Convergence rate of spectral semi-Galerkin approximations for the equations of a liquid crystal*

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Abstract

We study the convergence rate of solutions of spectral semi-Galerkin approximations for the equations of a liquid crystal in a bounded domain. We find error estimates that are optimal in the L^2 and H^1 norms

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1 Introduction

In this paper, we focus on mathematical aspects of a simplified model of the liquid crystal proposed by Lin [14]. This model inclues as a particular case the classical Navier-Stokes equations, wich has been widely studied (see for instance, Ladyzhenskaya [10], Lions [11], Temam [20] and the referencees there in).

We study convergence rates in several norms of spectral semi-Galerkin approximations. Although this is not a too interesting case from the point view of applications, we hope that the techniques that we introduce may be adapted in the important case where full discretization is used. This question is presently under investigation. We point out that the exact knowledge of the eigenfunctions of the Stokes operator is possible in certain domains see [18], [19].

Rautmann [15] gave a systematic development of error estimates for spectral Galerkin approximations of the classical Navier-Stokes equations(see also [8], [16]). Boldrini and Rojas-Medar gave analogous error estimates for the model of nonhomogeneous viscous incompressible fluids [2], [3].

The paper is organized as follows: in Section 2 we state some preliminaries results that will be useful in the rest of the paper; we describe the approximation method and state the results of existence and uniqueness as also some estimates apriori that form the theoretical basis for the problem. In Section 3 we derive a L^2-error estimate for the velocity and a H^1-error estimate for the optical director. In Section 4 we derive H^1-error estimates for velocity and H^2-error estimates for the optical director and Section 5 is devoted to the H^3- error estimates for optical director.

Finally, we would like to say that, as it usual in this context, to simplify the notation we will denote by C, M generic finite positive constants depending only on Ω and the other fixed parameters of the problem (like the initial data) that may have different values in different expressions. Sometimes, to emphasize the fact that the constants are different, we use $C_1, C_2, \ldots, M_1, M_2, \ldots$, and so on.

2 Preliminaries

The equations for the flow of liquid crystals are as follows. Being $\Omega \subset I\!R^n, n = 2$ or 3, a $C^{1,1}$-regular bounded domain, $T > 0$, these equations are (see [13])

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\triangle\mathbf{v} + \nabla p &= -\lambda\nabla\cdot(\nabla\mathbf{d}\circledcirc\nabla\mathbf{d}) \qquad \text{in } \ \Omega,\\
\nabla\mathbf{v} &= \mathbf{0} \qquad \text{in } \ \Omega,\\
\frac{\partial \mathbf{d}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{d} &= \gamma(\triangle\mathbf{d} - f(\mathbf{d})) \qquad \text{in } \ \Omega, \qquad (1)\\
\mathbf{v} &= 0 \qquad \text{on} \qquad \partial\Omega\times]0,T[,\\
\mathbf{d}(x,t) &= \mathbf{d}_0(x) \qquad \text{on} \qquad \partial\Omega\times]0,T[,\\
\mathbf{v}(x,0) &= \mathbf{v}_0(x) \qquad \text{in } \ \Omega,\\
\mathbf{d}(x,0) &= \mathbf{d}_0(x) \qquad \text{in } \ \Omega.
\end{aligned}$$

Here $[0,T[$ is the interval of time under consideration; Ω is the container where the fluid is; $\mathbf{v}(x,t)\in I\!R^n$ denotes the velocity of the fluid at point $x\in\Omega$ and time $t\in[0,T[;\mathbf{d}(x,t)\in I\!R^n$ and $p(x,t)\in I\!R$ denote, respectively, the optical director and the hydrostatic pressure of the fluid; $\mathbf{v}_0(x)$ and $\mathbf{d}_0(x)$ are the initial velocity and optical director respectively; ν,λ,γ are positive constants and $f(\mathbf{d})$ is a vector valued, smooth, bounded function defined for all $\mathbf{d}\in I\!R^n$,the fluid adheres to the wall $\partial\Omega$ of the container which is at rest. The expressions ∇,Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote $\dfrac{\partial\mathbf{v}}{\partial t}$ by $\mathbf{v}_t$); the i^{th} component of $\mathbf{v}\cdot\nabla\mathbf{v}$ is given by $[(\mathbf{v}\cdot\nabla)\mathbf{v}]_i = \sum_j v_j\dfrac{\partial v_i}{\partial x_j}$ and $[(\mathbf{v}\cdot\nabla)\mathbf{d}]_i = \sum_j v_j\dfrac{\partial d_i}{\partial x_j}$. The unusual term $\nabla\mathbf{d}\circledcirc\nabla\mathbf{d}$ denotes the $n\times n$ matrix whose (i,j)-th entry is given by $\mathbf{d}_{x_i}\cdot\mathbf{d}_{x_j}$, for $1\le i,j\le n$. The first equation in (1.1) corresponds to the balance of linear momentum; the third equation to the optical director and the second one states that fluid is incompressible. The unknowns in the problem are $\mathbf{v},\mathbf{d}$ and p.

The fisrt tentative of mathematical study to this model was given by Lin and Liu [13], their approach adop being inspired mainly in the pioneer work of Ladyzhenskaya [10]. But this trial has some gaps as we will show in our work [17]. The method used in [13] is similar to the one utilized by Kazhikov [9] in another context and was called "semi-Galerkin method " by Lions [12]. It consists in using the eigenfunctions of the Stokes operator to approximate the velocity field and one infinite-dimensional approximation for the optical director. To profile some gaps in [13], we follow the beautiful work of Heywood [6], [7]; moreover, we establishe the regularity of solutions (see [17]).

Let $\Omega \subset I\!R^n$, $n = 2$ or 3, be a bounded domain with smooth boundary Γ (class $C^{1,1}$ is enough), and we will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D),\ ||\partial^\alpha f||_{L^q(D)} < +\infty,\ |\alpha| \leq m\}$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq +\infty$, $D = \Omega$ or $\Omega\times]0, T[$, $0 < T < +\infty$, with their usual norms. When $q = 2$, we denote by $H^m(D) = W^{m,2}(D)$ and $H_0^m(D)$ = closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach space, we denote by $L^q(0, T; B)$ the Banach space of the B-valued functions defined in the interval $[0, T]$ that are L^q-integrable in the sense of Bochner. We shall consider the following spaces of divergence free functions

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{v \in (C_0^\infty(\Omega))^n \ / \ \text{div } v = 0 \quad \text{in} \quad \Omega\}, \\ H = &\quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \quad \text{in} \quad (L^2(\Omega))^n, \\ V = &\quad \text{closure of} \quad C_{0,\sigma}^\infty(\Omega) \quad \text{in} \quad (H^1(\Omega))^n. \end{aligned}$$

Throughout the paper, P denotes the orthogonal projection from $(L^2(\Omega))^n$ into H and $A = -P\Delta$ with $D(A) = V \cap H^2(\Omega)$ is the usual Stokes operator.

We will denote by $\varphi^n(x)$ and λ_n the eigenfunction and eigenvalue of A. It is well know that $\{\varphi^n\}_{n=1}^\infty$ is an orthogonal, complete, system in the spaces H, V and $H^2(\Omega) \cap V$, equipped with their usual inner products $(\mathbf{v}, \mathbf{u})$, $(\nabla\mathbf{v}, \nabla\mathbf{u})$ and $(A\mathbf{v}, A\mathbf{u})$ respectively.

For each $n \in I\!N$, we denote by P_n the orthogonal projection from $L^2(\Omega)$ onto $V_n = span\{\varphi^1(x), ..., \varphi^n(x)\}$. For more details on the Stokes operator see Temam [20].

We observe that for the regularity of the Stokes operator, it is usually assumed that Ω is of class C^3; this being in order to use Cattabriga's results [5]. We use instead the stronger results of Amrouche and Girault [1] which imply, in particular, that when $A\mathbf{v} \in L^2(\Omega)$, then $\mathbf{v} \in H^2(\Omega)$ and $||\mathbf{v}||_{H^2}$ and $||A\mathbf{v}||$ are equivalent norms when Ω is of Class $C^{1,1}$. Here $|| \cdot ||$ denotes the L^2-norm; also in this paper we will denote the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$.

We can rewrite the problem (2.1), by using the orthogonal projection P, as follows

$$\begin{aligned} \frac{\partial \mathbf{d}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{d} &= \gamma(\triangle \mathbf{d} - f(\mathbf{d})) \quad (\text{a.e.} \quad (x, t) \in \Omega\times]0, T[\,), \\ \frac{\partial \mathbf{v}}{\partial t} + \nu A\mathbf{v} &= P(-\mathbf{v} \cdot \nabla \mathbf{v} - \lambda \nabla \cdot (\nabla \mathbf{d} \circledcirc \nabla \mathbf{d})),\ t > 0 \end{aligned} \tag{2}$$

$$\mathbf{v}(0)=\mathbf{v}_0(x), \qquad \mathbf{d}(x,0)=\mathbf{d}_0(x),\ x\in\Omega$$
$$\mathbf{d}(x,t) = \mathbf{d}_0(x),\ (x,t)\in\partial\Omega\times(0,T).$$

or, equivalently,

$$\begin{aligned}
\mathbf{d}_t+\mathbf{v}\cdot\nabla\mathbf{d} &= \gamma(\triangle\mathbf{d}-f(\mathbf{d})) \ \text{ for } \ (x,t)\in\Omega\times(0,T),\\
(\mathbf{v}_t,\varphi)+(\mathbf{v}\cdot\nabla\mathbf{v},\varphi)+\nu(\nabla\mathbf{v},\nabla\varphi) &= -\lambda(\nabla\cdot(\nabla\mathbf{d}\circledcirc\nabla\mathbf{d}),\varphi)\\
\text{for } \ 0 &< t<T,\ \forall\,\varphi\in V, \qquad (3)\\
\mathbf{v}(0) &= \mathbf{v}_0(x),\mathbf{d}(x,0)=\mathbf{d}_0(x),\ x\in\Omega,\\
\mathbf{d}(x,t) &= \mathbf{d}_0(x),\ (x,t)\in\partial\Omega\times(0,T).
\end{aligned}$$

Concerning the existence of solutions for equations (2.1), they can be obtained by using a semi-Galerkin approximation. That is, we consider a Galerkin approximation

$$\mathbf{v}^n(x,t)=\sum_{i=1}^{n}C_{in}(t)\varphi^i(x)$$

for the velocity and an infinite dimensional approximation $\mathbf{d}^n(x,t)$ for the optical director satisfying the following equations:

$$\begin{aligned}
\mathbf{d}_t^n+\mathbf{v}^n\cdot\nabla\mathbf{d}^n &= \gamma(\triangle\mathbf{d}^n-f(\mathbf{d}^n)) \ \text{ for } \ (x,t)\in\Omega\times(0,T),\\
(\mathbf{v}_t^n,\varphi)+(\mathbf{v}^n\cdot\nabla\mathbf{v}^n,\varphi)+\nu(\nabla\mathbf{v}^n,\nabla\varphi) &= -\lambda(\nabla\cdot(\nabla\mathbf{d}^n\circledcirc\nabla\mathbf{d}^n),\varphi)\\
\text{for } \ 0 &< t<T,\ \forall\,\varphi\in V_n, \qquad (4)\\
\mathbf{v}^n(0) &= P_n\mathbf{v}_0(x),\quad \mathbf{d}^n(x,0)=\mathbf{d}_0(x),x\in\Omega,\\
\mathbf{d}^n(x,t) &= \mathbf{d}_0(x),\ (x,t)\in\partial\Omega\times(0,T).
\end{aligned}$$

It can be proved that $(\mathbf{v}^n,\mathbf{d}^n)$ converges in the appropriate sense to a solution $(\mathbf{v},\mathbf{d})$ of (2.1). As we said in the Introduction, we are interested in deriving error bounds, that is, estimates for $||\mathbf{v}(t)-\mathbf{v}^n(t)||$, $||\mathbf{d}(t)-\mathbf{d}^n(t)||$ in terms of powers of $\frac{1}{\lambda_{n+1}}$. These errors estimates will be derived in the following Section and we will based on the next result.

We assume the data for problem (2.1) satisfy

$(A1)$ $\quad \mathbf{d}_0\in H^3(\Omega)$

$(A2)\qquad \mathbf{v}_0 \in D(A)$
$(A3)$ $f = \nabla F(\mathbf{d})$, for some smooth, bounded function $F : I\!R^n \to I\!R$.

Under these hypotheses, we proved the following result [17]:

Theorem 1 *We assume* $(A1), (A2)$ *and* $(A3)$. *The dimension of* Ω *may be 2 or 3. There exists to* $0 < T_0 \leq T$ *such that the problem (2.2) (or (2.3)) has a unique strong solution that satisfies*

$$\begin{array}{rcl} \mathbf{v} & \in & L^\infty([0,T_0];\ D(A)), \\ \mathbf{v}_t & \in & L^\infty([0,T_0]; H) \cap L^2(0,T_0; V), \\ \mathbf{d} & \in & L^\infty([0,T_0]; H^3(\Omega)), \\ \mathbf{d}_t & \in & L^\infty([0,T_0]; H^1(\Omega)) \cap L^2([0,T_0], H^2(\Omega)), \\ \mathbf{d}_{tt} & \in & L^2([0,T_0]; L^2(\Omega)). \end{array}$$

Moreover, there holds the following estimates uniformly in n *for the approximations:*

$$\begin{array}{rcl} \|\mathbf{v}_t^n(t)\|^2 + \displaystyle\int_0^t \|\nabla \mathbf{v}_t^n(\tau)\|^2 d\tau & \leq & \Psi_1(t), \\ \sup\limits_t \|A\mathbf{v}^n(t)\|^2 & \leq & \Psi_2(t), \\ \|\nabla \mathbf{d}_t^n(t)\|^2 + \displaystyle\int_0^t \|\mathbf{d}_{tt}^n(\tau)\|^2 d\tau & \leq & \Psi_3(t), \\ \sup\limits_t \|\mathbf{d}^n(\tau)\|_{H^3}^2 & \leq & \Psi_4(t), \\ \displaystyle\int_0^t \|\triangle \mathbf{d}_t^n(\tau)\|^2 d\tau & \leq & \Psi_5(t), \end{array}$$

the functions on the right sides depend on their argument t, T_0 and $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$. On the interval under consideration these functions are continuosly differentiable with respect to t.

Remark 1 *Actually it is possible to prove that the strong solution of Theorem 2.1 is global either if* $n = 2$ *or if we take small enough initial data when* $n = 3$ *(Rojas-Medar and Ferreira [17]).*

Lemma 2 *Let* $\mathbf{v} \in V$. *Then the following estimate holds (see [7])*

$$\|\mathbf{v}\|_{L^3} \leq C\|\mathbf{v}\|^{1/2}\|\nabla \mathbf{v}\|^{1/2}.$$

Moreover, if $u \in V \cap H^2(\Omega)$ (see [4])

$$\|\mathbf{v}\|_{L^\infty} \leq C\|\nabla \mathbf{v}\|^{1/2}\|A\mathbf{v}\|^{1/2}.$$

The following result can be found in Rautmann [15].

Lemma 3 *If* $\mathbf{v} \in V$ *, then*

$$\|\mathbf{v} - P_n\mathbf{v}\|^2 \leq \frac{1}{\lambda_{n+1}}\|\nabla \mathbf{v}\|^2.$$

Also, if $\mathbf{v} \in V \cap H^2(\Omega)$ *, we have*

$$\begin{aligned} \|\mathbf{v} - P_n\mathbf{v}\|^2 &\leq \frac{1}{\lambda_{n+1}^2}\|A\mathbf{v}\|^2, \\ \|\nabla\mathbf{v} - \nabla P_n\mathbf{v}\|^2 &\leq \frac{1}{\lambda_{n+1}}\|A\mathbf{v}\|^2. \end{aligned}$$

3 L^2-error estimates for velocity and H^1-error estimates for optical director

From now on, as a matter of notation, we will denote T_0 simply by T. The sizes of the viscosity constants ν, γ, λ do not play important roles in the following and we shall therefore assume, for simplicity, that $\nu = \gamma = \lambda = 1$.

In this section we give the L^2-error estimate for the velocity and H^1-error estimate for density.

Let $(\mathbf{v}, \mathbf{d})$ be the strong solution of problem (2.2) (or (2.3)) given by Theorem 2.1 and $(\mathbf{v}^n, \mathbf{d}^n)$ the approximate solution of problem (2.4).

We define

$$\mathbf{w}^n = P_n\mathbf{v} - \mathbf{v}^n, \quad \sigma^n = \mathbf{d} - \mathbf{d}^n, \quad \eta^n = \mathbf{v} - P_n\mathbf{v}.$$

With these notations, we observe that $\mathbf{v}^n$ and σ^n satisfy the following equations

$$\begin{aligned} (\mathbf{w}_t^n, \phi) + (\nabla\mathbf{w}^n, \nabla\phi) &= -(\mathbf{w}^n \cdot \nabla\mathbf{v}^n, \phi) - (\eta^n \cdot \nabla\mathbf{v}^n, \phi) - (\mathbf{v} \cdot \nabla\mathbf{w}^n, \phi) \\ &\quad -(\mathbf{v}^n \cdot \nabla\eta^n, \phi) - (\triangle\sigma^n\nabla\mathbf{d}^n, \phi) - (\triangle\mathbf{d}\nabla\sigma^n, \phi), \end{aligned}$$

$$\begin{aligned}
\mathbf{w}^n(0) &= 0 \qquad (5)\\
\sigma^n_t - \triangle\sigma^n &= -\mathbf{w}^n\cdot\nabla\mathbf{d} - \eta^n\cdot\nabla\mathbf{d} - \mathbf{v}^n\cdot\nabla\sigma^n - (f(\mathbf{d}) - f(\mathbf{d}^n))\\
\sigma^n(0) &= 0\\
\sigma^n(x,t) &= 0 \text{ for } (x,t)\in\Omega\times(0,T)
\end{aligned}$$

To obtain our first estimate, we will need the following Proposition.

Proposition 4 *Under the hypotheses of Theorem 2.1, we have*

$$\|\mathbf{w}^n(t)\|^2 + \|\nabla\sigma^n(t)\|^2 + \int_0^t (\|\nabla\mathbf{w}^n(s)\|^2 + \|\triangle\sigma^n(s)\|^2)ds \le \frac{G_1(t)}{\lambda^2_{n+1}}, \qquad (6)$$

for any $t\in[0,T]$. The continuous function G_1 of the variable t depends on T and the norms $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. Setting $\phi = \mathbf{w}^n(t)$ in (3.1), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{w}^n\|^2 + \|\nabla\mathbf{w}^n\|^2 &= -(\mathbf{w}^n\cdot\nabla\mathbf{v}^n,\mathbf{w}^n) + (\eta^n\cdot\nabla\mathbf{v}^n,\mathbf{w}^n) - (\mathbf{v}^n\cdot\nabla\eta^n,\mathbf{w}^n)\\
&\quad -(\triangle\sigma^n\nabla\mathbf{d}^n,\mathbf{w}^n) - (\triangle\mathbf{d}\nabla\sigma^n,\mathbf{w}^n)
\end{aligned}$$

By using Hölder's inequality and Sobolev imbedding $H^2\hookrightarrow L^\infty$, $H^1\hookrightarrow L^6$, $H^1\hookrightarrow L^3$ and Lemma 2.2, we obtain the following estimates

$$\begin{aligned}
|(\mathbf{w}^n\cdot\nabla\mathbf{v}^n,\mathbf{w}^n)| &\le \|\mathbf{w}^n\|\|\nabla\mathbf{v}^n\|_{L^6}\|\mathbf{w}^n\|_{L^3}\\
&\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\mathbf{w}^n\|^2 + \varepsilon\|\nabla\mathbf{w}^n\|^2\\
|(\eta^n\cdot\nabla\mathbf{v}^n,\mathbf{w}^n)| &\le \|\eta^n\|_{L^2}\|\nabla\mathbf{v}^n\|_{L^6}\|\mathbf{w}^n\|_{L^3}\\
&\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\eta^n\|^2 + \varepsilon\|\nabla\mathbf{w}^n\|^2,\\
|(\mathbf{v}^n\cdot\nabla\eta^n,\mathbf{w}^n)| &= |(\mathbf{v}^n\cdot\nabla\mathbf{w}^n,\eta^n)|\\
&\le \|\mathbf{v}^n\|_{L^\infty}\|\nabla\mathbf{w}^n\|\|\eta^n\|^2\\
&\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\eta^n\|^2 + \varepsilon\|\nabla\mathbf{w}^n\|^2,\\
|(\triangle\sigma^n\nabla\mathbf{d}^n,\mathbf{w}^n)| &\le \|\triangle\sigma^n\|\|\nabla\mathbf{d}^n\|_{L^6}\|\mathbf{w}^n\|_{L^3}\\
&\le C_\delta\|\nabla\mathbf{d}^n\|^2_{L^6}\|\nabla\mathbf{w}^n\|\|\mathbf{w}^n\| + \delta\|\triangle\sigma^n\|^2\\
&\le C_{\varepsilon,\delta}\|\mathbf{d}^n\|^4_{H^2}\|\mathbf{w}^n\|^2 + \varepsilon\|\nabla\mathbf{w}^n\|^2 + \delta\|\triangle\sigma^n\|^2,
\end{aligned}$$

$$
\begin{aligned}
|(\triangle \mathbf{d} \nabla \sigma^n, \mathbf{w}^n)| &\leq \|\triangle \mathbf{d}\| \|\nabla \sigma^n\|_{L^6} \|\mathbf{w}^n\|_{L^3} \\
&\leq C \|\triangle \mathbf{d}\| \|\triangle \sigma^n\| \|\mathbf{w}^n\|^{1/2} \|\nabla \mathbf{w}^n\|^{1/2} \\
&\leq C_\delta \|\triangle \mathbf{d}\|^2 \|\mathbf{w}^n\| \|\nabla \mathbf{w}^n\| + \delta \|\triangle \sigma^n\|^2 \\
&\leq C_{\varepsilon,\delta} \|\mathbf{d}^n\|_{H^2}^4 \|\mathbf{w}^n\|^2 + \varepsilon \|\triangle \mathbf{w}^n\|^2 + \delta \|\triangle \sigma^n\|^2.
\end{aligned}
$$

By using the above estimates, we obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{w}^n\|^2 + \|\nabla \mathbf{w}^n\|^2 &\leq (C_\varepsilon \|A\mathbf{v}^n\|^2 + C_{\varepsilon,\delta} \|\mathbf{d}\|_{H^2}^4) \|\mathbf{w}^n\|^2 + C_\varepsilon \|A\mathbf{v}^n\|^2 \|\eta^n\|^2 \\
&\quad + 5\varepsilon \|\nabla \mathbf{w}^n\|^2 + 2\delta \|\triangle \sigma^n\|^2. \qquad (7)
\end{aligned}
$$

Multiplying equation $(3.1)_{iii}$ by $-\triangle \sigma^n$ and integrating over Ω, we have

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla \sigma^n\|^2 + \|\triangle \sigma^n\|^2 &= (\mathbf{w}^n \cdot \nabla \mathbf{d}, \triangle \sigma^n) + (\eta^n \cdot \nabla \mathbf{d}, \triangle \sigma^n) + (\mathbf{v}^n \cdot \nabla \sigma^n, \triangle \sigma^n) \\
&\quad - ((f(\mathbf{d}) - f(\mathbf{d}^n)), \triangle \sigma^n),
\end{aligned}
$$

since $\sigma_t^n = 0$ on $\partial \Omega$. Estimating, as usual, we have

$$
\begin{aligned}
|(\mathbf{v}^n \cdot \nabla \sigma^n, \triangle \sigma^n)| &\leq \|\mathbf{v}^n\|_{L^\infty} \|\nabla \sigma^n\| \|\triangle \sigma^n\| \\
&\leq C_\delta \|A\mathbf{v}^n\|^2 \|\nabla \sigma^n\|^2 + \delta \|\triangle \sigma^n\|^2, \\
|(\mathbf{w}^n \cdot \nabla \mathbf{d}, \triangle \sigma^n)| &\leq \|\mathbf{w}^n\|_{L^3} \|\nabla \mathbf{d}\|_{L^6} \|\triangle \sigma^n\| \\
&\leq C_\delta \|\mathbf{w}^n\| \|\nabla \mathbf{w}^n\| \|\mathbf{d}\|_{H^2}^2 + \delta \|\triangle \sigma^n\|^2 \\
&\leq C_{\varepsilon,\delta} \|\mathbf{d}\|_{H^2}^4 \|\mathbf{w}^n\|^2 + \varepsilon \|\nabla \mathbf{w}^n\|^2 + \delta \|\triangle \sigma^n\|^2, \\
|(\eta^n \cdot \nabla \mathbf{d}, \triangle \sigma^n)| &\leq \|\eta^n\| \|\nabla \mathbf{d}\|_{L^\infty} \|\triangle \sigma^n\|^2 \\
&\leq C_\delta \|\mathbf{d}\|_{H^3}^2 \|\eta^n\|^2 + \delta \|\triangle \sigma^n\|^2, \\
|(f(\mathbf{d}) - f(\mathbf{d}^n), \triangle \sigma^n)| &\leq \|f(\mathbf{d}) - f(\mathbf{d}^n)\| \|\triangle \sigma^n\| \\
&\leq C_\delta \|f(\mathbf{d}) - f(\mathbf{d}^n)\|^2 + \delta \|\triangle \sigma^n\|^2 \\
&\leq C_\delta \|\nabla F(\mathbf{d}) - \nabla F(\mathbf{d}^n)\|^2 + \delta \|\triangle \sigma^n\|^2 \\
&\leq C_\delta \|\nabla \sigma^n\|^2 + \delta \|\triangle \sigma^n\|^2.
\end{aligned}
$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\sigma^n\|^2+\|\triangle\sigma^n\|^2 \leq C_{\varepsilon,\delta}\|\mathbf{d}\|_{H^2}^4\|\mathbf{w}^n\|^2+C_\delta\|\mathbf{d}\|_{H^3}^2\|\eta^n\|^2 \qquad (8)$$
$$+(C_\delta+C_\delta\|A\mathbf{v}^n\|^2)\|\nabla\sigma^n\|^2+\varepsilon\|\nabla\mathbf{w}^n\|^2+3\delta\|\triangle\sigma^n\|^2.$$

Adding the inequalities (3.3) and (3.4), taking $\varepsilon>0$ and $\delta>0$ sufficiently small, after of integrate in t , we get

$$\|\mathbf{w}^n(t)\|^2+\|\nabla\sigma^n(t)\|^2+\int_0^t(\|\nabla\mathbf{w}^n(s)\|^2+\|\triangle\sigma^n(s)\|^2)ds$$
$$\leq \int_0^t g_1(s)(\|\mathbf{w}^n(s)\|^2+\|\nabla\sigma^n(s)\|^2)ds+\int_0^t g_2(s)(\|\eta^n(s)\|^2ds,$$

where $g_1(s)=C(\|\mathbf{d}(s)\|_{H^2}^4+C+\|A\mathbf{v}^n(s)\|^2)$, $g_2(s)=C(\|A\mathbf{v}^n(s)\|^2+\|\mathbf{d}(s)\|_{H^3}^2)$, we observe that from Theorem 2.1 $g_1,\ g_2\in L^\infty(0,T)$.

By using Gronwall's inequality and recalling the content of Lemma 2.3, we obtain estimate (3.2) with

$$G_1(t)=\sup_t\Psi_2(t)(\Psi_2(t)+\Psi_4(t))\times\exp(t(\Psi_2(t)+\Psi_4(t)^2)).$$

Theorem 5 *Suppose the assumptions of Theorem 2.1 hold. Then, the approximations* $\mathbf{v}^n$ *and* $\mathbf{d}^n$ *satisfy*

$$||\mathbf{v}(t)-\mathbf{v}^n(t)||^2\leq\frac{1}{\lambda_{n+1}^2}G_2(t), \qquad (9)$$

$$||\nabla\mathbf{d}(t)-\nabla\mathbf{d}^n(t)||^2+\int_0^t||\triangle\mathbf{d}(s)-\triangle\mathbf{d}^n(s)||^2ds\leq\frac{1}{\lambda_{n+1}^2}G_1(t), \qquad (10)$$

for any $t\in[0,T]$. *The continuous function* $G_2(t)$ *of the variable* t *depends on* T and the norms $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. We have from Lemma 2.3 and Proposition 3.1,

$$||\mathbf{v}(t)-\mathbf{v}^n(t)||^2 \leq ||\mathbf{w}^n(t)||^2+||\eta^n(t)||^2$$
$$\leq (G_1(t)+\|A\mathbf{v}\|^2)\frac{1}{\lambda_{n+1}^2}$$
$$\equiv \frac{1}{\lambda_{n+1}^2}G_2(t).$$

The estimate (3.6) is direct from Proposition 3.1.

4 $\mathbf{H}^1$-error estimates for velocity and $\mathbf{H}^2$-error estimates for optical director

Proposition 6 *Under the hypotheses of Theorem 2.1, we have*

$$\|\nabla \mathbf{w}^n(t)\|^2 + \int_0^t \|A\mathbf{w}^n(\tau)\|^2 d\tau \le \frac{G_3(t)}{\lambda_{n+1}}, \tag{4.11}$$

for any $t \in [0,T]$. *The continuous function* $G_3(t)$ *of the variable* t *depends on* T and the norms $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. Setting $\phi = A\mathbf{w}^n(t)$ in (3.1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}^n\|^2 + \|A\mathbf{w}^n\|^2 &= -(\mathbf{w}^n\cdot\nabla\mathbf{v}^n, A\mathbf{w}^n) - (\eta^n\cdot\nabla\mathbf{v}^n, A\mathbf{w}^n) - (\mathbf{v}^n\cdot\nabla\mathbf{w}^n, A\mathbf{w}^n) \\ &\quad -(\mathbf{v}^n\cdot\nabla\mathbf{w}^n, A\mathbf{w}^n) - (\triangle\sigma^n\nabla\mathbf{d}^n, A\mathbf{w}^n) - (\triangle\mathbf{d}\nabla\sigma^n, A\mathbf{w}^n).\end{aligned} \tag{4.12}$$

Now, we estimate the hand-right side of the above inequality of the following manner:

$$\begin{aligned}|(\mathbf{w}^n\cdot\nabla\mathbf{v}^n, A\mathbf{w}^n)| &\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\nabla\mathbf{w}^n\|^2 + \varepsilon\|A\mathbf{w}^n\|^2,\\ |(\eta^n\cdot\nabla\mathbf{v}^n, A\mathbf{w}^n)| &\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\nabla\eta^n\|^2 + \varepsilon\|A\mathbf{w}^n\|^2,\\ |(\mathbf{v}\cdot\nabla\mathbf{w}^n, A\mathbf{w}^n)| &\le C_\varepsilon\|A\mathbf{v}\|^2\|\nabla\mathbf{w}^n\|^2 + \varepsilon\|A\mathbf{w}^n\|^2,\\ |(\mathbf{v}^n\cdot\nabla\eta^n, A\mathbf{w}^n)| &\le C_\varepsilon\|A\mathbf{v}^n\|^2\|\nabla\eta^n\|^2 + \varepsilon\|A\mathbf{w}^n\|^2,\\ |(\triangle\sigma^n\nabla\mathbf{d}^n, A\mathbf{w}^n)| &\le C_\varepsilon\|\triangle\sigma^n\|\|\sigma^n\|_{H^3}\|\mathbf{d}^n\|_{H^2}^2 + \varepsilon\|A\mathbf{w}^n\|^2,\\ |(\triangle\mathbf{d}\nabla\sigma^n, A\mathbf{w}^n)| &\le C_\varepsilon\|\triangle\sigma^n\|\|\nabla\sigma^n\|\|\mathbf{d}\|_{H^3}^2 + \varepsilon\|A\mathbf{w}^n\|^2.\end{aligned}$$

The above estimates together with (4.2) imply the following differential inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{w}^n\|^2+\|A\mathbf{w}^n\|^2 &\leq (C\|A\mathbf{v}\|^2+C\|A\mathbf{v}^n\|^2)\|\nabla\mathbf{w}^n\|^2\\ &+(C\|A\mathbf{v}\|^2+C\|A\mathbf{v}^n\|^2)\|\nabla\eta^n\|^2\\ &+C\|\triangle\sigma^n\|\|\sigma^n\|_{H^3}\|\triangle\mathbf{d}^n\|^2\\ &+C\|\triangle\sigma^n\|\|\nabla\sigma^n\|\|\mathbf{d}\|_{H^3}^2.\end{aligned}$$

Which, upon Integration from 0 to t, yields

$$\begin{aligned}||\nabla\mathbf{w}^n(t)||^2+\int_0^t||A\mathbf{w}^n(s)||^2ds &\leq \int_0^t h_1(s)\|\nabla\mathbf{w}^n(s)\|^2ds\\ &+\int_0^t h_1(s)||\nabla\eta^n(s)||^2ds \qquad (4.13)\\ &+\int_0^t C\|\triangle\sigma^n\|\|\sigma^n\|_{H^3}\|\triangle\mathbf{d}^n\|^2ds\\ &+\int_0^t C\|\triangle\sigma^n\|\|\nabla\sigma^n\|\|\mathbf{d}\|_{H^3}^2ds,\end{aligned}$$

where $h_1(s)=C\|A\mathbf{v}\|^2+C\|A\mathbf{v}^n\|^2$. We observe that $h_1\in L^\infty(0,T)$.
On other hand,

$$\int_0^t C\|\triangle\sigma^n\|\|\sigma^n\|_{H^3}\|\triangle\mathbf{d}^n\|^2ds\leq\frac{G_4(t)}{\lambda_{n+1}}, \tag{4.14}$$

where

$$G_4(t)=C(G_1(t))^{1/2}(\Psi_4(t)+2(\Psi_4(t))^{1/2})t^{1/2},$$

and

$$\int_0^t C\|\triangle\sigma^n\|\|\nabla\sigma^n\|\|\mathbf{d}\|_{H^3}^2ds\leq\frac{G_5(t)}{\lambda_{n+1}}, \tag{4.15}$$

where

$$G_5(t)=C\int_0^t(G_1(\tau))^{1/2}d\tau(\Psi_4(t)+2(\Psi_4(t))^{1/2}).$$

By using the Lemma 2.3 and estimates (4.3), (4.4) and (4.5), we get

$$
\begin{aligned}
||\nabla \mathbf{w}^n(t)||^2 + \int_0^t ||A\mathbf{w}^n(s)||^2 ds \ &\leq \ \int_0^t h_1(s)\|\nabla \mathbf{w}^n(s)\|^2 ds \\
&+\frac{1}{\lambda_{n+1}}(\Psi_2(t))^2 \\
&+\frac{G_4(t)}{\lambda_{n+1}} + \frac{G_5(t)}{\lambda_{n+1}},
\end{aligned}
$$

and Gronwall inequality implies the desired result with

$$
G_3(t) = (\frac{1}{\lambda_{n+1}}\Psi_2(t)^2 + \frac{G_4(t)}{\lambda_{n+1}} + \frac{G_5(t)}{\lambda_{n+1}}) \times \exp(2C\Psi_2(t)t).
$$

Theorem 7 *Under the hypotheses of Theorem 2.1, we have*

$$
||\nabla(\mathbf{v} - \mathbf{v}^n)(t)||^2 \leq \frac{G_6(t)}{\lambda_{n+1}}. \tag{4.16}
$$

for any $t \in [0,T]$. *The continuous function* $G_6(t)$ *of the variable* t *depends on* T *and the norms* $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. *We have from Lemma 2.3 and Proposition 4.1,*

$$
\begin{aligned}
||\nabla \mathbf{v}(t) - \nabla \mathbf{v}^n(t)||^2 \ &\leq \ ||\nabla \mathbf{w}^n(t)||^2 + \|\nabla \eta^n(t)\|^2 \\
&\leq \ \frac{1}{\lambda_{n+1}}\|Au(t)\|^2 + \frac{G_3(t)}{\lambda_{n+1}}.
\end{aligned}
$$

Thus, we obtain (4.6) with $G_6(t) = G_3(t) + \Psi_2(t)$.

Corollary 8 *Under the hypotheses of the Theorem 2.1, we have*

$$
\int_0^t \|\mathbf{v}_t(\tau) - \mathbf{v}_t^n(\tau)\|^2 d\tau \leq \frac{G_7(t)}{\lambda_{n+1}}, \tag{4.17}
$$

$$
\int_0^t \|A\mathbf{v}(\tau) - A\mathbf{v}^n(\tau)\|^2 d\tau \leq \frac{G_8(t)}{\lambda_{n+1}}. \tag{4.18}
$$

for any $t \in [0,T]$. *The continuous functions* $G_7(t)$, $G_8(t)$ *of the variable* t *depend on* T *and the norms* $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. *To prove the estimate (4.7), we observe that*

$$\int_0^t ||\mathbf{v}_t^n(\tau) - \mathbf{v}_t^n(\tau)||^2 d\tau \leq \int_0^t (||\mathbf{w}_t^n(\tau)||^2 + \|\eta_t^n(\tau)\|^2) d\tau.$$

Thus, it is sufficient estimate the integral of the right-hand side in order to obtain the estimate desired.

By using Lemma 2.3, we have

$$\begin{aligned}\int_0^t ||\eta_t^n(\tau)||^2 d\tau &\leq \frac{1}{\lambda_{n+1}} \int_0^t ||\nabla \mathbf{v}_t^n(\tau)||^2 \\ &\leq \frac{\Psi_1(t)}{\lambda_{n+1}},\end{aligned}$$

where we used the estimate given in Theorem 2.1.

To estimate the other integral , we observe that equation $(3.1)_i$ implies

$$\begin{aligned}\int_0^t ||\mathbf{w}_t^n(\tau)||^2 d\tau &\leq C(\int_0^t ||A\mathbf{w}^n(\tau)||^2 + ||\mathbf{w}^n(\tau)\cdot\nabla\mathbf{v}^n(\tau)||^2 + ||\eta^n(\tau)\cdot\nabla\mathbf{v}^n(\tau)||^2 \\ &\quad +||\mathbf{v}(\boldsymbol{\tau})\cdot\nabla\mathbf{w}^n(\tau)||^2 + ||\mathbf{v}^n(\tau)\cdot\nabla\eta^n(\tau)||^2 \\ &\quad +||\triangle\sigma^n(\tau)\nabla\mathbf{d}^n(\tau)||^2 + ||\triangle\mathbf{d}(\tau)\nabla\sigma^n(\tau)||^2)d\tau \\ &\leq C(\int_0^t ||A\mathbf{w}^n(\tau)||^2 + ||\nabla\mathbf{w}^n(\tau)||^2||A\mathbf{v}^n(\tau)||^2 \\ &\quad +||\nabla\eta^n(\tau)||^2||A\mathbf{v}^n(\tau)||^2 + ||A\mathbf{v}(\tau)||^2||\nabla\mathbf{w}^n(\tau)||^2 \\ &\quad +||A\mathbf{v}^n(\tau)||^2||\nabla\eta^n(\tau)||^2 + ||\triangle\sigma^n(\tau)||^2||\mathbf{d}^n(\tau)||_{H^3}^2 \\ &\quad +||\mathbf{d}(\tau)||_{H^3}||\triangle\sigma^n(\tau)||^2)d\tau \\ &\leq \frac{G_9(t)}{\lambda_{n+1}},\end{aligned} \tag{4.19}$$

where $G_9(t) = G_3(t) + 2\Psi_2(t)\int_0^t G_3(\tau)d\tau + 2t(\Psi_2(t))^2 + 2\Psi_4(t)\int_0^t G_1(\tau)d\tau.$

To prove the estimate (4.8) it is sufficient to comente the difference between equations $(2.3)_{ii}$ and $(2.4)_{ii}$ and use the estimates already proved.

Theorem 9 *Under the hypotheses of Theorem 2.1, we have*

$$\|\mathbf{d}_t(t)-\mathbf{d}_t^n(t)\|^2+\int_0^t||\nabla\mathbf{d}_t^n(\tau)-\nabla\mathbf{d}_t^n(\tau)||^2d\tau\leq\frac{G_{10}(t)}{\lambda_{n+1}}, \tag{4.20}$$

$$\|\triangle\mathbf{d}(t)-\triangle\mathbf{d}^n(t)\|^2d\tau\leq\frac{G_{11}(t)}{\lambda_{n+1}}, \tag{4.21}$$

for any $t\in[0,T]$. *The continuous functions* $G_{10}(t)$, $G_{11}(t)$ *of the variable* t *depend on* T and the norms $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. Differentiating equation $(3.1)_{iii}$ with respect to t after multiplication by σ_t^n and integrating the result in Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sigma_t^n\|^2+\|\nabla\sigma_t^n\|^2 &= -(\mathbf{w}_t^n\cdot\nabla\mathbf{d},\sigma_t^n)-(\mathbf{w}^n\cdot\nabla\mathbf{d}_t,\sigma_t^n)\\ &\quad-(\eta_t^n\cdot\nabla\mathbf{d},\sigma_t^n)-(\eta^n\cdot\nabla\mathbf{d}_t,\sigma_t^n)\\ &\quad-(\mathbf{v}_t^n\cdot\nabla\sigma^n,\sigma_t^n)-((f(\mathbf{d})-f(\mathbf{d}^n))_t,\sigma_t^n).\end{aligned} \tag{4.22}$$

Now, we estimate the right-hand side of the above inequality as follows

$$\begin{aligned}|(\phi_t\cdot\nabla\mathbf{d},\sigma_t^n)| &\leq C_\varepsilon\|\phi_t\|^2\|\mathbf{d}\|_{H^2}^2+\varepsilon\|\nabla\sigma_t^n\|^2,\\ |(\phi\cdot\nabla\mathbf{d}_t,\sigma_t^n)| &\leq C_\varepsilon\|\nabla\phi\|^2\|\nabla\mathbf{d}_t^n\|^2+\varepsilon\|\nabla\sigma_t^n\|^2,\\ |(\mathbf{v}_t^n\cdot\nabla\sigma^n,\sigma_t^n)| &\leq C_\varepsilon\|\nabla\mathbf{v}_t^n\|^2\|\nabla\sigma^n\|^2+\varepsilon\|\nabla\sigma_t^n\|^2,\end{aligned}$$

for $\phi=\eta^n$ or $\mathbf{v}_t^n$.

The last term in the equality (4.12) is treated of the following manner:

$$\begin{aligned}|((f(\mathbf{d})-f(\mathbf{d})_t,\sigma_t^n)| &\leq |(\frac{\partial f(\mathbf{d})}{\partial t}\sigma_t^n,\sigma_t^n)|+|(\frac{\partial\mathbf{d}^n}{\partial t}[\frac{\partial f(\mathbf{d})}{\partial t}-\frac{\partial f(\mathbf{d}^n)}{\partial t}],\sigma_t^n)\\ &\leq C\|\sigma_t^n\|^2+C_\varepsilon\|\mathbf{d}_t^n\|^2\|\nabla\sigma^n\|^2+\varepsilon\|\sigma_t^n\|^2.\end{aligned}$$

Now, choosing $\varepsilon>0$ sufficiently small, the above estimates imply the following integral inequality

$$\|\sigma_t^n\|^2+\int_0^t\|\nabla\sigma_t^n\|^2\leq C\int_0^t(\|\mathbf{w}_t^n\|^2+\|\eta_t^n\|^2+\|\nabla\sigma^n\|^2+\|\nabla\sigma^n\|^2\|\mathbf{v}_t^n\|^2)ds,$$

where we used the estimates given in the Theorem 2.1. The above inequality implies the result (from Lemma 2.3, Proposition 3.1, Theorem 2.1 and (4.9)), with

$$G_{10}(t) = G_9(t) + \Psi_1(t) + t \sup_t G_1(t)(1 + \Psi_1(t)).$$

To prove the estimate (4.11), we observe that the equality $(3.1)_{iii}$ implies

$$\begin{aligned} \|\triangle\sigma^n\|^2 \quad &\leq \quad C(\|\nabla \mathbf{w}^n\|^2 \|\mathbf{d}\|_{H^2} + \|\nabla\eta^n\|^2\|\mathbf{d}\|^2_{H^2} + \|A\mathbf{v}^n\|^2\|\nabla\sigma^n\|^2 \\ &\quad +\|\sigma^n_t\|^2 + \|\sigma^n\|^2). \end{aligned}$$

From the estimates established in the Theorem 2.1, Lemma 2.3, Proposition 3.1 and (4.10) we get the result with

$$G_{11}(t) = G_1(t) + G_1(t)\Psi_2(t) + G_3(t)\Psi_4(t) + G_{10}(t) + \Psi_2(t)\Psi_4(t).$$

5 $\mathbf{H}^3$- error estimates for optical director

Theorem 10 *Under the hypotheses of Theorem 2.1, we have*

$$\|\nabla \mathbf{d}_t(t) - \nabla \mathbf{d}^n_t(t)\|^2 + \int_0^t ||\mathbf{d}^n_{tt}(\tau) - \mathbf{d}^n_{tt}(\tau)||^2 d\tau \leq \frac{G_{12}(t)}{\lambda_{n+1}}, \tag{5.1}$$

$$\|\mathbf{d}(t) - \mathbf{d}^n(t)\|^2_{H^3} \leq \frac{G_{13}(t)}{\lambda_{n+1}}, \tag{5.2}$$

for any $t \in [0,T]$. The continuous functions $G_{12}(t)$, $G_{13}(t)$ of the variable t depends on T and the norms $\|\mathbf{v}_0\|_{H^2(\Omega)}$, $\|\mathbf{d}_0\|_{H^3(\Omega)}$.

Proof. Differentiating the equation $(3.1)_{iii}$ with respect to t, after multiplication by σ^n_{tt} and integrating of result with respect to Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\sigma^n_t\|^2 + \|\sigma^n_{tt}\|^2 \quad &= \quad -(\mathbf{w}^n_t \cdot \nabla \mathbf{d}, \sigma^n_{tt}) - (\mathbf{w}^n \cdot \nabla \mathbf{d}_t, \sigma^n_{tt}) \\ &\quad -(\eta^n_t \cdot \nabla \mathbf{d}, \sigma^n_{tt}) - (\eta^n \cdot \nabla \mathbf{d}_t, \sigma^n_{tt}) \\ &\quad -(\mathbf{v}^n_t \cdot \nabla\sigma^n, \sigma^n_{tt}) - ((f(\mathbf{d}) - f(\mathbf{d}^n))_t, \sigma^n_{tt}). \end{aligned}$$

Now, we estimate as is usual to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sigma_t^n\|^2+\|\nabla\sigma_t^n\|^2 &\leq C(\|\mathbf{w}_t^n\|^2+\|\eta_t^n\|^2)\|\mathbf{d}\|_{H^3}^2\\ &+C(\|\nabla\eta^n\|^2+\|\nabla\mathbf{w}^n\|^2)\|\nabla\mathbf{d}_t\|\|\mathbf{d}_t\|_{H^2}\\ &+\|\nabla\mathbf{v}_t^n\|^2\|\nabla\sigma^n\|\|\triangle\sigma^n\|+C\|A\mathbf{v}^n\|^2\|\nabla\sigma_t^n\|^2\\ &+\|(f(\mathbf{d})-f(\mathbf{d}^n))_t\|^2.\end{aligned}$$

The above differential inequality impliesd the following integral inequality

$$\|\nabla\sigma_t^n(t)\|^2+\int_0^t ||\boldsymbol{\sigma}_{tt}^n(\tau)||^2 d\tau \leq \frac{G_{12}(t)}{\lambda_{n+1}},$$

where

$$\begin{aligned}G_{12}(t) &= C\Psi_4(t)(\Psi_1(t)+G_9(t))+C(\Psi_3(t))^{1/2}\Psi_5(t)(\Psi_2(t)+G_3(t))\\ &+C\Psi_2(t)G_{10}(t)+C(G_{11}(t)G_1(t))^{1/2}+G_{10}(t)+G_9(t)\Psi_3(t)t.\end{aligned}$$

To obtain the estimate (5.2), we take ∇ from the equation $(3.1)_{iii}$, in order to obtain

$$\begin{aligned}\|\nabla\triangle\sigma^n\|^2 &\leq C(\|\nabla\sigma_t^n\|^2+\|\nabla\mathbf{w}^n\nabla\mathbf{d}\|^2+\|\mathbf{w}^n\triangle\mathbf{d}\|^2+\|\nabla\eta^n\nabla\mathbf{d}\|^2\\ &+\|\eta^n\triangle\mathbf{d}\|^2+\|\nabla\mathbf{v}^n\nabla\sigma^n\|^2+\|\mathbf{v}^n\triangle\sigma^n\|^2+\|\nabla(f(\mathbf{d})-f(\mathbf{d}^n))_t\|^2.\end{aligned} \tag{5.3}$$

We observe that

$$\|\sigma^n\|_{H^3}^2 \leq C\|\nabla\triangle\sigma^n\|^2$$

and estimating the second member of the inequality (5.3) as is usual and using the estimates early proved, we get the desired result.

Remark 2 *We observe that the estimates obtained in this work are local in time. In another publication we will study the stability and error estimates uniform in time for the model (2.1)-(2.4).*

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