TOURNAMENTS AND GEOMETRY OF FULL FLAG MANIFOLDS

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We relate the study of complex geometry of full flag manifold $F(n)$ and harmonic maps into $F(n)$ with the tournament theory. In particular, we will sketch the proof that the usual Killing form metric on $F(n)$ is $(1,2)$-symplectic if and only if $n \leq 3$.

1 Introduction

This is concerned with questions arising from the intimate relationship between almost complex structures on flag manifolds and the tournament theory.

We follow Burstall-Salamon [3] and establish a one-to-one correspondence between tournaments and almost complex structures on $F(n)$. We describe the score vector characterization of the tournaments. We derive the holomorphic map equations.

In the late 1960, Calabi [4], Chern [5] and Eells [6] published many papers which are the modern basis for the study of harmonic maps. Then physicists Glaser-Stora and Din-Zakerewski complexified the problem and called the attention that the tight problem should be to study harmonic maps in $\mathbf{C}P^n$ instead of $\mathbf{R}P^n$.

Inspired by these ideas, Eells-Wood [7] gave a complete classification for harmonic maps from $\mathbf{C}P^1 \approx S^2$ to $\mathbf{C}P^{n-1}$ and some important partial results for arbitrary genus.

Then Chern-Wolfson and independently Burstall-Wood, classified the set of harmonic maps $\phi : S^2 \to G_k(\mathbf{C}^n)$ for $k = 2, 3, 4, 5$ and 6 in terms of full holomorphic maps between such manifolds. More generally, Uhlenbeck [14] classified harmonic maps $\phi : S^2 \to G_k(\mathbf{C}^n)$ for an arbitrary value of k in terms of holomorphic data.

We derive here the harmonic map equations for maps $\phi : M^2 \to F(n)$. According to the theorem of Sacks-Uhlenbeck [13] each homotopy class in

$\pi_2(F(n)) \cong \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n\text{-times}}$ has a harmonic representative. But according to Lichnerowicz's theorem [9] one possible way, is to find $(1,2)$-symplectic metrics on $F(n)$.

We will give an idea of the fact that the usual Killing metric of $F(n)$ is $(1,2)$-symplectic if and only if $n \leq 3$.

2 Almost complex structures on $F(n)$

We consider full complex flag manifold

$$F(n) = \{(L_1, \ldots, L_n);\, \dim L_i = 1,\, \oplus_{i=1}^{n} L_i = \mathbf{C}^n\}. \tag{2.1}$$

Algebraically, $F(n) = U(n)/T$, where

$$U(n) = \{A \in M(n \times n, \mathbf{C}) =: \mathbf{C}_n;\, A\bar{A}^t = AA^* = I\} \tag{2.2}$$

and T is the maximal torus of $U(n)$, i.e.

$$T = \underbrace{U(1) \times \cdots \times U(1)}_{n\text{-times}}. \tag{2.3}$$

Let p denote the tangent space of $F(n)$ at the identity coset, then

$$u(n) = \{X \in \mathbf{C}_n;\, X + X^* = 0\} = p \oplus \underbrace{u(1) \oplus \cdots \oplus u(1)}_{n\text{-times}}. \tag{2.4}$$

We now discuss invariant almost complex structures $J : p \to p$; $J^2 = -I$. Borel-Hirzebruch in [2] showed that there are $2^{\binom{n}{2}}$ such structures.

Example 2.1 Consider $n = 3$ and $J : p \to p$ defined by

$$J \begin{bmatrix} 0 & a_{12} & a_{13} \\ -\bar{a}_{12} & 0 & a_{23} \\ -\bar{a}_{13} & -\bar{a}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1\sqrt{-1}a_{12} & \varepsilon_2\sqrt{-1}a_{13} \\ \varepsilon_1\sqrt{-1}\bar{a}_{12} & 0 & \varepsilon_3\sqrt{-1}a_{23} \\ \varepsilon_2\sqrt{-1}\bar{a}_{13} & \varepsilon_3\sqrt{-1}\bar{a}_{23} & 0 \end{bmatrix},$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$ and 3. There are $2^{\binom{3}{2}}$ invariant almost complex structures.

Such choice of an almost complex structure clearly defines a tournament τ_J with n players $\{1, 2, \ldots, n\}$. More precisely, if

$$J(a_{ij}) = (a'_{ij}), \tag{2.5}$$

then τ_J is determined by

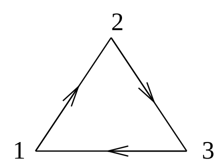
$$i \to j (i < j) \Leftrightarrow a'_{ij} = \sqrt{-1} a_{ij}, \text{ or} \tag{2.6}$$

$$j \to i (i < j) \Leftrightarrow a'_{ij} = -\sqrt{-1} a_{ij}. \tag{2.7}$$

Example 2.2 If J is the almost complex structure defined by

$$J \begin{bmatrix} 0 & a_{12} & a_{13} \\ -\bar{a}_{12} & 0 & a_{23} \\ -\bar{a}_{13} & -\bar{a}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{-1} a_{12} & -\sqrt{-1} a_{13} \\ \sqrt{-1} \bar{a}_{12} & 0 & \sqrt{-1} a_{23} \\ -\sqrt{-1} \bar{a}_{13} & \sqrt{-1} \bar{a}_{23} & 0 \end{bmatrix},$$

then the associated tournament τ_J is:

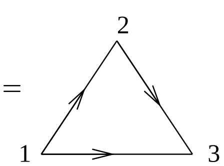


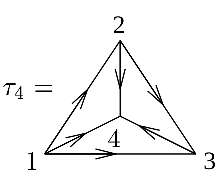
Definition 2.3 *a*) Let T_1 and T_2 be two tournaments. A map $\phi : T_1 \to T_2$ is said to be a *homomorphism* if

$$i \to j \Leftrightarrow \phi(i) \to \phi(j) \quad \text{or} \quad \phi(i) = \phi(j). \tag{2.8}$$

b) We define the *canonical tournament* τ_n in the following way:

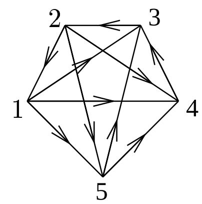
$$i \to j \Leftrightarrow i < j. \tag{2.9}$$

$\tau_3 =$ 

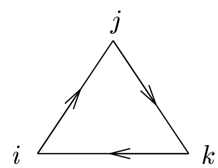
$\tau_4 =$ 

Proposition 2.4 ([10]) *Tournaments enjoy the following two basic and important properties:*

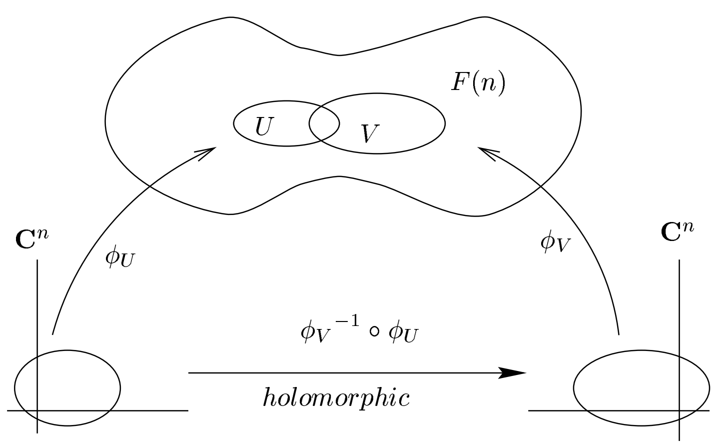
1) *If two tournaments T_1 and T_2 are isomorphic, then they have the same score vector (i.e., the vector each of whose components is equal to the number of games won by each player). For example, the following tournament has score vector* $(1, 1, 2, 3, 3)$.



2) *If a tournament T has a circuit $i_1 \to i_2 \to \cdots \to i_r \to i_1$, then it has a 3-cycle.*



An almost complex manifold $(F(n), J)$ is said to be a complex manifold if J is integrable. In this case, each transfer map between local complex coordinate systems is holomorphic.



Theorem 2.5 ([3], [10]) *There is a one-to-one correspondence between invariant almost complex structures J on $F(n)$ and n-tournaments τ_J such that J is integrable if and only if τ_J contains no 3-cycles.*

3 Harmonic and holomorphic maps

Let $\pi : U(n) \to F(n) = U(n)/T$ be the canonical projection and consider $\phi : M^2 \to F(n)$, where M^2 denotes a closed Riemann surface. We consider the lift map $\tilde{\phi} : M^2 \to U(n)$ of ϕ; i.e., $\phi = \pi \circ \tilde{\phi}$. Set

$$\pi_j = \tilde{\phi} E_j \tilde{\phi}^*, \tag{3.1}$$

where E_j denotes the matrix which has 1 in the (j,j)-position and zero elsewhere. Then we see that

$$\pi_j : M^2 \to \mathbf{C}_n \tag{3.2}$$

satisfies that

$$\pi_i^2 = \pi_i, \quad \pi_i \cdot \pi_j = 0, \quad i \neq j. \tag{3.3}$$

Put $A_z^{ij} = \pi_i(\partial \pi_j / \partial z)$. For $V \in \Gamma(\phi^* T(F(n)))$, we set $q = \phi^* \beta(V)$, where

$$\phi^* \beta : \phi^*(TF(n)) \to M \times u(n) \tag{3.4}$$

is the pull-back of the Maurer-Cartan form. Define the following variation of ϕ:

$$\phi_t(x) := \pi(\exp(-tq)\tilde{\phi}). \tag{3.5}$$

We denote the associated objects by $\pi_j(t), A_z^{ij}(t), \ldots$. Then we have

Lemma 3.1 1) $[\partial \pi_j(t)/\partial t]_{t=0} = [\pi_j, q]$.

2) $\partial[\pi_j, q]/\partial z = [\partial \pi_j / \partial z, q] + [\pi_j, \partial q/\partial z]$.

3) $[\partial A_z^{ij}(t)/\partial t]_{t=0} = [A_z^{ij}, q] - \pi_i(\partial q/\partial z)\pi_j$.

Proof. 1) By (3.1) we get

$$\pi_j(t) = e^{-tq} \pi_j e^{tq}. \tag{3.6}$$

Hence

$$[\partial \pi_j(t)/\partial t]_{t=0} = -q\pi_j + \pi_j q = [\pi_j, q]. \tag{3.7}$$

2) It is clear.

3) Using 1) and 2), we have

$$\begin{aligned}
[\partial A_z^{ij}(t)/\partial t]_{t=0} &= \{\partial[\pi_i(t)(\partial \pi_j/\partial z)]/\partial t\}_{t=0} \\
&= [\partial \pi_i(t)/\partial t]_{t=0} \cdot \partial \pi_j/\partial z + \pi_i \partial([\partial \pi_i(t)/\partial t]_{t=0})/\partial z \\
&= [\pi_i, q]\partial \pi_j/\partial z + \pi_i \partial[\pi_j, q]/\partial z \\
&= [\pi_i, q]\partial \pi_j/\partial z + \pi_i([\partial \pi_j/\partial z, q] + [\pi_j, \partial q/\partial z]) \\
&= [A_z^{ij}, q] - \pi_i(\partial q/\partial z)\pi_j,
\end{aligned} \tag{3.8}$$

since $\pi_i \cdot \pi_j = 0$ whenever $i \neq j$. Q.E.D.

We can define the usual Killing form inner product on $\mathbf{C}_n$ by

$$\langle A, B\rangle := \mathrm{tr}(AB^*), \quad A, B \in \mathbf{C}_n. \tag{3.9}$$

It is easy to check that $\langle A, B\rangle = \overline{\langle B, A\rangle}$ and

$$\langle A, [B, C]\rangle = \langle [B^*, A], C\rangle. \tag{3.10}$$

This inner product induces a natural invariant Riemannian metric on $F(n) = U(n)/T$ via the projection on the Killing inner product. We call such a metric Killing metric.

Definition 3.2 Consider $\phi = (\pi_1, \ldots, \pi_n) : M^2 \to F(n)$. We define the *energy* of ϕ by

$$E(\phi) = \int_M \sum_{i,j} |A_z^{ij}|^2 v_g. \tag{3.11}$$

Definition 3.3 Consider

$$\phi = (\pi_1, \ldots, \pi_n) : (M^2,\, g) \to (F(n), \text{Killing metric}). \tag{3.12}$$

We say that ϕ is *harmonic* if $[dE(\phi_t)/dt]_{t=0} = 0$ for any variation ϕ_t of ϕ.

Proposition 3.4 *Consider*

$$\phi = (\pi_1, \ldots, \pi_n) : (M^2, g) \to (F(n), \text{Killing metric}). \tag{3.13}$$

Then ϕ is harmonic if and only if $\partial A_z/\partial \bar{z} = 0$ if and only if $\partial A_x/\partial \bar{x} + \partial A_y/\partial \bar{y} = 0$, where $A_x = \sum \pi_i(\partial \pi_j/\partial x)$ and $A_y = \sum \pi_i(\partial \pi_j/\partial y)$.

Idea of the proof. Use Lemma 3.1 and the Stokes theorem. See [10] and [12] for more details. Q.E.D.

As we notice, for ϕ to be harmonic, it must satisfy a set of non-linear partial differential equations of second order known as the Euler-Lagrange equations.

The modern theory of harmonic maps is basically due to Calabi [4], Chern [5], Eells [7] and Uhlenbeck [13].

We will now consider the following fibration in the sense of Serre:

$$\underbrace{U(1)\times\cdots\times U(1)}_{n\text{-times}}\to U(n)\to F(n). \tag{3.14}$$

Using the homotopy long exact sequence, we have

$$\begin{aligned}\cdots &\to \pi_2(U(1)\times\cdots\times U(1))\to\pi_2(U(n))\to\pi_2(F(n))\\ &\to\pi_1(U(1)\times\cdots\times U(1))\to\pi_1(U(n))\to\pi_1(F(n))\to\cdots.\end{aligned} \tag{3.15}$$

But we know that $\pi_1(F(n))=0, \pi_1(U(n))\cong\mathbf{Z}$ and $\pi_2(U(n))=0$ (we can see these for n large enough simply by using the Bott periodicity). In any case we obtain that $\pi_2(F(n))=\underbrace{\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}}_{(n-1)\text{-times}}$.

According to the theorem of Sacks-Uhlenbeck [13] we can find harmonic maps representing each homotopy class in $\pi_2(F(n))$ which is isomorphic to $H_2(F(n);\mathbf{Z})$ by Hurewicz's theorem. This theorem can be seen as a generalization of the Hodge theorem for differentiable forms.

4 $(1,2)$-symplectic metrics and consequences

Definition 4.1 An almost complex manifold $(F(n),J,\langle\ ,\ \rangle)$ is said to be *Hermitian* if $\langle JX,JY\rangle=\langle X,Y\rangle$ for all $X,Y\in p$.

Definition 4.2 *a*) We define the *Keahler* 2*-form* for an arbitrary almost complex Hermitian manifold $(M,J,\langle\ ,\ \rangle)$ as

$$\Omega(X,Y):=\langle X,JY\rangle,\quad X,Y\in TM. \tag{4.1}$$

b) Such a manifold is said to be *almost Keahler* if $d\Omega=0$. If it is integrable, then an almost Keahler manifold is said to be *Keahler*.

Remark 4.3 $(F(n),J,\langle\ ,\ \rangle)$ is not an almost Keahler manifold. However, there are infinite number of Hermitian metrics h such that $(F(n),J,h)$ is a Keahler manifold. See [10] or [12] for details about this fact.

Definition 4.4 *a*) A map $\phi:(M,J_1)\to(F(n),J)$ is *J-holomorphic* if $d\phi\circ J_1=J\circ d\phi$; i.e., if ϕ satisfies the Cauchy-Riemann equations.

b) An almost Hermitian manifold $(F(n),J,\langle\ ,\ \rangle)$ is said to be $(1,2)$*-symplectic* if $d\Omega^{(1,2)}\equiv 0$, where $d\Omega^{(1,2)}$ denotes the $(1,2)$-component of $d\Omega$.

Theorem 4.5 ([9]) *Let* $\phi : (M^2, J_1, g) \to (F(n), J, \langle\, , \rangle)$ *be a* J*-holomorphic map and we also assume that the target manifold is* $(1,2)$*-symplectic. Then* ϕ *is harmonic.*

Hence we can combine the theorem of Sacks-Uhlenbeck [13] with the above one proved by Lichnerowicz in order to obtain harmonic representatives for elements in $\pi_2(F(n)) \cong H_2(F(n), \mathbf{Z})$ simply by considering holomorphic maps. Nevertheless, we have the following result.

Theorem 4.6 *The Killing metric on* $F(n)$ *is* $(1,2)$*-symplectic if and only if* $n \leq 3$.

Idea of the proof. We can see that ([10])

$$(1/4)d\Omega = \sum_{i<j<k} C_{ijk}\Psi_{ijk}. \tag{4.2}$$

Hence, $ds^2_{\Lambda=(\lambda_{ij})}$ is $(1,2)$-symplectic if and only if $d\Omega^{(1,2)} = 0$ if and only if

$$C_{ijk} = 0 \quad \text{whenever} \quad \Psi_{ijk} \in \mathbf{C}^{1,2} \oplus \mathbf{C}^{2,1}. \tag{4.3}$$

But we can show that $C_{ijk} = \varepsilon_{ij} - \varepsilon_{ik} + \varepsilon_{jk} \neq 0$. Hence (4.3) is equivalent to $\Psi_{ijk} \in \mathbf{C}^{0,3} \oplus \mathbf{C}^{3,0}$ for any $i < j < k$. But we can prove that the number of 3-cycles in the tournament τ_J is equal to $\binom{n}{3}$. It is impossible because, if $n > 3$, using Gale's inequality [8], [11], we have that the number of 3-cycles in τ_J is less than or equal to $(1/24)(n^2 - n)$ if n is odd or $(1/24)(n^3 - 4n)$ if n is even. Q.E.D.

Acknowledgments

The first author was supported by the National Natural Science Foundation of China (19871001). Both authors would like to thank FAPESP for the financial support. The first author wishes to thank IMECC for their hospitality. The second author wants to express his gratitude to Professor Karen Uhlenbeck for her immense support throughout these years.

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