The Equations of Nonhomogeneous Asymmetric Fluids: An Iterative Approach

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Abstract

We study the existence and uniqueness of strong solutions for the equations of nonhomogeneous asymmetric fluids. We use an iterative approach and we prove that the approximate solutions constructed by this method converge to the strong solution of these equations. We also give convergence rate bounds.

1 Introduction

In this paper, we study the equations of a nonhomogeneous viscous incompressible asymmetric fluid. These equations are considered in a bounded domain $\Omega \subset I\!R^3$, with boundary Γ, in a time interval $[0,T]$. Let $u(x,t) \in I\!R^3$, $w(x,t) \in I\!R^3$, $\rho(x,t) \in I\!R$ and $p(x,t) \in I\!R$, denote respectively, the velocity, the angular velocity of internal rotation, the density and the pressure at a point $x \in \Omega$ and at time $t \in [0,T]$. Then, the governing equations are

$$
\begin{aligned}
&\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u - (\mu+\mu_r)\Delta u + \nabla p = 2\mu_r \mathrm{rot}\, w + \rho f,\\
&\mathrm{div}\ u = 0, \qquad\qquad (1.1)\\
&\rho\frac{\partial w}{\partial t} + \rho(u\cdot\nabla)w - (c_a+c_d)\Delta w - (c_0+c_d-c_a)\nabla\ \mathrm{div}\ w + 4\mu_r w = 2\mu_r \mathrm{rot}\, u + \rho g,\\
&\frac{\partial \rho}{\partial t} + (u\cdot\nabla)\rho = 0,
\end{aligned}
$$

in $\Omega \times (0,T)$, with the following boundary and initial conditions

$$u(x,t) = 0,\ w(x,t) = 0 \ \text{ on } \ \Gamma \times (0,T),$$

$$u(x,0) = u_0(x),\ w(x,0) = w_0(x), \text{ in } \Omega, \tag{1.2}$$
$$\rho(x,0) = \rho_0(x) \text{ in } \Omega.$$

Here $f(x,t)$ and $g(x,t)$ are densities of linear and angular momentum, respectively. The positive constants μ, μ_r, c_0, c_a, c_d characterize isotropic properties of the fluid; μ is the usual Newtonian viscosity; μ_r, c_0, c_a, c_d are new positive viscosities related to the asymmetry of the stress tensor and in consequence related to the appearance of the field of internal rotation w; these constants satisfy $c_0 + c_d > c_a$. The expressions ∇, Δ, div and rot denote, the gradient, Laplacian, divergence and rotational operators respectively (we also denote $\frac{\partial u}{\partial t}$ by u_t); the i$\underline{th}$ component of $(u \cdot \nabla)v$ in cartesian coordinates is given by $[(u \cdot \nabla)v]_i = \sum_{j=1}^3 u_j \dfrac{\partial v_i}{\partial x_j}$; also $u \cdot \nabla\rho = \sum_{j=1}^3 u_j \dfrac{\partial \rho}{\partial x_j}$.

For the derivation and physical discussion of equations (1.1)-(1.2) see D.W. Condiff and J.S. Dahler (1964), L.G. Petrosyan (1984) and the recent book of G. Lukaszewicz (1999). We observe that this model of fluids includes as a particular case the classical Navier-Stokes equations, which has been thoroughly studied (see for instance the classical books of O. Ladyzhenskaya (1969), J.L. Lions (1969) and R. Temam (1979) and the references there in).

It also includes the reduced model of the nonhomogeneous Navier-Stokes equations, which has been less studied than the previous cases (see for instance, S. Antontsev, A. Kazhikov and V. Monakhov (1990), J. Simon (1990), J. Kim (1987), O. Ladyzhenskaya and V. Solonnikov (1976), R. Salvi (1991), J.L. Boldrini and M.A. Rojas-Medar (1992), (1997) and P.L. Lions (1996).

Concerning the generalized model of an asymmetric fluid, considered in this paper, G. Lukaszewicz (1990) established the existence of local weak solutions for (1.1)-(1.2) under certain assumptions, using linearization and an almost fixed point theorem. In the same paper Lukaszewicz remarked the possibility of proving the existence of strong solutions (under the hypothesis that the initial density is separated from zero) by the techniques used in Lukaszewicz (1988) and (1989) (linearization and fixed point theorems; Lukaszewicz (1988) and (1989) assume constant density).

The first result on the existence and uniqueness of strong solution (local and global) for problem (1.1)-(1.2) was proved by J.L. Boldrini and M.A. Rojas-Medar (1998) using the spectral semi-Galerkin method and compactness arguments. The convergence rate of this method was established by J.L. Boldrini and M.A. Rojas-Medar (1996).

In this work, we use another approach to obtain the existence and uniqueness of a strong solution. We use here an iterative process, considering a sequence of linear problems. For each linear problem it is easy to show the existence and uniqueness of a strong solution (for instance, using the spectral Semi-Galerkin method as in J.L. Boldrini and M.A. Rojas-Medar (1998)). Then, we obtain a priori estimates for the sequence generated by the iterative process. In the next step we show that the sequence is a Cauchy sequence in an appropriate Banach space, and consequently, we obtain strong convergence. With these convergences, the strong solution for the full original nonlinear problem is easily obtained. As by-product we obtain bounds for the convergence rate of the method.

We hope that the techniques developed here could be adapted to the case where full discretization is used. This question is presently under investigation.

We remark that, from the technical point of view, the hyperbolic character of the transport equation in (1.1) and the extra nonlinearities of the problem make the technical arguments more elaborated than those used in the case of constant density (compare with Ortega-Torres and Rojas-Medar (1997)).

This paper is organized as follows: in Section 2, we state some preliminary results that will be useful in the rest of the paper; we describe the approximation method and state the result of existence and uniqueness of a strong solution and the bound for the convergence rate. In Section 3, we derive some a priori estimates that form the theoretical basis in this problem. In Section 4, we stablish that the solutions of a sequence of linearized problems is a Cauchy sequence and we prove our main result. In Section 5, we give error estimates for the pressure.

Finally, as it is usual in this context, in order to simplify the notation we will denote by c, C_1, ..., M, M_1,... generic positive constants depending only on the domain and the fixed data of the problem.

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2 Preliminaries

Let Ω be a bounded domain in $I\!R^3$ with a smooth boundary $\partial\Omega$, $T > 0$ an arbitrary real number. The functions in this paper are either $I\!R$ or $I\!R^3$-valued, and sometimes we will not distinguish between them in our notation. This be will clear from the context. We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{ \ f \ \in L^q(\Omega) \ / \ \|\partial^\alpha f\|_{L^q(D)}) < \infty, \ |\alpha| \leq m \ \},$$

for $m \in I\!N$, $1 \leq p < \infty$ $D = \Omega$ or $D = \Omega \times (0,T)$, with the usual norm. When $q = 2$ we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(D)$ in $H^m(D)$. We put

$$C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) / \ \ \mathrm{div} \ v = 0 \quad \mathrm{in} \quad \Omega\}$$

$$
\begin{aligned}
H = &\quad \text{closure of} \quad C^\infty_{0,\sigma}(\Omega) \quad \text{in} \quad L^2(\Omega),\\
V = &\quad \text{closure of} \quad C^\infty_{0,\sigma}(\Omega) \quad \text{in} \quad H^1(\Omega).
\end{aligned}
$$

It is possible to show that

$$
V = \{v \in H^1_0(\Omega) \ / \ \operatorname{div} v = 0 \ \text{ in } \ \Omega\}.
$$

We denote by V^* the dual space of V and by H^{-1} the dual space of $H^1_0(\Omega)$.

We recall the Helmholtz decomposition of vector fields $L^2(\Omega) = H \oplus G$, were $G = \{\phi \ / \ \phi = \nabla p, \ p \in H^1(\Omega)\}$.

Throughout the paper P will denote the orthogonal projection from $L^2(\Omega)$ onto H. Then, the operator $A : D(A) \hookrightarrow H \longrightarrow H$ given by $A = -P\Delta$ with domain $D(A) = V \cap H^2(\Omega)$ is called the Stokes Operator. It is well known that A is a positive definite, self-adjoint operator and it is characterized by the relation

$$
(Aw, v) = (\nabla w, \nabla v), \quad \forall\, w \in D(A), \quad v \in V.
$$

When Ω is of class $C^{1,1}$ the norms $\|u\|_{H^2}$ and $\|Au\|$ are equivalent in $D(A)$ (see C. Amrouche and V. Girault (1991)). We assume known other properties of A, see for instance, O. Ladyzhenskaya (1969), J.L. Lions (1969) or R. Temam (1979). The same remark is valid for the Laplacian operator $B = -\Delta$ with Dirichlet boundary conditions and domain $D(B) = H^1_0(\Omega) \cap H^2(\Omega)$.

Using the properties of P, we can reformulate the problem (1.1)-(1.2) as follow:

Find u, w, ρ in suitable spaces (which will be defined later on), satisfying

$$
\begin{aligned}
& P(\rho u_t) + (\mu + \mu_r)Au + P(\rho u \cdot \nabla u) = 2\mu_r P(\operatorname{rot} w) + P(\rho f), && (2.1)\\
& \rho w_t + (c_a + c_d)Bw + \rho u \cdot \nabla w - (c_0 + c_d - c_a)\nabla \operatorname{div} w + 4\mu_r w && \\
& \qquad\qquad\qquad = 2\mu_r \operatorname{rot} u + \rho g, && (2.2)\\
& \rho_t + u \cdot \nabla \rho = 0, && (2.3)\\
& u(x,0) = u_0(x), \ w(x,0) = w_0(x), \ \rho(x,0) = \rho_0(x) \ \text{ in } \Omega. && (2.4)
\end{aligned}
$$

We consider the following iterative process for the approximate solution of problem (2.1)-(2.4). Setting

$$
u^1(t) = e^{-t(\mu+\mu_r)A}u_0, \quad w^1(t) = e^{-t(c_a+c_d)B}w_0, \quad \rho^1(x,t) = \rho_0(x)
$$

where $e^{-t(\mu+\mu_r)A}$ and $e^{-t(c_a+c_d)B}$ are the semigroup generated by the Stokes and Laplace operator, respectively. And if u^n, w^n and ρ^n are given, we define u^{n+1}, w^{n+1} and ρ^{n+1} as the unique solution of the following system of linear equations,

$$P(\rho^n u_t^{n+1}) + (\mu+\mu_r)Au^{n+1} + P(\rho^n u^n\cdot\nabla u^{n+1}) = 2\mu_r P(\mathrm{rot}\, w^n) + P(\rho^n f), \tag{2.5}$$

$$\begin{aligned}\rho^n w_t^{n+1} + (c_a+c_d)Bw^{n+1} + \rho^n u^n\cdot\nabla w^{n+1} - (c_o+c_d-c_a)\nabla\ \mathrm{div}\ w^{n+1} + 4\mu_r w^{n+1}\\ = 2\mu_r \mathrm{rot}\, u^n + \rho^n g,\end{aligned} \tag{2.6}$$

$$\rho_t^{n+1} + u^{n+1}\cdot\nabla\rho^{n+1} = 0, \tag{2.7}$$

$$u^{n+1}(x,0) = u_0(x), \quad w^{n+1}(x,0) = w_0(x), \quad \rho^{n+1}(x,0) = \rho_0(x) \ \text{ in } \Omega. \tag{2.8}$$

For simplicity, from now on we consider $u_0(x) = 0$ and $w_0(x) = 0$. Let first present the following results obtained for the approximate solutions. In this case, it is clear that the first iterate is $(u^1, w^1, \rho^1) = (0, 0, \rho_0)$.

Lemma 2.1 *Let assume that $\rho_0 \in C^1(\Omega)$ and $0 < \alpha \le \rho_0(x) \le \beta$, for all $x \in \bar\Omega$. If $f,\, g \in L^2(0,T;L^2(\Omega))$ are sufficiently small in the sense of the $L^2(Q)$ norm, then, for each n, the problem (2.5)-(2.8), has a unique strong solution (u^n, w^n, ρ^n). Furthermore, we have $u^n \in L^\infty(0,T;V)\cap L^2(0,T;D(A))$, $w^n \in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;D(B))$, $\rho^n \in L^\infty(0,T;L^\infty(\Omega))$, $u_t^n \in L^2(0,T;H)$, $w_t^n \in L^2(0,T;L^2(\Omega))$, $Au^n \in L^2(0,T;L^2(\Omega))$, $Bw^n \in L^2(0,T;L^2(\Omega))$, and all these sequences are uniformly bounded in the respective spaces.*

Lemma 2.2 *If the hypotheses of Lemma 2.1 are verified and assuming that $f, g \in L^2(0,T;H^1(\Omega))$ and $f_t,\, g_t \in L^2(0,T;L^2(\Omega))$, then the solution (u^n, w^n, ρ^n) of the problem (2.5)-(2.8) satisfies $u^n \in L^\infty(0,T;D(A)), w^n \in L^\infty(0,T;D(B))$ and $u_t^n \in L^\infty(0,T;H)\cap L^2(0,T;V)$, $w_t^n \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, for each n. Moreover, we obtain the following estimates uniformly in n:*

$$\begin{aligned}
\sup_t(\|u_t^n(t)\|^2 + \|w_t^n(t)\|^2) &\le C,\\
\int_0^t (\|\nabla u_t^n(\tau)\|^2 + \|\nabla w_t^n(\tau)\|^2)d\tau &\le C,\\
\sup_t(\|Au^n(t)\|^2 + \|Bw^n(t)\|^2) &\le C,\\
\int_0^t (\|\nabla u^n(\tau)\|^2_{L^\infty} + \|\nabla w^n(\tau)\|^2_{L^\infty})d\tau &\le C,\\
\sup_t \|\nabla\rho^n(t)\|^2_{L^\infty} &\le C,\\
\sup_t \|\rho_t^n(t)\|^2_{L^\infty} &\le C,\\
\sup_t \sigma(t)(\|\nabla u_t^n(t)\|^2 + \|\nabla w_t^n(t)\|^2) &\le C,\\
\int_0^t \sigma(\tau)(\|u_{tt}^n(\tau)\|^2 + \|w_{tt}^n(\tau)\|^2)d\tau &\le C,
\end{aligned}$$

$$\int_0^t \sigma(\tau)(\|Au_t^n(\tau)\|^2 + \|Bw_t^n(\tau)\|^2)d\tau \quad \leq \quad C,$$

for all $t \in [0,T]$, where $C > 0$ is a constant independent of n and $\sigma(t) = \min\{1, t\}$.

Theorem 2.3 *Let the conditions of Lemmas 2.1 and 2.2 be satisfied. Then the approximate solutions (u^n, w^n, ρ^n) converge to the limiting element (u, w, ρ) in the following senses*

$$\begin{aligned}
& u^n \longrightarrow u \quad \text{strongly in} \quad L^\infty(0,T; V \cap H^2(\Omega)), \\
& w^n \longrightarrow w \quad \text{strongly in} \quad L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \\
& u_t^n \longrightarrow u_t \quad \text{strongly in} \quad L^\infty(0,T; H), \\
& w_t^n \longrightarrow w_t \quad \text{strongly in} \quad L^\infty(0,T; L^2(\Omega)), \\
& u_t^n \longrightarrow u_t \quad \text{weakly in} \quad L^2(0,T; V) \cap L^2(\varepsilon,T; V \cap H^2(\Omega)), \quad \forall\, \varepsilon > 0, \\
& u_{tt}^n \longrightarrow u_{tt} \quad \text{weakly in} \quad L^2(\varepsilon,T; H), \quad \forall\, \varepsilon > 0, \\
& w_t^n \longrightarrow w_t \quad \text{weakly in} \quad L^2(0,T; H_0^1(\Omega)) \cap L^2(\varepsilon,T; H_0^1(\Omega) \cap H^2(\Omega)), \quad \forall\, \varepsilon > 0, \\
& w_{tt}^n \longrightarrow w_{tt} \quad \text{weakly in} \quad L^2(\varepsilon,T; L^2(\Omega)), \quad \forall\, \varepsilon > 0.
\end{aligned}$$

The limiting element (u, w, ρ) is the unique solution of problem (2.1)-(2.4) and

$$\begin{aligned}
\sup_t \{\|\nabla u^n(t) - \nabla u(t)\|^2 + \|\nabla w^n(t) - \nabla w(t)\|^2 &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\int_0^t (\|u_t^n(\tau) - u_t(\tau)\|^2 + \|w_t^n(\tau) - w_t(\tau)\|^2)d\tau &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\int_0^t (\|Au^n(\tau) - Au(\tau)\|^2 + \|Bw^n(\tau) - Bw(\tau)\|^2)d\tau &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\sup_t \|\rho^n(t) - \rho(t)\|_{L^\infty}^2 &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\sup_t \sigma(t)(\|u_t^n(t) - u_t(t)\|^2 + \|w_t^n(t) - w_t(t)\|^2) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\int_0^t \sigma(\tau)(\|\nabla u_t^n(\tau) - \nabla u_t(\tau)\|^2 + \|\nabla w_t^n(\tau) - \nabla w_t(\tau)\|^2)d\tau &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\sup_t \sigma(t)(\|Au^n(t) - Au(t)\|^2 + \|Bw^n(t) - Bw(t)\|^2) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\sup_t \sigma(t)(\|u^n(t) - u(t)\|_{L^\infty}^2 + \|w^n(t) - w(t)\|_{L^\infty}^2) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\int_0^t \sigma(\tau)(\|\nabla u^n(\tau) - \nabla u(\tau)\|_{L^\infty}^2 + \|\nabla w^n(\tau) - \nabla w(\tau)\|_{L^\infty}^2)d\tau &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}.
\end{aligned}$$

Moreover,

$$
\begin{aligned}
&u \in C^1([0,T];H) \cap C([0,T];D(A)),\\
&w \in C^1([0,T];L^2(\Omega)) \cap C([0,T];D(B)),\\
&\rho \in C^1(\Omega \times (0,T)).
\end{aligned}
$$

3 A priori estimates

In this section, we will prove uniform a priori estimates in n for the approximate solutions.

3.1 Proof of Lemma 2.1

The existence of a unique strong solution for the linear system (2.5)-(2.8), for each n, can be proved using the spectral Galerkin method (see for instance Boldrini and Rojas-Medar (1998)). The regularity asserted for the approximate solution in the lemma are firstly obtained for Galerkin approximations and then obtained for (u^n, w^n, ρ^n) by taking the limit. We will only prove the estimates, uniform in n, for the approximate solutions (u^n, w^n, ρ^n).

¿From the method of characteristics applied to the continuity equation (2.7) it follows immediately that whenever ρ^n exists, it satisfies $0 < \alpha \leq \rho^n \leq \beta$, then

$$
\{\rho^n\} \text{ is uniformly bounded in } L^\infty(0,T;L^\infty(\Omega)). \tag{3.1}
$$

¿From (2.7), we have also that $(\rho^n_t v, v) = -(\operatorname{div}(\rho^n u^n) v, v) = 2(\rho^n u^n \cdot \nabla v, v)$ and consequently

$$
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n} v\|^2 = \frac{1}{2}(\rho^n_t v, v) + (\rho^n v_t, v) = (\rho^n u^n \cdot \nabla v, v) + (\rho^n v_t, v),\ \forall\, v \in H^1_0,\ v_t \in L^2(\Omega).
$$

With this identity in mind, multiply (2.5) by u^{n+1} and (2.6) by w^{n+1}, to obtain respectively:

$$
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n} u^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u^{n+1}\|^2 = 2\mu_r(\operatorname{rot} w^n, u^{n+1}) + (\rho^n f, u^{n+1}), \tag{3.2}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n} w^{n+1}\|^2 + (c_a+c_d)\|\nabla w^{n+1}\|^2 + (c_0+c_d-c_a)\|\operatorname{div}\, w^{n+1}\|^2 + 4\mu_r\|w^{n+1}\|^2&\\
= 2\mu_r(\operatorname{rot} u^n, w^{n+1}) + (\rho^n g, w^{n+1}).&
\end{aligned} \tag{3.3}
$$

We observe that for $u \in H^1_0(\Omega)$, we have

$$\|\mathrm{rot}\, u\| \le \|\nabla u\|, \quad \|u\|_{L^4} \le 2^{1/2}\|u\|^{1/4}\|\nabla u\|^{3/4} \quad \text{and} \quad \|u\|^2 \le \lambda^{-1}\|\nabla u\|^2, \tag{3.4}$$

where λ is the smallest eigenvalue of the Laplace operator $B = -\Delta$ (see for instance Ladyzhenskaya (1969)).

By Hölder, Young and (3.4) inequalities, we get from (3.2) and (3.3) the following differential inequalities

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho^n}u^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u^{n+1}\|^2 \le \frac{8\mu_r^2}{\mu+\mu_r}\|w^n\|^2 + \frac{2\beta^2\,\lambda^{-1}}{\mu+\mu_r}\|f\|^2, \\
&\frac{d}{dt}\|\sqrt{\rho^n}w^{n+1}\|^2 + (c_a+c_d)\|\nabla w^{n+1}\|^2 + 2(c_0+c_d-c_a)\|\mathrm{div}\ w^{n+1}\|^2 \\
&\qquad\qquad \le \frac{4\mu_r^2}{c_a+c_d}\|u^n\|^2 + \frac{\beta^2}{8\mu_r}\|g\|^2.
\end{aligned}$$

Therefore, adding both inequalities and integrating both sides from 0 to t, we get the following integral inequality (recall that $u_0 = w_0 = 0$):

$$\begin{aligned}
&\alpha(\|u^{n+1}(t)\|^2 + \|w^{n+1}(t)\|^2) + (\mu+\mu_r)\int_0^t \|\nabla u^{n+1}(\tau)\|^2 d\tau \\
&+(c_a+c_d)\int_0^t \|\nabla w^{n+1}(\tau)\|^2 d\tau + 2(c_0+c_d-c_a)\int_0^t \|\mathrm{div}\ w^{n+1}(\tau)\|^2 d\tau \\
&\le \frac{8\mu_r^2}{\mu+\mu_r}\int_0^t \|w^n(\tau)\|^2 d\tau + \frac{4\mu_r^2}{c_a+c_d}\int_0^t \|u^n(\tau)\|^2 d\tau + \frac{2\beta^2\,\lambda^{-1}}{\mu+\mu_r}\|f\|^2_{L^2(Q)} + \frac{\beta^2}{8\mu_r}\|g\|^2_{L^2(Q)}
\end{aligned}$$

Then, there exist constants M and C such that

$$\begin{aligned}
&\|u^{n+1}(t)\|^2 + \|w^{n+1}(t)\|^2 + \frac{\mu+\mu_r}{\alpha}\int_0^t \|\nabla u^{n+1}(\tau)\|^2 d\tau + \frac{c_a+c_d}{\alpha}\int_0^t \|\nabla w^{n+1}(\tau)\|^2 d\tau \\
&\qquad \le C\int_0^t (\|u^n(\tau)\|^2 + \|w^n(\tau)\|^2) d\tau + M.
\end{aligned} \tag{3.5}$$

Choose, for example

$$C = \max\{\frac{8\mu_r^2}{\alpha(\mu+\mu_r)}, \frac{4\mu_r^2}{\alpha(c_a+c_d)}\} \quad \text{and} \quad M = \frac{2\beta^2\,\lambda^{-1}}{\alpha(\mu+\mu_r)}\|f\|^2_{L^2(Q)} + \frac{\beta^2}{8\alpha\,\mu_r}\|g\|^2_{L^2(Q)}.$$

Thus, setting $\varphi_n(t) = \|u^n(t)\|^2 + \|w^n(t)\|^2$, the last inequality implies

$$\varphi_{n+1}(t) \le M + C\int_0^t \varphi_n(\tau) d\tau.$$

Observing that $\varphi_1(t) = 0$, a straightforward induction argument shows that, for all n,

$$\begin{aligned} \varphi_n(t) &\leq M \sum_{k=0}^{n-1} \frac{(Ct)^k}{k!} \\ &\leq M \exp(Ct). \end{aligned}$$

We conclude that for all n we have

$$\sup_{t\in[0,T]} \left(\|u^n(t)\|^2 + \|w^n(t)\|^2 \right) \leq \sup_{t\in[0,T]} M\exp(Ct) = M\exp(CT) \equiv M_1. \tag{3.6}$$

Notice that M_1 does not depend on n and it can be made as small as needed, assuming that $\|f\|^2_{L^2(Q)}$ and $\|g\|^2_{L^2(Q)}$ are small enough.

Combining (3.5) and (3.6), we get

$$\|u^{n+1}\|^2_{L^2(0,T;V)} \leq \frac{\alpha M_1}{\mu+\mu_r} \quad \text{and} \quad \|w^{n+1}\|^2_{L^2(0,T;H^1_0(\Omega))} \leq \frac{\alpha M_1}{c_a+c_d} \tag{3.7}$$

where the bounds are independent of n.

Multiplying (2.5) by $\delta\, Au^{n+1}$, and then by u_t^{n+1} and integrating in Ω, we obtain respectively

$$\begin{aligned} \delta(\mu+\mu_r)\|Au^{n+1}\|^2 &= -\delta\,(\rho^n u_t^{n+1}, Au^{n+1}) + 2\mu_r\,\delta\,(\operatorname{rot} w^n, Au^{n+1}) \\ &\quad +\delta\,(\rho^n f, Au^{n+1}) - \delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1}) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|\sqrt{\rho^n}u_t^{n+1}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n+1}\|^2 &= 2\mu_r(\operatorname{rot} w^n, u_t^{n+1}) + (\rho^n f, u_t^{n+1}) \\ &\quad -(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1}). \end{aligned} \tag{3.9}$$

Then, since $\alpha \leq \rho^n \leq \beta$, we get

$$\begin{aligned} &\alpha\|u_t^{n+1}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n+1}\|^2 + \delta(\mu+\mu_r)\|Au^{n+1}\|^2 \\ &\leq |\delta\,(\rho^n u_t^{n+1}, Au^{n+1})| + |2\mu_r\,\delta\,(\operatorname{rot} w^n, Au^{n+1})| + |2\mu_r(\operatorname{rot} w^n, u_t^{n+1})| + |(\rho^n f, u_t^{n+1})| \\ &\quad +|\delta\,(\rho^n f, Au^{n+1})| + |\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| + |(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1})|. \end{aligned} \tag{3.10}$$

Now, using the Hölder, Young's inequalities and (3.4), we get

$$\begin{aligned}|\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| &\leq \delta\,\beta\|u^n\|_{L^4}\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\| \\ &\leq \delta\,\beta\sqrt{2}\,\|u^n\|^{1/4}\|\nabla u^n\|^{3/4}\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\| \\ &\leq \delta\,\beta\sqrt{2}\,\lambda^{-1/8}\|\nabla u^n\|\|\nabla u^{n+1}\|_{L^4}\|Au^{n+1}\|. \end{aligned} \tag{3.11}$$

Since $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$, for $u \in D(A)$, we have

$$\|\nabla u\|_{L^4} \leq \|u\|_{W^{1,4}} \leq C_\Omega\|u\|_{H^2} \leq C_\Omega M_3\|Au\|, \tag{3.12}$$

where C_Ω and M_3 are positive constants, independent of u. Thus, from (3.11) and (3.12), we obtain

$$|\delta(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| \leq \delta\,\beta\sqrt{2}\,\lambda^{-1/8}C_\Omega M_3\|\nabla u^n\|\|Au^{n+1}\|^2. \tag{3.13}$$

Similarly,

$$\begin{aligned}|(\rho^n u^n.\nabla u^{n+1}, u_t^{n+1})| &\leq \beta\,\|u^n\|_{L^4}\|\nabla u^{n+1}\|_{L^4}\|u_t^{n+1}\| \\ &\leq \delta\,\beta\sqrt{2}\;\lambda^{-1/8}C_\Omega M_3\|\nabla u^n\|\|Au^{n+1}\|^2 \\ &\quad + \frac{\beta\sqrt{2}\;\lambda^{-1/8}C_\Omega M_3}{4\delta}\|\nabla u^n\|\|u_t^{n+1}\|^2. \end{aligned} \tag{3.14}$$

Therefore, with these estimates in (3.10), and for any $\eta > 0$, we obtain

$$\begin{aligned}&(\mu+\mu_r)\frac{d}{dt}\|\nabla u^{n+1}\|^2 + 2\Big(\alpha - 3\eta - \frac{\beta\, 2^{1/2}\,\lambda^{-1/8}C_\Omega M_3}{4\delta}\|\nabla u^n\|\Big)\|u_t^{n+1}\|^2 \\ &+2\delta\Big((\mu+\mu_r) - \frac{3\delta\,\beta^2}{4\eta} - \beta\, 2^{3/2}\,\lambda^{-1/8}C_\Omega M_3\|\nabla u^n\|\Big)\|Au^{n+1}\|^2 \\ &\leq (\frac{8\mu_r^2\,\eta}{\beta^2} + \frac{2\mu_r^2}{\eta})\|\nabla w^n\|^2 + (2\eta + \frac{\beta^2}{2\eta})\|f\|^2.\end{aligned}$$

Integrating from 0 to t the last inequality, (recall that $u_0 = 0$) we obtain

$$\begin{aligned}&(\mu+\mu_r)\|\nabla u^{n+1}(t)\|^2 + 2\int_0^t\Big(\alpha - 3\eta - \frac{\beta\, 2^{1/2}\,\lambda^{-1/8}C_\Omega M_3}{4\delta}\|\nabla u^n(\tau)\|\Big)\|u_t^{n+1}(\tau)\|^2 d\tau \\ &+2\delta\int_0^t\Big((\mu+\mu_r) - \frac{3\delta\,\beta^2}{4\eta} - \beta\, 2^{3/2}\,\lambda^{-1/8}C_\Omega M_3\|\nabla u^n(\tau)\|\Big)\|Au^{n+1}(\tau)\|^2 d\tau \\ &\leq (\frac{8\mu_r^2\,\eta}{\beta^2} + \frac{2\mu_r^2}{\eta})\int_0^t\|\nabla w^n(\tau)\|^2 d\tau_+ (2\eta + \frac{\beta^2}{2\eta})\|f\|^2_{L^2(Q)} \\ &\leq (\frac{4\eta}{\beta^2} + \frac{1}{\eta})\frac{2\mu_r^2\alpha M_1}{c_a + c_d} + (2\eta + \frac{\beta^2}{2\eta})\|f\|^2_{L^2(Q)}.\end{aligned}$$

Then, choosing $\eta = \dfrac{\alpha}{4}$ and $\delta = \dfrac{\alpha(\mu+\mu_r)}{4\,\beta^2}$, we have

$$
\begin{aligned}
&(\mu+\mu_r)\|\nabla u^{n+1}(t)\|^2 + 2\int_0^t \left(\frac{\alpha}{4} - \frac{\beta^3\,2^{1/2}\,\lambda^{-1/8}C_\Omega M_3}{\alpha(\mu+\mu_r)}\|\nabla u^n(\tau)\|\right)\|u_t^{n+1}(\tau)\|^2 d\tau \\
&+\frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \left(\frac{\mu+\mu_r}{4} - \beta\,2^{3/2}\,\lambda^{-1/8}C_\Omega M_3\|\nabla u^n(\tau)\|\right)\|Au^{n+1}(\tau)\|^2 d\tau \\
&\leq 2(\alpha^2+4\beta^2)\frac{\mu_r^2}{\beta^2(c_a+c_d)}M_1 + \frac{1}{2\alpha}(\alpha^2+4\beta^2)\|f\|^2_{L^2(Q)} \qquad (3.15)
\end{aligned}
$$

The right hand side can be made smaller than any ε^2, choosing $\|f\|^2_{L^2(Q)}$ and $\|g\|^2_{L^2(Q)}$ sufficiently small.

Setting $n=1$ in (3.15) and using that $u^1=0$, we get

$$
(\mu+\mu_r)\|\nabla u^2(t)\|^2 + 2\int_0^t \frac{\alpha}{4}\|u_t^2(\tau)\|^2 d\tau + \frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \frac{\mu+\mu_r}{4}\|Au^2(\tau)\|^2 d\tau \leq \varepsilon^2,
$$

then, for all $t\in[0,T]$, we have

$$
\|\nabla u^2(t)\| \leq \frac{\varepsilon}{(\mu+\mu_r)^{1/2}}. \qquad (3.16)
$$

Supposing that this inequality is valid for some n, that is,

$$
\|\nabla u^n(t)\| \leq \frac{\varepsilon}{(\mu+\mu_r)^{1/2}},
$$

we can prove it for $n+1$, given that the coefficients inside both integral in (3.15) are positives. Or, if we suppose the inequality valid for that n, we need simply to prove that

$$
\frac{\alpha}{4} - \frac{\beta^3\,2^{1/2}\,\lambda^{-1/8}C_\Omega M_3\,\varepsilon}{\alpha(\mu+\mu_r)^{3/2}} > 0 \quad \text{and} \quad \frac{\mu+\mu_r}{4} - \frac{\beta\,2^{3/2}\,\lambda^{-1/8}C_\Omega M_3\,\varepsilon}{(\mu+\mu_r)^{1/2}} > 0,
$$

which is clearly true for ε sufficiently small. Therefore, for all n, we have proved that

$$
\sup_{t\in[0,T]}\|\nabla u^n(t)\| \leq \frac{\varepsilon}{(\mu+\mu_r)^{1/2}}. \qquad (3.17)
$$

From (3.15) and (3.17), we have

$$
\begin{aligned}
&2\int_0^t \left(\frac{\alpha}{4} - \frac{\beta^3\,2^{1/2}\,\lambda^{-1/8}C_\Omega M_3\,\varepsilon}{\alpha(\mu+\mu_r)^{3/2}}\right)\|u_t^{n+1}(\tau)\|^2 d\tau \\
&\quad+\frac{\alpha(\mu+\mu_r)}{2\,\beta^2}\int_0^t \left(\frac{\mu+\mu_r}{4} - \frac{\beta\,2^{3/2}\,\lambda^{-1/8}C_\Omega M_3\,\varepsilon}{(\mu+\mu_r)^{1/2}}\right)\|Au^{n+1}(\tau)\|^2 d\tau \leq \varepsilon^2.
\end{aligned}
$$

Therefore, we conclude that there exists a constant C, independent of n, such that

$$\int_0^t \|u_t^{n+1}(\tau)\|^2 d\tau + \int_0^t \|Au^{n+1}(\tau)\|^2 d\tau \leq C. \tag{3.18}$$

Similarly, for all n, we obtain

$$(c_a + c_d)\|\nabla w^{n+1}(t)\|^2 + c_2 \int_0^t \|Bw^{n+1}(\tau)\|^2 d\tau + \alpha \int_0^t \|w_t^{n+1}(\tau)\|^2 d\tau \leq C, \tag{3.19}$$

and the proof is complete.

3.2 Proof of Lemma 2.2

Now, differentiating (2.5) with respect to t, we obtain

$$\begin{aligned} &P(\rho_t^n u_t^{n+1}) + P(\rho^n u_{tt}^{n+1}) + (\mu + \mu_r)Au_t^{n+1} \\ &\qquad = 2\,\mu_r\, P(\mathrm{rot}\, w_t^n) + P(\rho_t^n f) + P((\rho^n f_t) - P(\rho_t^n u^n \cdot \nabla u^{n+1}) \\ &\qquad\quad -P(\rho^n u_t^n \cdot \nabla u^{n+1}) - P(\rho^n u^n \cdot \nabla u_t^{n+1}). \end{aligned} \tag{3.20}$$

The multiplication of (3.20) by u_t^{n+1} results in:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n} u_t^{n+1}\|^2 + (\mu + \mu_r)\|\nabla u_t^{n+1}\|^2 \\ &\quad = -\frac{1}{2}(\rho_t^n u_t^{n+1}, u_t^{n+1}) + 2\,\mu_r\,(\mathrm{rot}\, w_t^n, u_t^{n+1}) + (\rho_t^n f, u_t^{n+1}) + (\rho^n f_t, u_t^{n+1})) \\ &\qquad -(\rho_t^n u^n \cdot \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u_t^n . \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1})) \\ &\quad = \frac{1}{2}(\mathrm{div}\,(\rho^n u^n) u_t^{n+1}, u_t^{n+1}) + 2\,\mu_r\,(w_t^n, \mathrm{rot}\, u_t^{n+1}) - (\mathrm{div}\,(\rho^n u^n) f, u_t^{n+1})) \\ &\qquad +(\rho^n f_t, u_t^{n+1}) + (\mathrm{div}\,(\rho^n u^n) u^n . \nabla u^{n+1}, u_t^{n+1}) - (\rho^n u_t^n \cdot \nabla u^{n+1}, u_t^{n+1})) \\ &\qquad -(\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1}) \end{aligned} \tag{3.21}$$

since from (2.3), $\rho_t^n = -\mathrm{div}\,(\rho^n u^n)$.

Now, let obtain bounds for each one of the right hand side terms of (3.21), beginning with the first one:

$$\begin{aligned} \frac{1}{2}(\mathrm{div}\,(\rho^n u^n) u_t^{n+1}, u_t^{n+1}) &= -(\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1}) \\ &\leq \|\rho^n\|_{L^\infty} \|u^n\|_{L^4} \|\nabla u_t^{n+1}\| \|u_t^{n+1}\|_{L^4} \end{aligned}$$

$$
\begin{aligned}
&\leq \beta\|\nabla u^n\|\|\nabla u_t^{n+1}\|\|u_t^{n+1}\|^{1/4}\|\nabla u_t^{n+1}\|^{3/4} \\
&\leq C\,\|\nabla u_t^{n+1}\|^{7/4}\|u_t^{n+1}\|^{1/4} \\
&\leq C_\eta\|u_t^{n+1}\|^2+\eta\|\nabla u_t^{n+1}\|^2.
\end{aligned}
$$

The second one is simply

$$
2\,\mu_r\,(w_t^n,\mathrm{rot}\,u_t^{n+1})\leq C_\eta\|w_t^n\|^2+\eta\|\nabla u_t^{n+1}\|^2. \tag{3.22}
$$

For the third one, integrating by parts:

$$
\begin{aligned}
-(\mathrm{div}\,(\rho^n u^n)f,u_t^{n+1}) &= (\rho^n u^n.\nabla f,u_t^{n+1})+(\rho^n u^n\cdot\nabla u_t^{n+1},f) \\
&\leq \beta\|u^n\|_{L^4}\|\nabla f\|\|u_t^{n+1}\|_{L^4}+\beta\|u^n\|_{L^4}\|\nabla u_t^{n+1}\|\|f\|_{L^4} \\
&\leq C\,\|f\|_{H^1}\|\nabla u_t^{n+1}\|\leq C_\eta\|f\|_{H^1}^2+\eta\|\nabla u_t^{n+1}\|^2.
\end{aligned}
$$

For the fourth term, an obvious bound gives:

$$
(\rho^n f_t,u_t^{n+1})\leq C_\eta\|f_t\|^2+\eta\|\nabla u_t^{n+1}\|^2. \tag{3.23}
$$

The fifth term, again integrating by parts:

$$
\begin{aligned}
(\mathrm{div}\,(\rho^n u^n)u^n\cdot\nabla u^{n+1},u_t^{n+1}) &= \sum_{i,j,k}\int_\Omega\frac{\partial}{\partial x_i}(\rho^n u_i^n)u_j^n(\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx \\
&= -\sum_{i,j,k}\int_\Omega\rho^n u_i^n(\frac{\partial}{\partial x_i}u_j^n)(\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx \\
&\quad -\sum_{i,j,k}\int_\Omega\rho^n u_i^n u_j^n(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_k^{n+1})u_{k,t}^{n+1}dx \\
&\quad -\sum_{i,j,k}\int_\Omega\rho^n u_i^n u_j^n(\frac{\partial}{\partial x_j}u_k^{n+1})(\frac{\partial}{\partial x_i}u_{k,t}^{n+1})dx \\
&\leq C\,\beta\,\|u^n\|_{L^6}\|\nabla u^n\|_{L^6}\|\nabla u^{n+1}\|\|u_t^{n+1}\|_{L^6} \\
&\quad +C\,\beta\|u^n\|_{L^6}^2\|Au^{n+1}\|\|u_t^{n+1}\|_{L^6} \\
&\quad +C\,\beta\|u^n\|_{L^6}^2\|\nabla u^{n+1}\|_{L^6}\|\nabla u_t^{n+1}\| \\
&\leq C\,\|Au^n\|\|\nabla u_t^{n+1}\|+C\,\|Au^{n+1}\|\|\nabla u_t^{n+1}\| \\
&\leq C_\eta(\|Au^n\|^2+\|Au^{n+1}\|^2)+2\,\eta\|\nabla u_t^{n+1}\|^2.
\end{aligned}
$$

The sixth term:

$$\begin{aligned}
(\rho^n u_t^n \cdot \nabla u^{n+1}, u_t^{n+1}) &\leq \beta \|u_t^n\| \|\nabla u^{n+1}\|_{L^3} \|u_t^{n+1}\|_{L^6} \\
&\leq C\, \|u_t^n\| \|Au^{n+1}\| \|u_t^{n+1}\|_{H^1} \\
&\leq C_\eta \|u_t^n\|^2 \|Au^{n+1}\|^2 + \eta \|\nabla u_t^{n+1}\|^2.
\end{aligned}$$

And the seventh one:

$$\begin{aligned}
(\rho^n u^n \cdot \nabla u_t^{n+1}, u_t^{n+1}) &\leq \beta \|u^n\|_{L^6} \|\nabla u_t^{n+1}\| \|u_t^{n+1}\|_{L^3} \\
&\leq C \|\nabla u_t^{n+1}\| \|u_t^{n+1}\|^{1/2} \|\nabla u_t^{n+1}\|^{1/2} \\
&\leq C_\eta \|u_t^{n+1}\|^2 + \eta \|\nabla u_t^{n+1}\|^2.
\end{aligned}$$

With all these bounds we can transform (3.21) in the following inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^n} u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2 \\
&\quad \leq C_\eta \|u_t^{n+1}\|^2 + C_\eta \|w_t^n\|^2 + C_\eta \|f\|_{H^1}^2 + C_\eta \|f_t\|^2 + C_\eta \|Au^n\|^2 \\
&\qquad + C_\eta \|Au^{n+1}\|^2 + C_\eta \|u_t^n\|^2 \|Au^{n+1}\|^2 + C_\eta \|u_t^{n+1}\|^2 + 8\eta\, \|\nabla u_t^{n+1}\|^2.
\end{aligned}$$

Choosing $\eta = \dfrac{\mu+\mu_r}{16}$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho^n} u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2 \\
&\qquad \leq C\, \|u_t^{n+1}\|^2 + C\, \|w_t^n\|^2 + C\, \|f\|_{H^1}^2 + C\, \|f_t\|^2 + C\, \|Au^n\|^2 + C\, \|Au^{n+1}\|^2 \\
&\qquad\quad + C\, \|u_t^n\|^2 \|Au^{n+1}\|^2 + C\,. \qquad (3.24)
\end{aligned}$$

In order to get a bound for $\|Au^{n+1}\|^2$, multiply (2.5) by Au^{n+1}. We obtain

$$\begin{aligned}
(\mu+\mu_r)\|Au^{n+1}\|^2 &= -(\rho^n u_t^{n+1}, Au^{n+1}) + 2\,\mu_r\,(\mathrm{rot}\, w^n, Au^{n+1}) + (\rho^n f, Au^{n+1}) \\
&\quad - (\rho^n u^n.\nabla u^{n+1}, Au^{n+1})
\end{aligned}$$

and since

$$\begin{aligned}
|(\rho^n u^n.\nabla u^{n+1}, Au^{n+1})| &\leq \beta\, \|u^n\|_{L^4} \|\nabla u^{n+1}\|_{L^4} \|Au^{n+1}\| \\
&\leq C\, \|\nabla u^{n+1}\|^{1/4} \|Au^{n+1}\|^{7/4} \leq C_\delta \|\nabla u^{n+1}\|^2 + \delta \|Au^{n+1}\|^2,
\end{aligned}$$

we get, using obvious bounds for the remaining terms:

$$(\mu+\mu_r)\|Au^{n+1}\|^2 \le C_\delta\|u_t^{n+1}\|^2 + C_\delta\|\nabla w^n\|^2 + C_\delta\|f\|^2 + C_\delta\|\nabla u^{n+1}\|^2 + 4\,\delta\,\|Au^{n+1}\|^2.$$

Then, taking $\delta > 0$ sufficiently small, from the last inequality, we obtain the bound:

$$\|Au^{n+1}\|^2 \le C\,\|u_t^{n+1}\|^2 + C. \tag{3.25}$$

Thus, rewriting (3.24), we obtain now

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho^n}u_t^{n+1}\|^2 + (\mu+\mu_r)\|\nabla u_t^{n+1}\|^2 \\
&\qquad \le C\,\|u_t^{n+1}\|^2 + C\,\|w_t^n\|^2 + C\,\|f\|_{H^1}^2 + C\,\|f_t\|^2 + C\,\|Au^n\|^2 + C\,\|Au^{n+1}\|^2 \\
&\qquad\quad + C\,\|u_t^n\|^2\|u_t^{n+1}\|^2 + C\,\|u_t^n\|^2 + C.
\end{aligned}$$

Integrating from 0 to t

$$\begin{aligned}
&\alpha\|u_t^{n+1}(t)\|^2 + (\mu+\mu_r)\int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \\
&\quad \le C\int_0^t (\|u_t^{n+1}(\tau)\|^2 + \|w_t^n(\tau)\|^2 + \|f(\tau)\|_{H^1}^2 + \|f_t(\tau)\|^2)d\tau \\
&\qquad + C\int_0^t (\|Au^n(\tau)\|^2 + \|Au^{n+1}(\tau)\|^2)d\tau + C\int_0^t \|u_t^n(\tau)\|^2 d\tau \\
&\qquad + C\int_0^t \|u_t^n(\tau)\|^2\,\|u_t^{n+1}(\tau)\|^2 d\tau + \alpha\|u_t^{n+1}(0)\|^2 + Ct.
\end{aligned}$$

¿From the equation (2.5), we can easily bound the rightmost term $\|u_t^{n+1}(0)\|^2$. In fact, $\|u_t^{n+1}(t)\|^2$ is non decreasing at $t=0$, because on that time, $\nabla u_t^{n+1}(0) = 0$. Applying (3.18), (3.19) and the hypotheses on f and f_t, we get

$$\|u_t^{n+1}(t)\|^2 + \int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \le C + C\int_0^t \|u_t^n(\tau)\|^2\|u_t^{n+1}(\tau)\|^2 d\tau.$$

If we denote $\varphi(t) = \|u_t^{n+1}(t)\|^2$, the above inequality can be written as

$$\varphi(t) \le C + C\int_0^t \|u_t^n(\tau)\|^2\varphi(t)\,d\tau$$

which, by Gronwall's lemma, tell us that

$$\varphi(t) \le C\exp(C\int_0^t \|u_t^n(\tau)\|^2\,d\tau).$$

Therefore, again using (3.18) we conclude that

$$\|u_t^{n+1}(t)\|^2 + \int_0^t \|\nabla u_t^{n+1}(\tau)\|^2 d\tau \leq C. \tag{3.26}$$

Moreover, from (3.25) we have for all n

$$\sup_t \|Au^{n+1}(t)\|^2 \leq C. \tag{3.27}$$

Similarly, for all n, we prove

$$\|w_t^{n+1}(t)\|^2 + \int_0^t \|\nabla w_t^{n+1}(\tau)\|^2 d\tau \leq C \quad \text{and} \quad \sup_t \|Bw^{n+1}(t)\|^2 \leq C. \tag{3.28}$$

¿From (2.5), we have

$$(\mu + \mu_r)Au^{n+1} = P(F) \tag{3.29}$$

where

$$F = 2\mu_r \mathrm{rot}\, w^n + \rho^n f - \rho^n u_t^{n+1} - \rho^n u^n \cdot \nabla u^{n+1}.$$

We observe that from the estimates given in the Lemma 2.1, together with the estimates (3.26) and (3.27), we have $F \in L^2(0,T;L^6(\Omega))$ and consequently by the Amrouche-Girault's results (1991), we obtain $u^n \in L^2(0,T;W^{2,6}(\Omega))$ uniformly in n. Also, by using the Sobolev embedding, $u^n \in L^2(0,T;W^{1,\infty}(\Omega))$ uniformly in n.

The estimate for the density ρ^n is obtained from the Ladyzhenskaya-Solonnikov's results [(1968), see Lemma 1.3, page 705].

Now, multiplying (3.20) by u_{tt}^{n+1}, using (3.1), (3.17), Lemma 2.1, the above estimates, the Hölder and Young's inequalities, we obtain

$$\begin{aligned} \alpha \|u_{tt}^{n+1}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u_t^{n+1}\|^2 \quad &\leq \quad C_\varepsilon \|u_t^{n+1}\|^2 + C_\varepsilon \|\nabla w_t^n\|^2 + C_\varepsilon \|f\|^2 + C_\varepsilon \|f_t\|^2 \\ &\quad + C_\varepsilon \|Au^{n+1}\|^2 + C_\varepsilon \|\nabla u_t^n\|^2 \|Au^{n+1}\|^2 \\ &\quad + C_\varepsilon \|\nabla u_t^{n+1}\|^2 + 7\varepsilon \|u_{tt}^{n+1}\|^2. \end{aligned}$$

Choosing $\varepsilon = \dfrac{\alpha}{14}$ and observing (3.26)-(3.27), we have

$$\begin{aligned} \alpha \|u_{tt}^{n+1}\|^2 + (\mu+\mu_r)\frac{d}{dt}\|\nabla u_t^{n+1}\|^2 \quad &\leq \quad c\,\|\nabla w_t^n\|^2 + c\,\|f\|^2 + c\,\|f_t\|^2 + c\,\|\nabla u_t^n\|^2 \\ &\quad + c\,\|\nabla u_t^{n+1}\|^2 + c \end{aligned}$$

and multiplying by $\sigma(t)=\min\{1,t\}$, result

$$\begin{aligned}\alpha\,\sigma(t)\|u_{tt}^{n+1}\|^2+(\mu+\mu_r)\frac{d}{dt}(\sigma(t)\|\nabla u_t^{n+1}\|^2)\\ \leq(\mu+\mu_r)\sigma'(t)\|\nabla u_t^{n+1}\|^2+c\,\sigma(t)(\|\nabla u_t^n\|^2+\|\nabla w_t^n\|^2)\\ +c\,\sigma(t)(\|f\|^2+\|f_t\|^2)+c\,\sigma(t)(\|\nabla u_t^{n+1}\|^2+1).\end{aligned}\tag{3.30}$$

In view of (3.26), there exists a sequence $\varepsilon_k\longrightarrow 0$, such that $\varepsilon_k\,\|\nabla u_t^{n+1}(\varepsilon_k)\|^2\leq C$.

Therefore, since $\sigma(t)\leq 1$ and $\sigma'(t)\leq 1$ a.e. in $[0,T]$, observing (3.26)-(3.28) and integrating (3.30) from ε_k to t, we find

$$\alpha\int_{\varepsilon_k}^t\sigma(\tau)\|u_{tt}^{n+1}(\tau)\|^2d\tau+(\mu+\mu_r)\sigma(t)\|\nabla u_t^{n+1}(t)\|^2\leq c+c\,(\mu+\mu_r)\sigma(\varepsilon_k)\,\|\nabla u_t^{n+1}(\varepsilon_k)\|^2+C$$

and letting $\varepsilon_k\longrightarrow 0$, for all n, result in

$$\int_0^t\sigma(\tau)\|u_{tt}^{n+1}(\tau)\|^2d\tau+\sigma(t)\|\nabla u_t^{n+1}(t)\|^2\leq C.$$

Analogously, for all n,

$$\int_0^t\sigma(\tau)\|w_{tt}^{n+1}(\tau)\|^2d\tau+\sigma(t)\|\nabla w_t^{n+1}(t)\|^2\leq C.$$

To prove the last estimate given in Lemma 2.2, we observe from (3.20) that

$$(\mu+\mu_r)\int_0^t\sigma(\tau)\|Au_t^{n+1}(\tau)\|^2d\tau\leq\int_0^t\sigma(\tau)\|G^n(\tau)\|^2d\tau$$

where

$$\begin{aligned}G^n\;&=\;2\mu_r\mathrm{rot}\,w_t^n+\rho_t^nf+\rho^nf_t-\rho_t^nu_t^{n+1}-\rho^nu_{tt}^{n+1}-\rho_t^nu^n\cdot\nabla u^{n+1}\\&\quad-\rho^nu_t^n.\nabla u^{n+1}-\rho^nu^n\cdot\nabla u_t^{n+1}.\end{aligned}$$

We observe that the above estimates imply that $\sigma^{1/2}(t)G^n\in L^2(0,T;L^2(\Omega))$ uniformly in n. Analogously, we prove the estimate for w^n.

Remark. Using arguments of compactness and the estimates given in Lemma 2.1 and Lemma 2.2, it is possible to prove that the approximate solutions (u^n,w^n,ρ^n) converge to a unique strong solution of the problem (1.1)-(1.2). This can be done in exactly the same way as in Boldrini and Rojas-Medar (1998). In the next section, we will prove the convergence of the approximate solutions by other arguments.

4 Proof of Theorem 2.3

For $n,\ s \geq 1$, let

$$u^{n,s}(t) = u^{n+s}(t) - u^n(t),\ w^{n,s}(t) = w^{n+s}(t) - w^n(t) \text{ and } \rho^{n,s}(t) = \rho^{n+s}(t) - \rho^n(t).$$

With these notations, we observe that $u^{n,s},\ w^{n,s}$ and $\rho^{n,s}$ satisfy the following equations

$$\begin{aligned}
P(\rho^{n-1+s}u_t^{n,s}) + (\mu+\mu_r)Au^{n,s} &= 2\mu_r P(\operatorname{rot} w^{n-1,s}) + P(\rho^{n-1,s}f) - P(\rho^{n-1,s}u_t^n) \\
&\quad -P(\rho^{n-1+s}u^{n-1+s}\cdot\nabla u^{n,s}) - P(\rho^{n-1+s}u^{n-1,s}\cdot\nabla u^n) \\
&\quad -P(\rho^{n-1,s}u^{n-1}\cdot\nabla u^n) \qquad (4.1)\\
\rho^{n-1+s}w_t^{n,s} + (c_a+c_d)Bw^{n,s} &- (c_0+c_d-c_a)\nabla \text{ div } w^{n,s} + 4\mu_r w^{n,s} \\
&= 2\mu_r(\operatorname{rot} u^{n-1,s}) + \rho^{n-1,s}g - \rho^{n-1,s}w_t^n - \rho^{n-1+s}u^{n-1+s}\cdot\nabla w^{n,s} \\
&\quad -\rho^{n-1+s}u^{n-1,s}\cdot\nabla w^n - \rho^{n-1,s}u^{n-1}\cdot\nabla w^n \qquad (4.2)\\
\rho_t^{n,s} + u^{n,s}\cdot\nabla\rho^{n+s} + u^n\cdot\nabla\rho^{n,s} &= 0. \qquad (4.3)
\end{aligned}$$

The following lemma will be fundamental in order to obtain error estimates.

Lemma 4.1 *Let $0 \leq \phi_1(t) \leq M$ for all $t \in [0,T]$ and assume that for all $n \geq 2,\ n \in N$, we have the following inequality*

$$0 \leq \phi_n(t) \leq C\int_0^t \phi_{n-1}(\tau)d\tau$$

where $C > 0$ is a constant independent of n. Then,

$$\phi_n(t) \leq M\frac{(Ct)^{n-1}}{(n-1)!} \leq M\frac{(CT)^{n-1}}{(n-1)!}$$

for all $t \in [0,T]$ and $n \geq 2$. Therefore, $\phi_n(t) \longrightarrow 0$ as $n \longrightarrow \infty,\ \forall\, t \in [0,T]$.

Lemma 4.2 *Under the hypotheses of Lemma 2.2, for $2 \leq r \leq 6$, we have*

$$\|\rho^{n,s}(t)\|_{L^r}^2 \leq C\int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau.$$

Proof.

For a bounded domain Ω, it is known that $L^r(\Omega) \hookrightarrow L^6(\Omega)$, for $2 \leq r \leq 6$, then we only need to prove the result in the case $r = 6$.

Multiplying (4.3) by $(\rho^{n,s})^5$ and integrating over Ω, we obtain

$$
\begin{aligned}
\frac{1}{6}\frac{d}{dt}\int_\Omega |\rho^{n,s}|^6 dx &= -\int_\Omega u^{n,s}\cdot\nabla\rho^{n+s}(\rho^{n,s})^5 dx - \frac{1}{6}\int_\Omega u^n\cdot\nabla(\rho^{n,s})^6 dx \\
&\leq \int_\Omega |u^{n,s}||\nabla\rho^{n+s}||\rho^{n,s}|^5 dx + \frac{1}{6}\int_\Omega \text{ div } u^n(\rho^{n,s})^6 dx \\
&\leq \|\nabla\rho^{n+s}\|_{L^\infty(0,T;L^\infty(\Omega))}\int_\Omega |u^{n,s}||\rho^{n,s}|^5 dx \\
&\leq C\left(\int_\Omega |u^{n,s}|^6 dx\right)^{1/6}\left(\int_\Omega |\rho^{n,s}|^6 dx\right)^{5/6}.
\end{aligned}
$$

This implies

$$
\frac{1}{6}\frac{d}{dt}\|\rho^{n,s}\|_{L^6}^6 \leq C\,\|u^{n,s}\|_{L^6}\|\rho^{n,s}\|_{L^6}^5
$$

but,

$$
\frac{1}{6}\frac{d}{dt}\|\rho^{n,s}\|_{L^6}^6 = \|\rho^{n,s}\|_{L^6}^5\frac{d}{dt}\|\rho^{n,s}\|_{L^6},
$$

then, since $H^1(\Omega)\hookrightarrow L^6(\Omega)$, we obtain

$$
\frac{d}{dt}\|\rho^{n,s}\|_{L^r} \leq C\,\|\nabla u^{n,s}\|.
$$

Integrating from 0 to t the last inequality and applying the Cauchy-Schwartz inequality, we have

$$
\|\rho^{n,s}(t)\|_{L^r} \leq C\int_0^t \|\nabla u^{n,s}(\tau)\|d\tau \leq c\Big(\int_0^t \|\nabla u^{n,s}(\tau)\|^2 d\tau\Big)^{1/2}. \tag{4.4}
$$

Lemma 4.3 *Under the hypotheses of Lemma 2.2, we have:*

$$
\begin{aligned}
\sup_t\{\|\nabla u^{n+s}(t)-\nabla u^n(t)\|^2+\|\nabla w^{n+s}(t)-\nabla w^n(t)\|^2 &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\int_0^t(\|u_t^{n+s}(\tau)-u_t^n(\tau)\|^2+\|w_t^{n+s}(\tau)-w_t^n(\tau)\|^2)d\tau &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\int_0^t(\|Au^{n+s}(\tau)-A^nu(\tau)\|^2+\|Bw^{n+s}(\tau)-Bw^n(\tau)\|^2)d\tau &\leq M\frac{(M_1T)^{n-1}}{(n-1)!}, \\
\sup_t\|\rho^{n+s}(t)-\rho^n(t)\|_{L^\infty}^2 &\leq M\frac{(M_1T)^{n-1}}{(n-1)!},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t (\|\nabla u^{n+s}(\tau) - \nabla u^n(\tau)\|^2 + \|\nabla w^{n+s}(\tau) - \nabla w^n(\tau)\|^2) d\tau &\leq M\frac{(M_1T)^n}{n!}, \\
for\ 2 \leq r \leq 6 \qquad \sup_t \|\rho^{n+s}(t) - \rho^n(t)\|^2_{L^r} &\leq M\frac{(M_1T)^n}{n!}, \\
\sup_t \sigma(t)(\|u_t^{n+s}(t) - u_t^n(t)\|^2 + \|w_t^{n+s}(t) - w_t^n(t)\|^2) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\int_0^t \sigma(\tau)(\|\nabla u_t^{n+s}(\tau) - \nabla u_t^n(\tau)\|^2 + \|\nabla w_t^{n+s}(\tau) - \nabla w_t^n(\tau)\|^2) d\tau &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\sup_t \sigma(t)(\|Au^{n+s}(t) - Au^n(t)\|^2 + \|Bw^{n+s}(t) - Bw^n(t)\|^2) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\sup_t \sigma(t)(\|u^{n+s}(t) - u^n(t)\|^2_{L^\infty} + \|w^{n+s}(t) - w^n(t)\|^2_{L^\infty}) &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}, \\
\int_0^t \sigma(\tau)(\|\nabla u^{n+s}(\tau) - \nabla u^n(\tau)\|^2_{L^\infty} + \|\nabla w^{n+s}(\tau) - \nabla w^n(\tau)\|^2_{L^\infty}) d\tau &\leq M\frac{(M_1T)^{n-2}}{(n-2)!}.
\end{aligned}
$$

Proof. Multiplying (4.1) by $\delta Au^{n,s}$, integrating over Ω and estimating as usual, we obtain,

$$
\begin{aligned}
\delta(\mu+\mu_r)\|Au^{n,s}\|^2 \leq\ & \eta\|u_t^{n,s}\|^2 + \eta\|\nabla w^{n-1,s}\|^2 + \eta\|\rho^{n-1,s}\|^2_{L^6}\|f\|^2_{L^3} \\
& +\eta\|\rho^{n-1,s}\|^2_{L^6}\|\nabla u_t^n\|^2 + \eta\|\nabla u^{n,s}\|^2 \\
& +\eta\|\nabla u^{n-1,s}\|^2 + \eta\|\rho^{n-1,s}\|^2_{L^6} + \frac{1}{4\eta}\delta^2 C\|Au^{n,s}\|^2. \qquad (4.5)
\end{aligned}
$$

Similarly, multiplying (4.1) by $u_t^{n,s}$, we get

$$
\begin{aligned}
& \alpha\|u_t^{n,s}\|^2 + \frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n,s}\|^2 \\
& \leq C_\eta\|\nabla w^{n-1,s}\|^2 + C_\eta\|\rho^{n-1,s}\|^2_{L^6}\|f\|^2_{L^3} + C_\eta\|\rho^{n-1,s}\|^2_{L^6}\|\nabla u_t^n\|^2 \\
& \quad +C_\eta\|\nabla u^{n,s}\|^2 + C_\eta\|\nabla u^{n-1,s}\|^2 + C_\eta\|\rho^{n-1,s}\|^2_{L^6} + 6\eta\|u_t^{n,s}\|^2. \qquad (4.6)
\end{aligned}
$$

¿From (4.2), we have

$$
\begin{aligned}
& \rho^{n-1+s}w_t^{n,s} + Lw^{n,s} + 4\mu_r w^{n,s} \\
& \qquad = 2\mu_r(\operatorname{rot} u^{n-1,s}) + \rho^{n-1,s}g - \rho^{n-1,s}w_t^n - \rho^{n-1+s}u^{n-1+s}\cdot\nabla w^{n,s} \\
& \qquad\quad -\rho^{n-1+s}u^{n-1,s}\cdot\nabla w^n - \rho^{n-1,s}u^{n-1}\cdot\nabla w^n \qquad (4.7)
\end{aligned}
$$

where $Lw^{n,s} = (c_a+c_d)Bw^{n,s} - (c_0+c_d-c_a)\nabla\ \mathrm{div}\ w^{n,s}$.

Since L is a strongly elliptic operator (see Ladyzhenskaya-Solonnikov-Uralceva, 1968 page 70), there exists a constant $N_0 > 0$ (N_0 only depending of $c_a + c_d,\ c_0 + c_d - c_a$ and $\partial\Omega$) such that

$$
(Lw^{n,s}, Bw^{n,s}) \geq (c_a+c_d)\|Bw^{n,s}\|^2 - N_0\|\nabla w^{n,s}\|^2. \qquad (4.8)
$$

Now, multiplying (4.7) by $\theta Bw^{n,s}$, using (4.8) and estimating as usual, we have

$$\begin{aligned}\theta(c_a+c_d)\|Bw^{n,s}\|^2 \quad \le \quad & \theta C_\zeta\|\nabla w^{n,s}\|^2+\zeta\|w_t^{n,s}\|^2+\zeta\|\nabla u^{n-1,s}\|^2+\zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2\\ &+\zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2+\zeta\|\nabla w^{n,s}\|+\zeta\|\nabla u^{n-1,s}\|\\ &+\zeta\|\rho^{n-1,s}\|_{L^6}^2+\frac{\theta^2 C}{4\zeta}\|Bw^{n,s}\|^2. \end{aligned} \tag{4.9}$$

Multiplying (4.2) by $w_t^{n,s}$ and estimating as usual, we obtain

$$\begin{aligned}\alpha\|w_t^{n,s}\|^2&+\frac{c_a+c_d}{2}\frac{d}{dt}\|\nabla w^{n,s}\|^2+\frac{c_0+c_d-c_a}{2}\frac{d}{dt}\|\text{div } w^{n,s}\|^2+2\mu_r\frac{d}{dt}\|w^{n,s}\|^2\\ &\le C_\zeta\|\nabla u^{n-1,s}\|^2+C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2+C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2\\ &\quad+C_\zeta\|\nabla w^{n,s}\|^2+C_\zeta\|\nabla u^{n-1,s}\|^2+C_\zeta\|\rho^{n-1,s}\|_{L^6}^2+6\zeta\|w_t^{n,s}\|^2. \end{aligned} \tag{4.10}$$

Adding (4.5) and (4.6), we get

$$\begin{aligned}\alpha\|u_t^{n,s}\|^2&+\frac{\mu+\mu_r}{2}\frac{d}{dt}\|\nabla u^{n,s}\|^2+\delta(\mu+\mu_r)\|Au^{n,s}\|^2\\ &\le 2C_\eta\|\nabla w^{n-1,s}\|^2+2C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2+2C_\eta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2\\ &\quad+2C_\eta\|\nabla u^{n,s}\|^2+2C_\eta\|\nabla u^{n-1,s}\|^2+2C_\eta\|\rho^{n-1,s}\|_{L^6}^2+6\eta\|u_t^{n,s}\|^2\\ &\quad+\frac{C\delta^2}{4\eta}\|Au^{n,s}\|^2. \end{aligned}$$

Choosing $\eta=\dfrac{\alpha}{12}$ and $\delta>0$ such that $(\mu+\mu_r)\delta-\dfrac{C\delta^2}{4\eta}>0$, we have

$$\begin{aligned}\alpha\|u_t^{n,s}\|^2&+(\mu+\mu_r)\frac{d}{dt}\|\nabla u^{n,s}\|^2+c_1\|Au^{n,s}\|^2\\ &\le C\|\nabla w^{n-1,s}\|^2+C\|\rho^{n-1,s}\|_{L^6}^2\|f\|_{L^3}^2+C\|\rho^{n-1,s}\|_{L^6}^2\|\nabla u_t^n\|^2\\ &\quad+C\|\nabla u^{n,s}\|^2+C\|\nabla u^{n-1,s}\|^2+C\|\rho^{n-1,s}\|_{L^6}^2 \end{aligned} \tag{4.11}$$

with positive constants $c_1,\ C$ independent of n

Adding (4.9) and (4.10), we get

$$\begin{aligned}\alpha\|w_t^{n,s}\|^2&+\frac{c_a+c_d}{2}\frac{d}{dt}\|\nabla w^{n,s}\|^2+\theta(c_a+c_d)\|Bw^{n,s}\|^2\\ &+\frac{c_0+c_d-c_a}{2}\frac{d}{dt}\|\text{div } w^{n,s}\|^2+2\mu_r\frac{d}{dt}\|w^{n,s}\|^2\\ &\le 4C_\zeta\|\nabla u^{n-1,s}\|^2+2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2+2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2\\ &\quad+(\theta+1)C_\zeta\|\nabla w^{n,s}\|^2+2C_\zeta\|\rho^{n-1,s}\|_{L^6}^2+6\zeta\|w_t^{n,s}\|^2+\frac{C\theta^2}{4\zeta}\|Bw^{n,s}\|^2. \end{aligned}$$

Now, choosing $\zeta = \dfrac{\alpha}{12}$ and $\theta = \dfrac{(c_a+c_d)\alpha}{25\beta^2}$, and noting $c_2 = \frac{(c_a+c_d)^2\alpha}{(25\beta)^2}$, we get

$$\begin{aligned}
&\alpha\|w_t^{n,s}\|^2 + (c_a+c_d)\frac{d}{dt}\|\nabla w^{n,s}\|^2 + c_2\|Bw^{n,s}\|^2 + (c_0+c_d-c_a)\frac{d}{dt}\|\text{div } w^{n,s}\|^2 \\
&\quad +4\mu_r\frac{d}{dt}\|w^{n,s}\|^2 \le C\|\nabla u^{n-1,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|g\|_{L^3}^2 + C\|\rho^{n-1,s}\|_{L^6}^2\|\nabla w_t^n\|^2 \\
&\qquad +C\|\nabla w^{n,s}\|^2 + C\|\rho^{n-1,s}\|_{L^6}^2. \qquad (4.12)
\end{aligned}$$

Adding (4.11) and (4.12), integrating from 0 to t, we obtain

$$\begin{aligned}
&(\mu+\mu_r)\|\nabla u^{n,s}(t)\|^2 + (c_a+c_d)\|\nabla w^{n,s}(t)\|^2 + \alpha\int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2)d\tau \\
&\quad +c_1\int_0^t \|Au^{n,s}(\tau)\|^2 d\tau + c_2\int_0^t \|Bw^{n,s}(\tau)\|^2 d\tau + (c_0+c_d-c_a)\|\text{div } w^{n,s}(t)\|^2 \\
&\quad \le C\int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau \\
&\qquad +C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2(\|f(\tau)\|_{L^3}^2 + \|g(\tau)\|_{L^3}^2)d\tau \\
&\qquad +C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2(\|\nabla u_t^n(\tau)\|^2 + \|\nabla w_t^n(\tau)\|^2)d\tau \\
&\qquad +C\int_0^t (\|\nabla u^{n,s}(\tau)\|^2 + \|\nabla w^{n,s}(\tau)\|^2)d\tau + C\int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2 d\tau. \qquad (4.13)
\end{aligned}$$

¿From (4.4), with $r=6$, $\forall\ \tau\in(0,t),\ 0<t<T$, we have

$$\|\rho^{n-1,s}(t)\|_{L^r}^2 \le C\int_0^\tau \|\nabla u^{n-1,s}(\tau)\|^2 d\tau \le C\int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 d\tau$$

and replacing this last inequality in (4.13), we obtain

$$\begin{aligned}
&c_3(\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2) + c_3\int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2)d\tau \\
&\quad +c_3\int_0^t (\|Au^{n,s}(\tau)\|^2 + \|Bw^{n,s}(\tau)\|^2)d\tau + c_3\|\text{div } w^{n,s}(t)\|^2 \\
&\quad \le C\int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau \\
&\qquad +C\int_0^t \|\nabla u^{n-1,s}(t_1)\|^2 dt_1\int_0^t (\|f(\tau)\|_{L^3}^2 + \|g(\tau)\|_{L^3}^2)d\tau \\
&\qquad +C\int_0^t \|\nabla u^{n-1,s}(t_1)\|^2 dt_1\int_0^t (\|\nabla u_t^n(\tau)\|^2 + \|\nabla w_t^n(\tau)\|^2)d\tau \\
&\qquad +C\int_0^t (\|\nabla u^{n,s}(\tau)\|^2 + \|\nabla w^{n,s}(\tau)\|^2)d\tau + CT\int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 d\tau.
\end{aligned}$$

where $c_3 = \min\{\ \mu + \mu_r,\ c_a + c_d,\ \alpha,\ c_1,\ c_2\ \}$.

Then,

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 + \int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2)d\tau + \int_0^t \|Au^{n,s}(\tau)\|^2 d\tau$$
$$+\int_0^t \|Bw^{n,s}(\tau)\|^2 d\tau \le c\int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau$$
$$+c\int_0^t (\|\nabla u^{n,s}(\tau)\|^2 + \|\nabla w^{n,s}(\tau)\|^2)d\tau.$$

Applying the Gronwall's inequality (see Varhorn (1994), Lemma 3.10 page 122), we get

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 + \int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2)d\tau + \int_0^t \|Au^{n,s}(\tau)\|^2 d\tau$$
$$+\int_0^t \|Bw^{n,s}(\tau)\|^2 d\tau \le M_1 \int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau \tag{4.14}$$

Thus, we have

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \le M_1 \int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau.$$

Since $\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \le M,\ \forall n, s$ and $t \in [0, T]$, using the Lemma 4.1, we obtain

$$\|\nabla u^{n,s}(t)\|^2 + \|\nabla w^{n,s}(t)\|^2 \le M\frac{(M_1 t)^{n-1}}{(n-1)!} \le M\frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{4.15}$$

We observe that

$$M_1 \int_0^t (\|\nabla u^{n-1,s}(\tau)\|^2 + \|\nabla w^{n-1,s}(\tau)\|^2)d\tau \le M_1 \int_0^t M\frac{(M_1 \tau)^{n-2}}{(n-2)!}d\tau \le M\frac{(M_1 t)^{n-1}}{(n-1)!}. \tag{4.16}$$

Therefore, from (4.14) and (4.16), we have

$$\int_0^t (\|u_t^{n,s}(\tau)\|^2 + \|w_t^{n,s}(\tau)\|^2)d\tau \le M\frac{(M_1 t)^{n-1}}{(n-1)!} \le M\frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{4.17}$$

$$\int_0^t (\|Au^{n,s}(\tau)\|^2 + \|Bw^{n,s}(\tau)\|^2)d\tau \le M\frac{(M_1 t)^{n-1}}{(n-1)!} \le M\frac{(M_1 T)^{n-1}}{(n-1)!}. \tag{4.18}$$

Now, from (4.3), we have

$$\begin{aligned} \rho_t^{n,s} + u^n.\nabla\rho^{n,s} &= -u^{n,s}\cdot\nabla\rho^{n+s} \\ \rho^{n,s}(0) &= 0. \end{aligned}$$

Let $z^n(x,t,\tau)$ be the solution of the Cauchy problem

$$\begin{aligned} z^n_t &= u^n(z^n,\tau) \\ z^n &= x \ \text{ for } \ \tau = t. \end{aligned}$$

Then, using the characteristic method, we have

$$\rho^{n,s}(x,t) = -\int_0^t u^{n,s}(z^n(\tau),\tau).\nabla\rho^{n+s}(z^n(\tau),\tau)d\tau.$$

Bearing in mind properties of z^n (see Ladyzhenskaya and Solonnikov (1978), p. 93-96), we get

$$\|\rho^{n,s}(t)\|_{L^\infty} \le \|\nabla\rho^{n+s}\|_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t \|u^{n,s}(\tau)\|_{L^\infty}d\tau \le c\int_0^t \|Au^{n,s}(\tau)\|d\tau.$$

Hence, applying the Cauchy-Schwartz's inequality and observing (4.18), we have

$$\|\rho^{n,s}(t)\|^2_{L^\infty} \le c\int_0^t \|Au^{n,s}(\tau)\|^2 d\tau \le M\frac{(M_1t)^{n-1}}{(n-1)!} \le M\frac{(M_1T)^{n-1}}{(n-1)!}. \tag{4.19}$$

Integrating from 0 to t the inequality (4.15), we obtain

$$\int_0^t (\|\nabla u^{n,s}(\tau)\|^2 + \|\nabla w^{n,s}(\tau)\|^2)d\tau \le M\frac{(M_1t)^n}{n!} \le M\frac{(M_1T)^n}{n!}. \tag{4.20}$$

Also, from (4.4) and (4.20), for $2 \le r \le 6$, we have

$$\|\rho^{n,s}(t)\|^2_{L^r} \le M\frac{(M_1T)^n}{n!}. \tag{4.21}$$

At this point we have proved the first six bounds of Lemma 4.3 which correspond to (4.15), (4.17-21). The following bounds in the lemma require some technical manipulation. Let differentiate (4.1) with respect to t, multiply the result multiplying by $u^{n,s}_t$ and integrate on Ω. We get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^{n-1+s}}u^{n,s}_t\|^2 + (\mu+\mu_r)\|\nabla u^{n,s}_t\|^2 \\
&\quad = -\frac{1}{2}(\rho^{n-1+s}_t u^{n,s}_t, u^{n,s}_t) + 2\mu_r(\operatorname{rot} w^{n-1,s}_t, u^{n,s}_t) + (\rho^{n-1,s}f_t, u^{n,s}_t) \\
&\qquad -(\rho^{n-1,s}u^n_{tt}, u^{n,s}_t) - (\rho^{n-1+s}_t u^{n-1+s}\cdot\nabla u^{n,s}, u^{n,s}_t) \\
&\qquad -(\rho^{n-1+s}u^{n-1+s}_t.\nabla u^{n,s}, u^{n,s}_t) - (\rho^{n-1+s}u^{n-1+s}\cdot\nabla u^{n,s}_t, u^{n,s}_t) \\
&\qquad -(\rho^{n-1+s}_t u^{n-1,s}\cdot\nabla u^n, u^{n,s}_t) - (\rho^{n-1+s}u^{n-1,s}_t\cdot\nabla u^n, u^{n,s}_t) \\
&\qquad -(\rho^{n-1+s}u^{n-1,s}\cdot\nabla u^n_t, u^{n,s}_t) - (\rho^{n-1,s}u^{n-1}_t.\nabla u^n, u^{n,s}_t) \\
&\qquad -(\rho^{n-1,s}u^{n-1}\cdot\nabla u^n_t, u^{n,s}_t) + (\rho^{n-1,s}_t f, u^{n,s}_t) \\
&\qquad -(\rho^{n-1,s}_t u^n_t, u^{n,s}_t) - (\rho^{n-1,s}_t u^{n-1}\cdot\nabla u^n, u^{n,s}_t).
\end{aligned}$$

Let group the terms containing $\rho_t^{n-1,s}$, namely

$$h_2 = (\rho_t^{n-1,s} f, u_t^{n,s}) - (\rho_t^{n-1,s} u_t^n, u_t^{n,s}) - (\rho_t^{n-1,s} u^{n-1} \cdot \nabla u^n, u_t^{n,s}),$$

and denote by $h1$ the remaining terms. Then, we have

$$\frac{d}{dt} \|\sqrt{\rho^{n-1+s}} u_t^{n,s}\|^2 + 2\,(\mu + \mu_r) \|\nabla u_t^{n,s}\|^2 = 2\,h_1 + 2\,h_2.$$

Multiplying this equation by $\sigma(t) = \min\{1, t\}$ and integrating the result from 0 to t, we get

$$\begin{aligned} &\sigma(t) \|\sqrt{\rho^{n-1+s}(t)} u_t^{n,s}(t)\|^2 + 2\,(\mu + \mu_r) \int_0^t \sigma(\tau) \|\nabla u_t^{n,s}(\tau)\|^2 d\tau \\ &\quad = \int_0^t \sigma'(t) \|\sqrt{\rho^{n-1+s}(\tau)} u_t^{n,s}(\tau)\|^2 d\tau + 2\,H_1(t) + 2\,H_2(t) \qquad (4.22) \end{aligned}$$

where $H_1(t) = \int_0^t \sigma(\tau)\, h_1(\tau)\, d\tau$ and $H_2(t) = \int_0^t \sigma(\tau)\, h_2(\tau)\, d\tau$.

Now, we estimate the right-hand side of the above equation. From the fact that $0 \le \sigma'(t) \le 1$ a. e. in $t \in [0, T]$, we have

$$\begin{aligned} \int_0^t \sigma'(t) \|\sqrt{\rho^{n-1+s}}(\tau) u_t^{n,s}(\tau)\|^2 d\tau &\le \beta \int_0^t \|u_t^{n,s}(\tau)\|^2 d\tau \\ &\le \beta\, M \frac{(M_1 T)^{n-1}}{(n-1)!} \qquad (4.23) \end{aligned}$$

as a consequence of (4.17).

It is easy to show that

$$H_1(t) \le c\, \frac{(M_1 T)^{n-2}}{(n-2)!} + \frac{\mu + \mu_r}{4} \int_0^t \sigma(\tau) \|\nabla u_t^{n,s}(\tau)\|^2 d\tau. \qquad (4.24)$$

For each term in $h_2(t)$, using (4.3) and integration by parts, we can obtain the same kind of bound. In fact,

$$\begin{aligned} (\rho_t^{n-1,s} f, u_t^{n,s}) &= -((u^{n-1,s} \cdot \nabla \rho^{n-1+s})\psi, u_t^{n,s}) - ((u^{n-1} \cdot \nabla \rho^{n-1,s})\psi, u_t^{n,s}) \\ &= -((u^{n-1,s} \cdot \nabla \rho^{n-1+s})\psi, u_t^{n,s}) + (\rho^{n-1,s} u^{n-1} \cdot \nabla \psi, u_t^{n,s}) \\ &\quad + (\rho^{n-1,s} u^{n-1} \cdot \nabla u_t^{n,s}, \psi) \\ &\le \|u^{n-1,s}\|_{L^4} \|\nabla \rho^{n-1+s}\|_{L^\infty} \|\psi\| \|u_t^{n,s}\|_{L^4} \\ &\quad + \|\rho^{n-1,s}\|_{L^6} \|u^{n-1}\|_{L^\infty} \|\nabla \psi\| \|u_t^{n,s}\|_{L^3} \\ &\quad + \|\rho^{n-1,s}\|_{L^6} \|u^{n-1}\|_{L^\infty} \|\nabla u_t^{n,s}\| \|\psi\|_{L^3} \\ &\le C \|\nabla u^{n-1,s}\| \|\psi\| \|\nabla u_t^{n,s}\| + C \|\rho^{n-1,s}\|_{L^6} \|\psi\|_{H^1} \|\nabla u_t^{n,s}\| \\ &\le C_\eta \|\nabla u^{n-1,s}\|^2 \|\psi\|^2 + C_\eta \|\rho^{n-1,s}\|_{L^6}^2 \|\psi\|_{H^1}^2 + 2\eta \|\nabla u_t^{n,s}\|^2. \qquad (4.25) \end{aligned}$$

Taking respectively $\psi = f$, $\psi = u_t^n$ and $\psi = u^{n-1} \cdot \nabla u_t^n$, choosing $\eta = \dfrac{\mu + \mu_r}{24}$, we have

$$\begin{aligned} H_2(t) \leq\ & c \int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 \|f(\tau)\|^2 d\tau + c \int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2 \|f(\tau)\|_{H^1}^2 d\tau \\ & + c \int_0^t \|\nabla u^{n-1,s}(\tau)\|^2 d\tau + c \int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2 \|\nabla u_t^n(\tau)\|^2 d\tau \\ & + c \int_0^t \|\rho^{n-1,s}(\tau)\|_{L^6}^2 d\tau + \frac{\mu + \mu_r}{4} \int_0^t \sigma(\tau) \|\nabla u_t^{n,s}(\tau)\|^2 d\tau. \end{aligned}$$

Now, using (4.15), (4.20) and (4.21), we obtain

$$H_2(t) \leq M \frac{(M_1 T)^{n-2}}{(n-2)!} + \frac{\mu + \mu_r}{4} \int_0^t \sigma(\tau) \|\nabla u_t^{n,s}(\tau)\|^2 d\tau. \tag{4.26}$$

Therefore, carrying (4.24) and (4.26) in (4.22), we obtain

$$\sigma(t) \|u_t^{n,s}\|^2 + \int_0^t \sigma(\tau) \|\nabla u_t^{n,s}(\tau)\|^2 d\tau \leq M \frac{(M_1 T)^{n-2}}{(n-2)!}. \tag{4.27}$$

which correspond to the bounds for $u_t^{n,s}$ in Lemma 4.3 (seventh and eighth bounds inequalities in lemma). For $w_t^{n,s}$ the arguments are similar.

The ninth bound in lemma is directly obtained from the first bound. Similarly, the tenth bound is consequence of the previous bound, because of the Sobolev embedding $L^\infty(\Omega) \subseteq H^2(\Omega)$.

Finally, the last bound of Lemma 4.3 is obtained, repeating the same argument used in Lemma 2.2 (see equation 3.29). That is, from (4.1) write

$$(\mu + \mu_r) A u^{n,s} = P(F)$$

where

$$\begin{aligned} F =\ & 2\mu_r \ rot\ w^{n-1,s} + \rho^{n-1,s} f - \rho^{n-1,s} u_t^n - \rho^{n-1+s} \cdot \nabla u^{n,s} \\ & - \rho^{n-1+s} u^{n-1,s} \cdot \nabla u^n - \rho^{n-1,s} u^{n-1} \cdot \nabla u^n - \rho^{n-1+s} u_t^{n,s}. \end{aligned}$$

Verify then that $F \in L^2(0,T; L^6(\Omega))$ and apply the Amrouche-Girault result (1991). This completes the proof of Lemma 4.3.

Now, we are ready to prove Theorem 2.3.

Firstly, we observe that $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega) \cap V)$ is a Banach space and consequently the Lemma 4.3 implies that there exists $u \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega) \cap V)$ such that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega) \cap V). \tag{4.28}$$

Analogously, the Lemma 4.3 implies that there exists $w \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;$ $H^2(\Omega) \cap H_0^1(\Omega))$ and $\rho \in L^\infty(0,T;L^\infty(\Omega))$ such that

$$\begin{aligned}
&w^n \longrightarrow w \quad \text{strongly in} \quad L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)),\\
&\rho^n \longrightarrow \rho \quad \text{strongly in} \quad L^\infty(0,T;L^\infty(\Omega)).
\end{aligned}$$

Also, the Lemma 4.3 implies that there exists $v \in L^2(0,T;H)$ such that

$$u_t^n \longrightarrow v \quad \text{strongly in} \quad L^2(0,T;H).$$

By standard arguments, (4.28) implies that $v = u_t$.

Similarly, $w_t^n \longrightarrow w_t$ strongly in $L^2(0,T;L^2(\Omega))$. Moreover, from Lemma 4.3, for every $\epsilon > 0$, we get

$$\begin{aligned}
&u_t^n \longrightarrow u_t \quad \text{strongly in} \quad L^\infty(\epsilon,T;H) \cap L^2(\epsilon,T;V),\\
&w_t^n \longrightarrow w_t \quad \text{strongly in} \quad L^\infty(\epsilon,T;L^2(\Omega)) \cap L^2(\epsilon,T;H_0^1(\Omega)),\\
&u^n \longrightarrow u \quad \text{strongly in} \quad L^\infty(\epsilon,T;H^2(\Omega) \cap V) \cap L^2(\epsilon,T;W^{1,\infty}(\Omega) \cap V),\\
&w^n \longrightarrow w \quad \text{strongly in} \quad L^\infty(\epsilon,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap L^2(\epsilon,T;W^{1,\infty}(\Omega) \cap H_0^1(\Omega)).
\end{aligned}$$

Now, the next step is to take limit. But, once the above convergences have been established, this is a standard procedure, and we obtain

$$\begin{aligned}
&\int_0^T \langle\, \rho\, u_t + \rho\, u \cdot \nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu + \mu_r)\Delta u, v\,\rangle \phi(t)\, dt = 0,\\
&\int_0^T \langle\, \rho\, w_t + \rho\, u \cdot \nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a + c_d)\Delta w\\
&\qquad\qquad - (c_0 + c_d - c_a)\nabla \mathrm{div}\, w, z\,\rangle \psi(t)\, dt = 0,
\end{aligned}$$

for all $z,\, v \in L^2(\Omega)$ and $\phi,\, \psi \in L^\infty(0,T)$.

These equalities together with the Du Bois - Reymond's Theorem imply

$$\langle\, \rho\, u_t + \rho\, u \cdot \nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu + \mu_r)\Delta u, v\,\rangle = 0,$$

$$\langle\, \rho\, w_t + \rho\, u \cdot \nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \mathrm{div}\, w, z\,\rangle = 0,$$

a. e. in $[0,T]$, for every $v \in H,\ z \in L^2(\Omega)$.

These two last equalities, imply

$$P(\, \rho\, u_t + \rho\, u \cdot \nabla u - \rho\, f - 2\,\mu_r \mathrm{rot}\, w - (\mu + \mu_r)\Delta u\,) = 0 \quad \text{and}$$

$$\rho\, w_t + \rho\, u \cdot \nabla w - \rho\, g - 2\,\mu_r \mathrm{rot}\, u + 4\,\mu_r\, w - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \mathrm{div}\, w = 0.$$

For the density, we observe from Lemma 4.3 that

$$\begin{aligned} &u^n \longrightarrow u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \\ &\rho_t^n \longrightarrow \rho_t, \quad \text{and} \quad \nabla\rho^n \longrightarrow \nabla\rho \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), \end{aligned}$$

Thus, when $n \to \infty$ in the approximated continuity equation, we obtain

$$\rho_t + u \cdot \nabla\rho = 0 \quad \text{in the} \quad L^2(0,T;L^2(\Omega)) - \text{sense}.$$

Now, we prove the continuity established in Theorem 2.3 for solution (u, w, ρ). firstly, given that $u \in L^\infty(0,T;D(A))$ and $u_t \in L^2(\varepsilon,T;D(A))$, then by interpolation (see Temam (1979), p. 260) u is a.e. equal to a continuous function from $[\varepsilon, T]$ into $D(A)$, i.e.,

$$u \in C([\varepsilon,T];D(A)) \quad \forall\, \varepsilon > 0.$$

On the other hand, since $u_t \in L^2(\varepsilon,T;D(A))$, $u_{tt} \in L^2(\varepsilon,T;H)$, by interpolation we have

$$u_t \in C([\varepsilon,T];V), \quad \forall\, \varepsilon > 0.$$

Therefore,

$$u \in C^1([\varepsilon,T];V) \cap C([\varepsilon,T];D(A)), \quad \forall\, \varepsilon > 0.$$

Analogously, we prove that

$$w \in C^1([\varepsilon,T];H_0^1(\Omega)) \cap C([\varepsilon,T];D(B)), \quad \forall\, \varepsilon > 0.$$

To prove the continuity in $t = 0$, we proceed as follows.

It is easy to show that

$$\lim_{t\to 0^+} \|u(t) - u(0)\| = 0, \qquad \lim_{t\to 0^+} \|\nabla u(t) - \nabla u(0)\| = 0.$$

We prove then that

$$\lim_{t\to 0^+} \|Au(t) - Au(0)\| = 0.$$

To prove this, we observe that is sufficient to show that

$$\lim_{t\to 0^+} \sup \|Au(t)\| \leq \|Au_0\|$$

as we already know that $u(t) \longrightarrow u_0$ in $H^1(\Omega)$.

Multiplying (2.5) by Au_t^{n+1} and integrating in Ω, we have

$$\begin{aligned}
&\frac{\mu+\mu_r}{2}\frac{d}{dt}\|Au^{n+1}\|^2+\|\sqrt{\rho^n}\nabla u_t^{n+1}\|^2\\
&\qquad=-(\rho^n u^n\cdot\nabla u^{n+1},Au_t^{n+1})+2\mu_r(\operatorname{rot} w^n,Au_t^{n+1})+(\rho^n f,Au_t^{n+1})\\
&\qquad\quad-(\nabla\rho^n\cdot\nabla u_t^{n+1},u_t^{n+1}).
\end{aligned}$$

Then, integrating from 0 to t, we get

$$\begin{aligned}
\|Au^{n+1}(t)\|^2 \;\le\; & \|Au_0\|^2+\frac{2}{\mu+\mu_r}[(-\rho^n(t)u^n(t)\cdot\nabla u^{n+1}(t)+2\mu_r\operatorname{rot} w^n(t)\\
&+\rho^n(t)f(t),Au^{n+1}(t))-(-\rho_0^n u^n(0)\cdot\nabla u_0^{n+1}+2\mu_r\operatorname{rot} w_0^n\\
&+\rho_0^n f(0),Au_0^{n+1})]+\frac{2}{\mu+\mu_r}N(t)
\end{aligned}$$

uniformly in n and where

$$\begin{aligned}
N(t) \;=\; & \int_0^t |(\rho_t^n u^n\cdot\nabla u^{n+1}+\rho^n u_t^n\cdot\nabla u^{n+1}+\rho^n u^n\cdot\nabla u_t^{n+1}-2\mu_r\operatorname{rot} w_t^n\\
&-\rho_t^n f-\rho^n f_t,Au^{n+1})|\,d\tau+\int_0^t|(\nabla\rho^n\cdot\nabla u_t^{n+1},u_t^{n+1})|\,d\tau\\
\le\; & c\int_0^t(\|\nabla u^{n+1}\|+\|\nabla u_t^n\|+\|\nabla u_t^{n+1}\|+\|\nabla w_t^n\|+\|f\|+\|f_t\|)\,d\tau\le c\,t^{1/2}
\end{aligned}$$

by virtue of Hölder's inequality and the estimates given in Lemma 2.2.

¿From this, we conclude

$$\begin{aligned}
\|Au(t)\|^2 \;\le\; & \|Au_0\|^2+c\,[(-\rho(t)u(t)\cdot\nabla u(t)+2\mu_r\operatorname{rot} w(t)+\rho(t)f(t),Au(t))\\
&-(-\rho_0 u(0)\cdot\nabla u_0+2\mu_r\operatorname{rot} w_0+\rho_0 f(0),Au_0)]+c\,t^{1/2}.
\end{aligned}$$

Since $\rho(t)u(t)\cdot\nabla u(t)\longrightarrow\rho_0 u_0\cdot\nabla u_0$, $\rho(t)f(t)\longrightarrow\rho_0 f(0)$, $\operatorname{rot} w(t)\longrightarrow\operatorname{rot} w_0$ in $L^2(\Omega)$ and $Au(t)\longrightarrow Au_0$ weakly in $L^2(\Omega)$ as $t\to 0^+$, we obtain the desired result. From this, it is easy to show

$$\lim_{t\to 0^+}\|u_t(t)-u_t(0)\|=0.$$

The results for w are proved in the same way.

We need only to argument the uniqueness of the solution in order to complete the proof of Theorem 2.3. Suppose that there is another solution (u_1,w_1,ρ_1) of (1.1)-(1.2) with the same regularity as stated in the Theorem. Define:

$$U = u_1 - u,\ W = w_1 - w \text{ and } R = \rho_1 - \rho.$$

These auxiliary functions verify a set of equations similar to (4.1)–(4.3). If we multiply the first equation by U, the second by W and the third by R and repeat the argument given in Lemma 2.1, we obtain for $\varphi(t) = \|U(t)\|^2 + \|W(t)\|^2 + \|R(t)\|^2$ an inequality of the following type:

$$\varphi(t) \leq C \int_0^t \varphi(\tau)\, d\tau$$

which, by Gronwall's inequality, is equivalent to assert $U = 0$, $W = 0$ and $R = 0$.

5 Results on the pressure

Lemma 5.1 *With the hypotheses of Lemma 2.1, for each n, there exists $p^n \in L^2(0,T;H^1(\Omega)/I\!R)$ such that (u^n, w^n, ρ^n, p^n) is an approximate solution of the problem (1.1)-(1.2), where (u^n, w^n, ρ^n) is given by Lemma 2.1.*

With the hypotheses of Lemma 2.2, $p^n \in L^\infty(0,T;H^1(\Omega)/I\!R)$.

Proof. It is easily derived from (3.29) and the Amrouche-Girault's results (1991).

Lemma 5.2 *Under the hypotheses of the Lemma 4.3, we have*

$$\int_0^t \|p^{n+s}(\tau) - p^n(\tau)\|^2_{H^1(\Omega)/I\!R} d\tau \leq M_{14} \frac{(M_1 T)^{n-1}}{(n-1)!},$$

$$\sup_t \sigma(t) \|p^{n+s}(t) - p^n(t)\|^2_{H^1(\Omega)/I\!R} \leq M_{15} \frac{(M_1 T)^{n-2}}{(n-2)!}.$$

for all $t \in [0,T]$.

Proof. We denote $p^{n,s} = p^{n+s} - p^n, \ \ \forall\, n \geq 1$. Then, from (2.5) and (4.1), we have

$$-(\mu + \mu_r) \Delta u^{n,s} + \nabla p^{n,s} = J \tag{5.1}$$

$$\begin{aligned} \text{where} \quad J \quad = \quad & 2\mu_r \mathrm{rot}\, w^{n-1,s} + \rho^{n-1,s} f - \rho^{n-1,s} u_t^n - \rho^{n-1+s} u^{n-1+s} \cdot \nabla u^{n,s} \\ & - \rho^{n-1+s} u^{n-1,s} \cdot \nabla u^n - \rho^{n-1,s} u^{n-1} \cdot \nabla u^n - \rho^{n-1+s} u_t^{n,s}. \end{aligned} \tag{5.2}$$

Moreover,

$$\begin{aligned} \|J\|^2 \quad \leq \quad & c\, \|\nabla w^{n-1,s}\|^2 + c\, \|\rho^{n-1,s}\|^2_{L^6} \|f\|^2_{L^3} + c\, \|\rho^{n-1,s}\|^2_{L^6} \|u_t^n\|^2_{L^3} + c\, \|\nabla u^{n,s}\|^2 \\ & + c\, \|\nabla u^{n-1,s}\|^2 + c\, \|\rho^{n-1,s}\|^2_{L^6} \|\nabla u^n\|^2_{L^3} + c\, \|u_t^{n,s}\|^2. \end{aligned} \tag{5.3}$$

Now, (5.1)-(5.3) and the Amrouche-Girault's results (1991), imply

$$\|p^{n,s}\|^2_{H^1(\Omega)/I\!R} \leq c\,\|J\|^2 \tag{5.4}$$

and integrating from 0 to t, we get

$$\begin{aligned}\int_0^t \|p^{n,s}(\tau)\|^2_{H^1(\Omega)/I\!R}d\tau \quad \leq \quad & c\,M_5\frac{(M_1T)^{n-1}}{(n-1)!} + c\,M_6\frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|f(\tau)\|^2_{L^3}d\tau \\ & + c\,M_6\frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|\nabla u^n_t(\tau)\|^2 d\tau + c\,M_5\frac{(M_1T)^n}{n!} \\ & + c\,M_6\frac{(M_1T)^{n-1}}{(n-1)!}\int_0^t \|Au^n(\tau)\|^2 d\tau + c\,M_3\frac{(M_1T)^{n-1}}{(n-1)!}\end{aligned}$$

by virtue (4.16), (4.17), (4.20) and (4.21). Therefore,

$$\int_0^t \|p^{n,s}(\tau)\|^2_{H^1(\Omega)/I\!R}d\tau \leq M_{14}\frac{(M_1T)^{n-1}}{(n-1)!}.$$

Also, from (5.2) and (5.4), with $\sigma(t) = \min\{1, t\}$, we have

$$\begin{aligned}\sigma(t)\|p^{n,s}\|^2_{H^1(\Omega)/I\!R} \quad \leq \quad & c\,\|\nabla w^{n-1,s}\|^2 + c\,\|\rho^{n-1,s}\|^2_{L^\infty}\|f\|^2 + c\,\|\rho^{n-1,s}\|^2_{L^\infty} \\ & + c\,\|\nabla u^{n,s}\|^2 + c\,\|\nabla u^{n-1,s}\|^2 + c\,\|\rho^{n-1,s}\|^2_{L^6} + c\,\sigma(t)\|u^{n,s}_t\|^2 \\ \leq \quad & c\,M_2\frac{(M_1T)^{n-2}}{(n-2)!} + c\,M_4\frac{(M_1T)^{n-2}}{(n-2)!}\|f\|^2 + c\,M_4\frac{(M_1T)^{n-1}}{(n-1)!} \\ & + c\,M_2\frac{(M_1T)^{n-1}}{(n-1)!} + c\,M_6\frac{(M_1T)^{n-1}}{(n-1)!} + c\,M_8\frac{(M_1T)^{n-2}}{(n-2)!}\end{aligned}$$

by virtue (4.15), (4.19), (4.21) and (4.29).

Therefore, since by interpolation $f \in C([0,T]; L^2(\Omega))$, of the last inequality, we conclude

$$\sigma(t)\|p^{n,s}(t)\|^2_{H^1(\Omega)/I\!R} \leq M_{15}\frac{(M_1T)^{n-2}}{(n-2)!}. \tag{5.5}$$

Theorem 5.3 *Under the hypotheses of Lemma 4.3, the approximate pressure p^n converge to the limiting element p in $L^2(0,T; H^1(\Omega)/I\!R)$ and (u, w, ρ, p) is the unique solution of (1.1)-(1.2), where (u, w, ρ) is the solution given in the Theorem 2.3. Moreover, we have the following error estimate*

$$\int_0^t \|p^n(\tau) - p(\tau)\|^2_{H^1(\Omega)/I\!R}d\tau \leq M_{14}\frac{(M_1T)^{n-1}}{(n-1)!}.$$

Also, p^n converge to p in $L^\infty(\varepsilon, T; H^1(\Omega)/I\!R)$, for all $\varepsilon > 0$ and is satisfied the following error estimate

$$\sup_t \sigma(t) \|p^n(t) - p(t)\|^2_{H^1(\Omega)/I\!R} \leq M_{15} \frac{(M_1 T)^{n-2}}{(n-2)!}.$$

References

Amrouche C. & Girault V., 1991, *On the existence and regularity of the solutions of Stokes problem in arbitrary dimension*, Proc. Japan Acad., 67, Ser A, 171-175.

Antontsev S.N., Kazhikov A.V. & Monakhov V.N., 1990, *Boundary value problems in mechanics of nonhomogeneous fluids*, North-Holland, Amsterdam.

Boldrini J.L. & Rojas-Medar M.A., 1992, *Global solutions to the equations for the motion of stratified incompressible fluids*, Mat. Contemp. 3, 1-8.

Boldrini J.L. & Rojas-Medar M.A., 1997, *Global strong solutions of the equations for the motion of nonhomogeneous incompressible fluids*, Numerical methods in Mechanics, Pitman Research Notes in Math. Ser., N$^{\underline{o}}$ 371, 35-45.

Boldrini J.L. & Rojas-Medar M.A., 1996, *On the convergence rate of spectral approximations for the equations for nonhomogeneous asymmetric fluids*, Math. Modell. and Num. Anal. (M^2AN), 30(2), 123-155.

Boldrini J.L. & Rojas-Medar M.A., 1998, *Strong solutions of the equations for nonhomogeneous asymmetric fluids*, Technical Report N$^{\underline{o}}$ 21/98, IMECC-UNICAMP, Brazil.

Condiff D.W. & Dahler J.S., 1964, *Fluid mechanical aspects of antisymmetric stress*, Phys. Fluid V.11, N$^{\underline{o}}$ 9, 842-854.

Kim J.U., 1987, *Weak solutions of an initial boundary value problem for an incompressible viscous fluids with nonnegative density*, SIAM J. Math. Anal. 18, 89-96.

Ladyzhenskaya O.A., Solonnikov V.A. & Uralceva N.N., 1968, *Linear and quasilinear equations of parabolic type*, Translation of Mathematical Monograph Vol 23, revised edition, 1988.

Ladyzhenskaya O.A., 1969, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York.

LADYZHENSKAYA O.A. & SOLONNIKOV V.A., 1975, *Unique solvability of an initial and boundary value problem for viscous nonhomogeneous incompressible fluids*, Zap. Nauĉn Sem. - Leningrado Otdel Mat. Inst. Steklov, 52, 52-109; English Transl., J. Soviet Math., 9 (1978), 697-749.

LIONS J.L., 1969, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villars, Paris.

LUKASZEWICZ G., 1988, *On nonstationary flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL, Mem. Mat., vol. XII, 35–44.

LUKASZEWICZ G., 1989, *On the existence, uniqueness and asymptotic properties for solutions of flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL, Mem. Mat., Vol. XIII, 105-120.

LUKASZEWICZ G., 1990, *On nonstationary flows of incompressible asymmetric fluids*, Math. Methods Appl. Sci. 13 (3), 219-232.

LUKASZEWICZ G., 1999, *Micropolar fluids: Theory and Applications*, Birkhauser.

ORTEGA-TORRES E.E. & ROJAS-MEDAR M.A., 1997, *The equations of a viscous asymmetric fluid: An iterational approach*, Technical Report N° 42, IMECC-UNICAMP, Brazil.

PETROSYAN L.G., 1984, *Some problems of mechanics of fluids with antisymmetric stress tensor*, Erevan (in Russian).

SALVI R., 1991, *The equations of viscous incompressible nonhomogeneous fluid: on the existence and regularity*, J. Australian Math. Soc. Series B - Applied Math., 33 (1), 94-110.

SIMON J., 1990, *Nonhomogeneous viscous incompressible fluids; existence of velocity, density and pressure*, SIAM J. Math. Anal., 21, 1093-1117.

TEMAM R., 1979, *Navier-Stokes equations*, revised ed., Amsterdam, North-Holland.

VARNHORN W., 1994, *The Stokes equations*, Math. Research Vol 76, Berlin, Academie-Verlag.