)(A_x^{ij} - \sqrt{-1}A_y^{ij}) = \frac{\partial A_x^{\Lambda}}{\partial x} + \frac{\partial A_y^{\Lambda}}{\partial y}$.

§4 Non−f−holomorphic equiharmonic tori

Suppose $\phi : R^2 \to F(n)$ is defined by

$$\phi = \pi \circ \tilde{\phi}, \quad \tilde{\phi}(x, y) = e^{Ax+By} \tag{4.1}$$

where $A, B \in u(n), \quad [A, B] = 0$. Then $\tilde{\phi}(x, y) = e^{By}e^{Ax}$ and

$$\frac{\partial \tilde{\phi}}{\partial x} = \tilde{\phi}A, \quad \frac{\partial \tilde{\phi}^*}{\partial x} = -A\tilde{\phi}^* \tag{4.2}$$

Combine with (3.3), we have $\frac{\partial \Pi_i}{\partial x} = \frac{\partial}{\partial x}(\tilde{\phi}E_i\tilde{\phi}^*) = \tilde{\phi}[A,\, E_i]\tilde{\phi}^*$. So

$$A_x^{ji} = \pi_j \frac{\partial \pi_i}{\partial x} = \tilde{\phi}E_j[A,\, E_i]\tilde{\phi}^* = \tilde{\phi}E_jAE_i\tilde{\phi}^* \tag{4.3}$$

Similarly we have $A_y^{ji} = \tilde{\phi}E_jBE_i\tilde{\phi}^*$. Hence the matrix of second fundamental forms of ϕ satisfy that

$$A_z^{ji} = \tilde{\phi}E_j\chi E_i\tilde{\phi}^* \tag{4.4}$$

where $\chi = \frac{1}{2}(A - \sqrt{-1}B)$. Now let $\mathcal{F}$ be an invariant f-structure with associated ϵ-matrix $(\mathcal{F}_{ij})$. Put $A = (a_{ij})$ and $B = (b_{ij})$. Using (3.1)(4.4) and Proposition 2.2 it is clear that ϕ is $f-$ holomorphic with respect to $\mathcal{F}$ if and only if

$$i \neq j, \quad \mathcal{F}_{ij} \neq 1 \quad \Longrightarrow \quad b_{ij} = \sqrt{-1}a_{ij}. \tag{4.5}$$

In fact, we have stronger conditions as following:

Proposition 4.1. *An equivalent condition of ϕ to be $f-$holomorphic with respect to f-structure $\mathcal{F}$ is*

1) $a_{ij} = b_{ij} = 0$ *if* $\mathcal{F}_{ij} = 0, \quad i \neq j$
2) $b_{ij} = -\sqrt{-1}a_{ij}$ *if* $\mathcal{F}_{ij} = 1$
3) $b_{ij} = \sqrt{-1}a_{ij}$ *if* $\mathcal{F}_{ij} = -1$

Proof. ¿From (4.5) it is enough to show the necessity. If $\mathcal{F}_{ij} = 0$ and $i \neq j$, then (4.5) implies that $b_{ij} = \sqrt{-1}a_{ij}$ and $b_{ji} = \sqrt{-1}a_{ji}$. Together with $A, B \in u(n)$ we get $a_{ij} = b_{ij} = 0$. If $\mathcal{F}_{ij} = 1$ then $\mathcal{F}_{ji} = -1$, so we get$b_{ji} = \sqrt{-1}a_{ji}$ Take conjugation we have $b_{ij} = -\sqrt{-1}a_{ij}$. Q.E.D.

Now we are in the position to investigate the harmonicity of ϕ (defined in (4.1) with double periods. ¿From (4.2) and (4.3) it is easy to see that

$$\frac{\partial A_x^{ij}}{\partial x} = \frac{\partial}{\partial x}\left(\tilde{\phi}E_iAE_j\tilde{\phi}^*\right) = \tilde{\phi}[A,\, E_iAE_j]\tilde{\phi}^* \tag{4.6}$$

Similarly we have

$$\frac{\partial A_y^{ij}}{\partial y} = \tilde{\phi}[B,\, E_iBE_j]\tilde{\phi}^* \tag{4.7}$$

Substitute (4.6) and (4.7) into (3.11), we have

Proposition 4.2. *Suppose that $\phi : R^2 \to F(n)$ defined in (4.1) has double periods. Then ϕ is harmonic with respect to ds^2_Λ if and only if*

$$[A,\, \Sigma\lambda_{ij}E_iAE_j] + [B,\, \Sigma\lambda_{ij}E_iBE_j] = 0 \tag{4.8}$$

Now we construct two classes of non$-f-$holomorphic equiharmonic tori on full complex flag manifolds. Let $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \in \mathbb{Q}\backslash\{0\}$ (where $\mathbb{Q}$ denotes the set of rational numbers) and for $j = 1, \cdots, k \leq \frac{n}{4}$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; A_j = \begin{pmatrix} \alpha_j X & 0 \\ 0 & \beta_j X \end{pmatrix},\; B_j = \begin{pmatrix} \beta_j X & 0 \\ 0 & \alpha_j X \end{pmatrix} \tag{4.9}$$

$$A = \sqrt{-1}diag(A_1, \cdots, A_k, 0 \cdots, 0) \tag{4.10}$$

$$B = \sqrt{-1}diag(B_1, \cdots, B_k, 0, \cdots, 0) \tag{4.11}$$

We have the following

Theorem 4.3. *1). $\phi : R^2 \to F(n)$ given by $\phi(x, y) = \pi(e^{Ax+By})$ has double periods; 2). $\psi : T^2 \to F(n)$ given by $\psi \circ p = \phi$ is equiharmonic but not $f-$holomorphic with respect to any invariant $f-$structure on $F(n)$ where $p : R^2 \to T^2$ is the natural projection.*

Proof. 1). $A^l = (\sqrt{-1})^l diag(\alpha_1^l X^l, \beta_1^l X^l, \cdots, \alpha_k^l X^l, \beta_k^l X^l, 0, \cdots, 0)$, for $l \in \{1, 2, \cdots\}$,where

$$X^l = \begin{cases} X & \text{if} \quad l = \text{odd} \\ I_2 & \text{if} \quad l = \text{even} \end{cases} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that

$$\begin{aligned} e^{Ax} &= I + Ax + \frac{A^2x^2}{2!} + \cdots \\ &= diag(\cos\alpha_1 x I_2,\, \cos\beta_1 x I_2,\, \cdots,\, \cos\alpha_k x I_2,\, \cos\beta_k x I_2,\, 0,\, \cdots,\, 0) \\ &\quad + \sqrt{-1}diag(\sin\alpha_1 x X,\, \sin\beta_1 x X\, \cdots,\, \sin\alpha_k x X,\, \sin\beta_k x X\,, 0,\, \cdots, 0) \end{aligned}$$

Combine with $[A, B] = 0$ and $\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \in \mathbb{Q}\backslash\{0\}$, there exists a $\nu \in \mathbb{Z}\backslash\{0\}$, such that $\phi(x + 2\pi n\nu, y + 2\pi m\nu) = \phi(x, y)$. Hence we have $\phi : T^2 = \frac{\mathbb{R}^2}{2\pi\nu(\mathbb{Z}\oplus\mathbb{Z})} \to F(n)$.

2). For any left-invariant ds^2_Λ on $F(n)$, from (4.9) and (4.10) we get

$$\begin{aligned}&\frac{1}{\sqrt{-1}}\Sigma\lambda_{ij}E_iAE_j\\ &\qquad = diag(\alpha_1\lambda_{12}X, \beta_1\lambda_{34}X, \cdots, \alpha_k\lambda_{4k-3,4k-2}X, \beta_k\lambda_{4k-1,4k}X, 0, \cdots, 0)\end{aligned}$$

so

$$\begin{aligned}&A\cdot\Sigma\lambda_{ij}E_iAE_j\\ &\quad = -diag(\alpha_1^2\lambda_{12}X^2, \beta_1^2\lambda_{34}X^2, \cdots, \alpha_k^2\lambda_{4k-3,4k-2}X^2, \beta_k^2\lambda_{4k-1,4k}X^2, 0, \cdots, 0)\\ &\quad = (\Sigma\lambda_{ij}E_iAE_j)\cdot A\end{aligned}$$

It follows that

$$[A, \Sigma\lambda_{ij}E_iAE_j] = 0 \tag{4.12}$$

Similarly we have

$$[B, \Sigma\lambda_{ij}E_iBE_j] = 0 \tag{4.13}$$

Subutitute (4.12) and (4.13) into (4.8), we see that ϕ is equiharmonic.

3). Suppose that ϕ is $f-$holomorphic for the invariant $f-$structure $\mathcal{F}$, and $(\mathcal{F}_{ij})$ is the ϵ-matrix of $\mathcal{F}$. From Proposition 4.1 one of following is true: i). $\sqrt{-1}\alpha_1 = \sqrt{-1}\beta_1 = 0$, ii). $\sqrt{-1}\beta_1 = \alpha_1$, iii). $\sqrt{-1}\beta_1 = -\alpha_1$. However this is impossible because $\alpha_1, \beta_1 \in \mathbb{Q}\backslash\{0\}$.

Similarly, let $\alpha_1, \cdots, \alpha_k \in \mathbb{Q}\backslash\{0\}, \quad 2k \le n$ and

$$A = \sqrt{-1}diag(\alpha_1X, \alpha_2X, \cdots, \alpha_kX, 0, \cdots, 0)$$

where X is defined in (4.9) Then we have

Theorem 4.4. *$\phi : T^2 \to F(n)$ defined by $(x, y) \to \pi(e^{A(x+y)})$ has double periods and the corresponding $\psi : T^2 \to F(n)$ is an equiharmonic map but not $f-$holomorphic with respect to any invariant $f-$structure on $F(n)$.*

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