Embedding of Level-Continuous Fuzzy Sets on Banach Spaces[1]

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Abstract. In this paper we present an extension of the Minkowski embedding theorem, showing the existence of an isometric embedding between the class $\mathcal{F}_c(\mathcal{X})$ of compact-convex and level-continuous fuzzy sets on a real separable Banach space $\mathcal{X}$ and $\mathcal{C}([0,1] \times B(\mathcal{X}^*))$, the Banach space of real continuous functions defined on the cartesian product between $[0,1]$ and the unit ball $B(\mathcal{X}^*)$ in the dual space $\mathcal{X}^*$. Also, by using of this embedding, we give some applications to the characterization of relatively compact subsets of $\mathcal{F}_c(\mathcal{X})$. In particular, an Ascoli-Arzelá type theorem is proved and applied to solving the Cauchy problem on $\mathcal{F}_c(\mathcal{X})$.

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1. INTRODUCTION

Let $\mathcal{F}_L(\mathcal{X})$ be the class of level-Lipschitz fuzzy sets on $\mathcal{X}$. This space has been exhaustively studied and used by many authors in the last years. For instance, the concept of Gaussian fuzzy random variables, limit theorems and fuzzy martingales was defined and studied by Puri&Ralescu in [15-21-22-23], respectively; while Kaleva [11-12] introduced the concept of fuzzy differential equation obtaining some results of existence and uniqueness on $\mathcal{F}_L(\mathcal{X})$. On the other hand, the space $\mathcal{F}_c(\mathcal{X})$ has been studied and used by the authors in recent papers, including Rojas&Bassanezi&Román [26] (embedding theorems), Rojas&Román [24] (convergence of fuzzy sets), Román-Flores [25] (compactness of spaces of fuzzy sets) and Román&Rojas [27] (differentiability of fuzzy mappings).

In particular, Puri&Ralescu in [15-21-23] showed that there is an embedding $j : \mathcal{F}_L(\mathcal{X}) \to \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$ which is an additive and positively homogeneus isometry. For the finite-dimensional case, Rojas&Bassanezi&Román in [26], by using multivalued Bernstein polynomials and its properties, proves that $\overline{\mathcal{F}_L(\mathcal{X})} = \mathcal{F}_c(\mathcal{X})$ and extend j to $\mathcal{F}_c(\mathcal{X})$.

Our purpose here, roughly speaking, is to present an extension of this last results to the infinite-dimensional case and to show some applications to the characterization of relative compact subsets of $\mathcal{F}_c(\mathcal{X})$, via demonstration of an Ascoli-Arzelá type theorem on $\mathcal{F}_c(\mathcal{X})$. Also, some examples are presented and, finally, we discuss some aspects of the Cauchy problem on $\mathcal{F}(\mathcal{X})$.

2. DEFINITIONS AND BASIC RESULTS

In the sequel, $\mathcal{X}$ will denote a real separable Banach space with norm $\| \ \|$ and with dual $\mathcal{X}^*$. We will also use the notation

$$\mathcal{K}(\mathcal{X}) = \{A \subseteq \mathcal{X} / \ A \text{ nonempty, compact and convex}\} .$$

A linear structure in $\mathcal{K}(\mathcal{X})$ is defined by

$$\begin{aligned} A + B &= \{a + b/a \in A, b \in B\} \\ \lambda A &= \{\lambda a/a \in A\} \end{aligned}$$

for all $A, B \in \mathcal{K}(\mathcal{X}),\ \lambda \in \mathbb{R}$.

The Hausdorff metric H on $\mathcal{K}(\mathcal{X})$ is defined by

$$H(A, B) = inf\,\{\epsilon > 0/ \ A \subseteq N(B, \epsilon) \ and \ B \subseteq N(A, \epsilon)\}$$

where $N(A, \epsilon) = \{x/ \ d(x, A) < \epsilon\}$ and $d(x, A) = \inf_{a \in A} \| \ x - a \ \|$.

It is well known that $(\mathcal{K}(\mathcal{X}), H)$ is complete and separable metric space (see [2-14-25]).

The norm of $A \in \mathcal{K}(\mathcal{X})$ is defined by $\| A \| = H(A, \{\mathbf{0}\})$.

For any $A \in \mathcal{K}(\mathcal{X})$, the support function s_A of A is defined on $\mathcal{X}^*$ as

$$s_A(x^*) = \sup_{a \in A} x^*(a), \ \forall x^* \in \mathcal{X}^*.$$

PROPOSITION 2.1. *If* $A, A_1, B, B_1, C \in \mathcal{K}(\mathcal{X})$ *then*

i) $H(\lambda A, \lambda B) = \lambda H(A, B)$, *for all* $\lambda \geq 0$.

ii) $H(A + B, A_1 + B_1) \leq H(A, A_1) + H(B, B_1)$

iii) $H(A + C, B + C) = H(A, B)$

iv) $H(A, B) = \sup_{\|x*\| \leq 1} \mid s_A(x^*) - s_B(x^*) \mid$

v) *For any* $x^* \in \mathcal{X}^*$

$$s_{A+B}(x^*) = s_A(x^*) + s_B(x^*).$$

For details see [1-10].

If we denote by $\mathcal{B}(\mathcal{X}^*) = \{x^* \in \mathcal{X}^* / \parallel x^* \parallel \leq 1\}$ the unit ball in the dual $\mathcal{X}^*$, then it is well known that $\mathcal{B}(\mathcal{X}^*)$ endowed with the $weak^*$-topology is compact and metrizable. Denote by ρ a metric which induces the $weak^*$-topology on $\mathcal{B}(\mathcal{X}^*)$. Also let $\mathcal{C}(\mathcal{B}(\mathcal{X}^*))$ denote the Banach space of functions $f : \mathcal{B}(\mathcal{X}^*) \to \mathbb{R}$ which are continuous with respect to the $weak^*$-topology in $\mathcal{B}(\mathcal{X}^*)$.

As a direct consequence of Proposition 2.1 and properties of support functions we obtain the following result:

COROLLARY 2.2. *The application* $\sigma : \mathcal{K}(\mathcal{X}) \to \mathcal{C}(\mathcal{B}(\mathcal{X}^*))$, $\sigma(A) = s_A$, *is such that*

i) σ *is an isometry (i.e.,* $H(A, B) = \sup_{x^* \in \mathcal{B}(\mathcal{X}^*)} \mid s_A(x^*) - s_B(x^*) \mid$*)*

ii) $\sigma(A + B) = \sigma(A) + \sigma(B)$

iii) $\sigma(\lambda A) = \lambda \sigma(A)$, $\forall\ \lambda \geq 0$.

For details see [15-23].

THEOREM 2.3. (Bernstein approximation). *Let* $F : [0, 1] \to \mathcal{K}(\mathcal{X})$ *be continuous. Define the* nth *Bernstein approximant* $B_n F$ *of* F *by*

$$B_nF(t) = \sum_{j=0}^{n} \begin{pmatrix} n \\ \\ j \end{pmatrix} t^j (1-t)^{n-j} F(j/n).$$

Then $\lim_{n\to\infty} B_nF = F$ *uniformly in* $[0,1]$, *i.e.,* $H(B_nF(t), F(t)) \to 0$ *uniformly in* $[0,1]$ *as* $n \to \infty$.

For more details on properties of Bernstein approximants see [4-30].

Now, we wil give the basic concepts and results on fuzzy sets theory which will be used in the rest of the paper.

A fuzzy set in $\mathcal{X}$ is a function $u : \mathcal{X} \to [0,1]$ (see [31]). We denote by $L_\alpha u = \{x \in \mathcal{X} \, / u(x) \geq \alpha\}$ for $0 < \alpha \leq 1$, the α-level of u, and $L_0 u = \mathbf{cl}\{x \in \mathcal{X} \, /u(x) > 0\} \ = supp(u)$ is called the support of u.

As an extension of $\mathcal{K}(\mathcal{X})$ we define the space $\mathcal{F}(\mathcal{X})$ of fuzzy sets $u : \mathcal{X} \to [0,1]$ with the following properties:

i) u is normal, i.e., $\{x \in \mathcal{X} / \ u(x) = 1\} \neq \emptyset$;

ii) u is fuzzy-convex, i.e., for all $x, y \in \mathcal{X}$ and $\lambda \in [0,1]$ we have

$$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\};$$

iii) u is upper semicontinuous;

iv) $L_0 u$ is compact.

A linear structure in $\mathcal{F}(\mathcal{X})$ is defined by

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda) & if \;\; \lambda \neq 0 \\ \chi_{\{\mathbf{0}\}}(x) & if \;\; \lambda = 0 \end{cases}$$

where $u, v \in \mathcal{F}(\mathcal{X})$ and $\lambda \in \mathbb{R}$.

With these definitions we obtain $L_\alpha(u+v) = L_\alpha u + L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$, for all $u, v \in \mathcal{F}(\mathcal{X})$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$ (see [18-22-26-27-28-29]).

We can be extend the Hausdorff metric to $\mathcal{F}(\mathcal{X})$ by mean

$$D(u,v) = \sup_{\alpha \in [0,1]} H(L_\alpha u, L_\alpha v), \; \forall u, v \in \mathcal{F}(\mathcal{X}).$$

We note that $(\mathcal{F}(\mathcal{X}), D)$ is complete (see [22]) but is not separable (see [3-24-25]).

The norm of $u \in \mathcal{F}(\mathcal{X})$ is defined as $\| u \| = D(u, \chi_{\{\mathbf{0}\}}) = \sup_{\alpha \in [0,1]} \| L_\alpha u \|$.

It is clear that $A \in \mathcal{K}(\mathcal{X})$ implies $\chi_A \in \mathcal{F}(\mathcal{X})$ and $\| A \| = \| \chi_A \|$.

Also, the support function of $u \in \mathcal{F}(\mathcal{X})$ is defined as

$$s_u : [0,1] \times \mathcal{X}^* \to \mathbb{R},\;\; s_u(\alpha, x^*) = s_{L_\alpha u}(x^*).$$

More details on properties and applications of support functions s_u can be found in [3-13-26].

PROPOSITION 2.4. *If* $u, v, w \in \mathcal{F}(\mathcal{X})$ *then*

i) $D(\lambda u, \lambda v) = \lambda D(u, v)$, *for all* $\lambda \geq 0$.

ii) $D(u + w, v + w) = D(u, v)$

iii) *For any* $x^* \in \mathcal{X}^*$

$$s_{u+v}(x^*) = s_u(x^*) + s_v(x^*).$$

For details see [3-26].

Define $\mathcal{F}_L(\mathcal{X})$ as the espace of fuzzy sets $u \in \mathcal{F}(\mathcal{X})$ with the property that the function $\alpha \mapsto L_\alpha u$ is Lipschitz with respect to the Hausdorff metric, i.e., there exists a constant $M > 0$ such that

$$H(L_\alpha u, L_\beta u) \leq M \mid \alpha - \beta \mid$$

for every $\alpha, \beta \in [0,1]$.

Analogously, we define $\mathcal{F}_c(\mathcal{X})$ as the space of fuzzy sets $u \in \mathcal{F}(\mathcal{X})$ with the property that the function $\alpha \mapsto L_\alpha u$ is H-continuous on $[0,1]$.

It is clear that $\mathcal{F}_L(\mathcal{X}) \subseteq \mathcal{F}_c(\mathcal{X})$.

Also let $\mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$ denote the Banach space of functions $f : [0,1] \times \mathcal{B}(\mathcal{X}^*) \to \mathbb{R}$ which are continuous with respect to the product of the topology in $\mathbb{R}$ and the *weak**-topology in $\mathcal{B}(\mathcal{X}^*)$.

THEOREM 2.5 ([15-21-23]). *The application* $j : \mathcal{F}_L(\mathcal{X}) \to \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$, $j(u) = s_u$, *is such that:*

i) *j is an isometry (i.e.,* $\| j(u) - j(v) \| = D(u,v)$)

ii) $j(u+v) = j(u) + j(v)$

iii) $j(\lambda u) = \lambda j(u),\ \lambda \geq 0$.

3. DENSITY OF $\mathcal{F}_L(\mathcal{X})$ IN $\mathcal{F}_c(\mathcal{X})$

The aims of this section is to show that $\overline{\mathcal{F}_L(\mathcal{X})} = \mathcal{F}_c(\mathcal{X})$ and, consequently, the isometric embedding j in Th.2.5 can be extended to $\mathcal{F}_c(\mathcal{X})$. Also, we want to remark that in [26] the authors obtain an analogous result in the setting of finite-dimensional spaces.

THEOREM 3.1. $(\mathcal{F}_c(\mathcal{X}), D)$ *is a complete metric space.*

Proof. Let (u_p) a D-Cauchy sequence in $\mathcal{F}_c(\mathcal{X})$. Then, because the space $(\mathcal{F}(\mathcal{X}), D)$ is complete, we deduce that there is $u \in \mathcal{F}(\mathcal{X})$ such that $u_p \overset{D}{\to} u$.

In continuation, we prove that $u \in \mathcal{F}_c(\mathcal{X})$. In fact, given $\epsilon > 0$ there is

$N \in \mathbb{N}$ such that $D(u_p, u) < \epsilon/3$ for all $p \geq N$. For a fixed $p_0 \geq N$ because $u_{p_0} \in \mathcal{F}_c(\mathcal{X})$, we have that there exist $\delta = \delta(\epsilon, p_0) > 0$ such that

$$\mid \alpha - \beta \mid < \delta \Rightarrow H(L_\alpha u_{p_0}, L_\beta u_{p_0}) < \epsilon/3. \tag{1}$$

Consequently, for $\mid \alpha - \beta \mid < \delta$ we have

$$\begin{aligned} H(L_\alpha u, L_\beta u) &\leq H(L_\alpha u, L_\alpha u_{p_0}) + H(L_\alpha u_{p_0}, L_\beta u_{p_0}) + H(L_\beta u_{p_0}, L_\beta u) \\ &\leq D(u, u_{p_0}) + \epsilon/3 + D(u_{p_0}, u) \qquad (by\ (1)) \\ &< \epsilon. \end{aligned}$$

So, $u \in \mathcal{F}_c(\mathcal{X})$ and the proof is complete.

To show the density of $\mathcal{F}_L(\mathcal{X})$ in $\mathcal{F}_c(\mathcal{X})$, we will use the Kuratowski limits and its connections with H-convergence.

If $(A_q)_{q\in\mathbb{N}}$ is a sequence of subsets of $\mathcal{X}$, we define the upper and lower limits in the Kuratowski sense as

$$\begin{aligned}\limsup_{q\to\infty} A_q &= \{x \in \mathcal{X} / \ x = \lim_{j\to\infty} x_{q_j}, \ x_{q_j} \in A_{q_j}\} \\ &= \bigcap_{q=1}^{\infty} \left(\overline{\bigcup_{m\geq q} A_m} \right)\end{aligned}$$

and,

$$\begin{aligned}\liminf_{q\to\infty} A_q &= \{x \in \mathcal{X} / \ x = \lim_{q\to\infty} x_q, \ x_q \in A_q\} \\ &= \bigcap_{H} \left(\overline{\bigcup_{m\in H} A_m} \right),\end{aligned}$$

respectively, where the last intersection is over all sets H cofinal in $\mathbb{N}$ (we recall that H is a cofinal subset of $\mathbb{N}$ if for all $n \in \mathbb{N}$ there is $h \in H$ such that $h > n$).

We say that the sequence (A_q) converges to A, $A \subseteq \mathcal{X}$, in the Kuratowski sense, if $\lim_{q\to\infty} \inf A_q = \lim_{q\to\infty} \sup A_q = A$; in this case, we write $A = \lim_{q\to\infty} A_q$ or $A_q \stackrel{K}{\to} A$, and we say that A_q K-converges to A.

The Kuratowski limits are closed sets. Moreover, the following relations are true:

$$\begin{aligned} \lim_{q\to\infty} \inf A_q &\subseteq \lim_{q\to\infty} \sup A_q \\ \lim_{q\to\infty} \inf \overline{A_q} &= \lim_{q\to\infty} \inf A_q \quad \text{and} \\ \lim_{q\to\infty} \sup \overline{A_q} &= \lim_{q\to\infty} \sup A_q. \end{aligned}$$

The following result is very useful (see [8-9-26]).

Lemma 3.2. *A sequence $(A_q) \subseteq \mathcal{X}$ converges to a compact set A respect to the Hausdorff metric if and only if there is K compact in $\mathcal{X}$ such that $A_q \subseteq K$ for all q and*

$$\lim_{q\to\infty} \inf A_q = \lim_{q\to\infty} \sup A_q = A.$$

We now state the following density result.

THEOREM 3.3. $\overline{\mathcal{F}_L(\mathcal{X})} = \mathcal{F}_c(\mathcal{X})$.

Proof. Let $u \in \mathcal{F}_c(\mathcal{X})$, then the set-valued function $F : [0,1] \to \mathcal{K}(\mathcal{X})$ given by $F(\alpha) = L_\alpha u$ is H-continuous on $[0,1]$. Now, if we consider the nth Bernstein polynomial $B_n F$ associated with F then, due Theor. 2.3, we have that $B_n F$ converges uniformly to F in H-metric on $[0,1]$.

We observe that $B_n F(\alpha) \in \mathcal{K}(\mathcal{X})$ for each $n \in \mathbb{N}$ and $\alpha \in [0,1]$. Now, we will verify the hypothesis of the Representation Theorem given by Negoita and Ralescu [19] to show that the family $N_\alpha = B_n F(\alpha)$, for each $n \in \mathbb{N}$, define an unique fuzzy set $u_n \in \mathcal{F}(\mathcal{X})$.

In fact, if $\alpha \leq \beta$, then $F(\alpha) \supseteq F(\beta)$ and, consequently, $B_n F(\alpha) \supseteq B_n F(\beta)$ (see Vitale [30, p. 312]. So, we only have to prove that, if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_l \nearrow \alpha \neq 0$ as $l \to \infty$, then

$$B_n F(\alpha) = \bigcap_{l=1}^{\infty} B_n F(\alpha_l). \tag{2}$$

We observe that $\alpha \mapsto B_n F(\alpha)$ is a H-continuous set-valued function (moreover, this map is H-Lipschitz as we will proves below) and, consequently, for each n, we have

$$B_nF(\alpha_l) \overset{H}{\to} B_nF(\alpha) \text{ as } l \to \infty.$$

So, because $(B_nF(\alpha_l))_l \subseteq \mathcal{K}(\mathcal{X})$ is an decreasing sequence we deduce, from Lemma 3.2, that $B_nF(\alpha_l) \overset{K}{\to} B_nF(\alpha)$ as $l \to \infty$.

Moreover, $B_nF(\alpha_l) \overset{K}{\to} \bigcap_{l=1}^{\infty} B_nF(\alpha_l)$ which implies that (2) is verified.

This completes the hypothesis of the Negoita-Ralescu Theorem.

We want to remark that the uniform convergence of B_nF to F on $[0, 1]$ ensures $u_n \overset{D}{\to} u$.

Finally, we want to prove that $(u_n) \subseteq \mathcal{F}_L(\mathcal{X})$ and, for this, it is sufficient to show that $\alpha \mapsto B_nF(\alpha)$ is a Lipschitzian application, for each $n \in \mathbb{N}$. In fact, due Proposition 2.1 iii) it is sufficient to show that

$$\sup_{||x^*|| \leq 1} |s_{B_nF(\alpha)}(x^*) - s_{B_nF(\beta)}(x^*)| \leq C|\alpha - \beta|$$

with $C > 0$ independent of α and β.

As the support function of Bernstein approximant of F is given by

$$s_{B_nF(\alpha)}(x^*) = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) \alpha^j (1-\alpha)^{q-j} s_{F(j/n)}(x^*)$$

with $x^* \in \mathcal{B}(\mathcal{X}^*)$, we have that

$$\begin{aligned} |s_{B_nF(\alpha)}(\mathrm{x}^*)\text{-}s_{B_nF(\beta)}(\mathrm{x}^*)| &\leq \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) |s_{F(j/n)}(x)|\ |\alpha^j (1\text{-}\alpha)^{q-j} - \beta^j (1\text{-}\beta)^{q-j}| \\ &\leq ||s_{F(0)}||_\infty \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) |\alpha^j (1\text{-}\alpha)^{q-j} - \beta^j (1\text{-}\beta)^{q-j}| \\ &\leq C|\alpha - \beta|, \end{aligned}$$

Since $F(0) \supseteq F(j/n) \supseteq F(1)$ for all $0 < j < q$, then

$$s_{F(1)}(x^*) \leq s_{F(j/n)}(x^*) \leq s_{F(0)}(x^*)$$

for all $x^* \in \mathcal{B}(\mathcal{X}^*)$. This completes the proof.

THEOREM 3.4. *The application* $j : \mathcal{F}_c(\mathcal{X}) \to \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$ *defined by*

$j(u) = s_u$ *is an additive and positively homogeneous isometry.*

Proof. Since $\mathcal{F}_L(\mathcal{X})$ is dense in $\mathcal{F}_c(\mathcal{X})$ then the isometry j in Theorem 2.5 has a unique uniformly continuous extension to $\mathcal{F}_c(\mathcal{X})$ and it is easy to show that this extension is also an isometry (see [5]).

COROLLARY 3.5. $D(u, v) = \sup_{(\alpha, x*) \in [0,1] \times \mathcal{B}(\mathcal{X}^*)} \mid s_u(\alpha, x^*) - s_v(\alpha, x^*) \mid$, for all $u, v \in \mathcal{F}_c(\mathcal{X})$.

COROLLARY 3.6. $(\mathcal{F}_c(\mathcal{X}), D)$ *is a separable metric space.*

Proof. It is well known that if M, N are metric spaces, M compact and N separable, then the space of continuous functions $\mathcal{C}(M, N)$, endowed with the uniform metric, is separable (see [17, p. 276]. Thus, because $[0, 1] \times \mathcal{B}(\mathcal{X}^*)$ is $\mid \mid \times weak^*$ compact then $\mathcal{C}([0, 1] \times \mathcal{B}(\mathcal{X}^*))$ is separable.

COROLLARY 3.7. *If* $u_p, u \in \mathcal{F}_c(\mathcal{X})$ *then* $u_p \overset{D}{\to} u$ *iff* $s_{u_p} \to s_u$ *uniformly on* $[0, 1] \times \mathcal{B}(\mathcal{X}^*)$.

The following result shows that $\mathcal{F}_c(\mathcal{X})$ is the maximal domain of j in $\mathcal{F}(\mathcal{X})$.

THEOREM 3.8. *If* $u \in \mathcal{F}(\mathcal{X})\backslash\mathcal{F}_c(\mathcal{X})$ *then* $j(u) \notin \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$.

Proof. The $|\ | \times weak^*-$topology on $[0,1] \times \mathcal{B}(\mathcal{X}^*)$ can be metrizable by mean

$$d\left((\alpha, x^*), (\beta, y^*)\right) = max\left\{\mid \alpha - \beta \mid, \rho(x^*, y^*)\right\}$$

(see, for instance, [5, p. 75], [6, p. 207]).

Let $j(u) \in \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$ be, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$d\left((\alpha, x^*), (\beta, y^*)\right) < \delta \Longrightarrow \mid\ j(u)(\alpha, x^*) - \ j(u)(\beta, y^*) \mid < \epsilon. \tag{3}$$

Now, if we suppose that $u \notin \mathcal{F}_c(\mathcal{X})$ with $j(u) \in \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$, then we have that the map $\alpha \mapsto L_\alpha u$ is not continuous; consequently there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ we can to choose $\alpha, \beta \in [0,1]$ with $\mid \alpha - \beta \mid < \delta$ and

$$H(L_\alpha u, L_\beta u) = \sup_{x^* \in \mathcal{B}(\mathcal{X}^*)} \mid j(u)(\alpha, x^*) - j(u)(\beta, x^*) \mid \geq \epsilon_0.$$

Thus, due to compactness of $([0,1] \times \mathcal{B}(\mathcal{X}^*), \rho)$ and the continuity of $j(u)$ there exists $x_0^* \in \mathcal{B}(\mathcal{X}^*)$ such that $\mid j(u)(\alpha, x_0^*) - j(u)(\beta, x_0^*) \mid \geq \epsilon_0$ which contradicts (3) and the proof is complete.

4. RELATIVE COMPACTNESS IN $\mathcal{F}_c(\mathcal{X})$

We recall that if M, N are metric spaces then $E \subseteq \mathcal{C}(M,N)$ is equicontinuous in $a \in M$ if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $d(x,a) < \delta$ in M implies $d(f(x), f(a)) \leq \epsilon$ in N, for every $f \in E$. We say that E is equicontinuous if E is equicontinuous in $a \in M$, $\forall a \in M$.

THEOREM 4.1. *(Ascoli-Arzelá). Let $E \subseteq \mathcal{C}(M,N)$ be, M compact. Then E is relatively compact if and only if*

i) *E equicontinuous*

ii) *for every $a \in M$, the set $E(a) = \{f(a)/f \in E\}$ is relatively compact in N.*

(see [17, Prop. 16, p. 244]).

By using the isometric embedding $j : \mathcal{F}_c(\mathcal{X}) \to \mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$ and $|\ | \times weak^*-$compactness of $[0,1] \times \mathcal{B}(\mathcal{X}^*)$, we want to give an characterization of the relatively compact subsets of $\mathcal{F}_c(\mathcal{X})$.

REMARK 4.2. It is clear that A is relatively compact in $\mathcal{F}_c(\mathcal{X})$ if and only if $j(A)$ is relatively compact in $\mathcal{C}([0,1] \times \mathcal{B}(\mathcal{X}^*))$.

In fact, if A relatively compact then $j(\overline{A})$ is compact. So, because $j(A) \subseteq j(\overline{A})$, we have $\overline{j(A)} \subseteq j(\overline{A})$ and, therefore, $\overline{j(A)}$ is compact.

Conversely, if $j(A)$ is relatively compact, because $\overline{A} = j^{-1}\left(\overline{j(A)}\right)$ and $j^{-1} : j\left(\mathcal{F}_c(\mathcal{X})\right) \to \mathcal{F}_c(\mathcal{X})$ is continuous, then $\overline{A}$ is compact.

DEFINITION 4.3. *A subset $A \subseteq \mathcal{F}_c(\mathcal{X})$ is said compact-supported if there exists a compact $K \subseteq \mathcal{X}$ such that $L_0 u \subseteq K$, $\forall u \in A$.*

DEFINITION 4.4. *A subset $A \subseteq \mathcal{F}_c(\mathcal{X})$ is said level-equicontinuous in $\alpha_0 \in [0,1]$,* if $\forall \epsilon > 0$ *there exists $\delta > 0$ such that*

$$| \alpha - \alpha_0 | < \delta \Rightarrow H(L_\alpha u, L_{\alpha_0} u) \leq \epsilon, \ \forall u \in A.$$

A is equicontinuous on $[0,1]$ if A is equicontinuous in α, for all $\alpha \in [0,1]$.

THEOREM 4.5. *Let A be a compact-supported subset of $\mathcal{F}_c(\mathcal{X})$. Then are equivalent:*

i) *A is a relatively compact subset of $(\mathcal{F}_c(\mathcal{X}), D)$*

ii) *A is level-equicontinuous on* $[0,1]$.

Proof. i)→ii). Because A is relatively compact in $\mathcal{F}_c(\mathcal{X})$ then $j(A)$ is relatively compact in $\mathcal{C}([0,1]\times\mathcal{B}(\mathcal{X}^*))$ (Remark 4.2), i.e.,

a) $\forall(\alpha,x^*)\in[0,1]\times\mathcal{B}(\mathcal{X}^*)$, the set $j(A)(\alpha,x^*)=\{s_u(\alpha,x^*)/u\in A\}$ is relatively compact,

b) $j(A)$ is equicontinuous.

Therefore, given any $(\alpha_0,x_0^*)\in[0,1]\times\mathcal{B}(\mathcal{X}^*)$ and $\epsilon>0$, there exists an open ball

$$B((\alpha_0,x_0^*),r)=\{((\alpha,x^*)\in[0,1]\times\mathcal{B}(\mathcal{X}^*)/d\left((\alpha,x^*),(\alpha_0,x_0^*)\right)<r\}$$

such that

$$(\alpha,x^*)\in B((\alpha_0,x_0^*),r)\Rightarrow\mid s_u(\alpha,x^*)-s_u(\alpha_0,x_0^*)\mid\leq\epsilon,\ \forall u\in A. \tag{4}$$

Now, if we suppose that A is not level-equicontinuous then there exists $\overline{\alpha}\in[0,1]$ and $\overline{\epsilon}>0$ such that $\forall\delta>0$ there is $\beta\in[0,1]$ with

$$\mid\overline{\alpha}-\beta\mid<\delta \text{ and } H(L_{\overline{\alpha}}u,L_\beta u)>\overline{\epsilon},\ \text{ for some } u\in A. \tag{5}$$

Due $H(L_\alpha u, L_\beta u) = \sup_{\|x*\|\leq 1} \mid s_{L_{\overline{\alpha}}u}(x^*) - s_{L_\beta u}(x^*) \mid$ (Prop. 2.1) and *weak**-compactness of $\mathcal{B}(\mathcal{X}^*)$, there exists $y^* \in \mathcal{B}(\mathcal{X}^*)$ such that

$$\mid s_{L_{\overline{\alpha}}u}(y^*) - s_{L_\beta u}(y^*) \mid = \mid s_u(\overline{\alpha}, y^*) - s_u(\beta, y^*) > \overline{\epsilon}. \tag{6}$$

Therefore, given if $\delta > 0$ we can to choose β, u and y^* verifying (5) and (6), and taking the open ball $B((\overline{\alpha}, y^*), \delta)$ we obtain that $(\beta, y^*) \in B((\overline{\alpha}, y^*), \delta)$ but $\mid s_u(\overline{\alpha}, y^*) - s_u(\beta, y^*) > \overline{\epsilon}$ which contradicts (4).

ii)$\rightarrow$i). Suppose that A is level-equicontinuous on $[0,1]$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mid \alpha - \alpha_0 \mid < \delta \Rightarrow H(L_\alpha u, L_{\alpha_0} u) \leq \epsilon/2,\ \forall u \in A. \tag{7}$$

On the other hand, we can to define another distance ρ_0 on $\mathcal{B}(\mathcal{X}^*)$ by

$$\rho_0(x^*, y^*) = \sup_{a \in K} \mid \langle x^* - y^*, a \rangle \mid$$

and note that ρ_0 is continuous in the *weak**-topology, therefore given $\epsilon >$

0 there exists $\delta > 0$ such that

$$\rho(x^*, y^*) < \delta \Rightarrow \rho_0(x^*, y^*) < \epsilon/2 \tag{8}$$

(see [23, p. 119]).

Because A is compact-supported then $j(A)(\alpha, x^*)$ is relatively compact, $\forall(\alpha, x^*) \in [0,1] \times \mathcal{B}(\mathcal{X}^*)$.

Now, if we suppose that A is not relatively compact then $j(A)$ is not relatively compact and, therefore, there exists $(\alpha_0, x_0^*) \in [0,1] \times \mathcal{B}(\mathcal{X}^*)$ and $\epsilon > 0$ such that for every open ball $B((\alpha_0, x_0^*), \delta)$ we have

$$\mid s_u(\alpha, x^*) - s_u(\alpha_0, x_0^*) \mid > \epsilon, \tag{9}$$

for some $u \in A$ and $(\alpha, x^*) \in B((\alpha_0, x_0^*), \delta)$.

Thus, due (9), taking $\delta = 1/n$ we can to found sequences $\{(\alpha_n, x_n^*)\}$ in $[0,1] \times \mathcal{B}(\mathcal{X}^*)$ and $\{u_n\}$ in A such that

$$d\left((\alpha_n, x_n^*), (\alpha_0, x_0^*)\right) < 1/n, \ \forall n, \tag{10}$$

and

$$| s_{u_n}(\alpha_n, x_n^*) - s_{u_n}(\alpha_0, x_0^*) | > \epsilon, \ \forall n. \tag{11}$$

Consequently, due (11) we obtain

$$| s_{L_{\alpha_n} u_n}(x_n^*) - s_{L_{\alpha_0} u_n}(x_n^*) | + | s_{L_{\alpha_0} u_n}(x_n^*) - s_{L_{\alpha_0} u_n}(x_0^*) | > \epsilon, \ \forall n,$$

and this implies that

$$H(L_{\alpha_n} u_n, L_{\alpha_0} u_n) + | s_{L_{\alpha_0} u_n}(x_n^*) - s_{L_{\alpha_0} u_n}(x_0^*) | > \epsilon, \ \forall n. \tag{12}$$

Also, a straightforward calculus show that

$$| s_{L_{\alpha_0} u_n}(x_n^*) - s_{L_{\alpha_0} u_n}(x_0^*) | \leq \sup_{a \in K} | \langle x_n^* - x_0^*, a \rangle | = \rho_0(x_n^*, x_0^*), \tag{13}$$

and, due (8) and (10), we have

$$\rho_0(x_n^*, x_0^*) < \epsilon/2, \text{ for every } n \text{ sufficiently large.} \tag{14}$$

Consequently, due (10), (12), (13) and (14) we conclude that

$$| \alpha_n - \alpha_0 |< 1/n \text{ and } H(L_{\alpha_n} u_n, L_{\alpha_0} u_n) > \epsilon/2, \ \forall n,$$

which contradicts (7), i.e., A is not level-equicontinuous and the proof is complete.

REMARK 4.6. Recently, in the setting of $\mathcal{X} = \mathbb{R}^n$, Greco [9, Th.2.3] proves an analogous result to Th.4.5 for the class $\mathcal{F}_{sc}(\mathbb{R}^n) = \{u \in \mathcal{F}(\mathbb{R}^n)/send(u) \ convex\}$, which is a proper subclass of $\mathcal{F}_c(\mathbb{R}^n)$. In fact, if $f \in \mathcal{F}_{sc}(\mathbb{R}^n)$ then $\overline{\{f > \alpha\}} = \{f \geq \alpha\}$ for all $\alpha \in (0,1)$ (see [9]), and this implies that $f \in \mathcal{F}_c(\mathbb{R}^n)$ (see [26]). On the other hand, taking $u \in \mathcal{F}(\mathbb{R}^1)$ defined by

$$u(x) = \begin{cases} x^2 & if \quad x \in [0,1] \\ 0 & if \quad x \notin [0,1] \end{cases}$$

then we have that $u \in \mathcal{F}_c(\mathbb{R}^n) \setminus \mathcal{F}_{sc}(\mathbb{R}^n)$.

We recall that $send(u) = (supp(u) \times [0,1]) \cap \{(x, \alpha) \in \mathbb{R}^n \times [0,1]/u(x) \geq \alpha\}$.

Example 4.7. Define $A = \{f_n\}_{n\geq 1}$ where:

$$f_n(x) = \begin{cases} \frac{x}{n} + 1 - \frac{1}{n} & if \quad x \in [0,1] \\ 0 & if \quad x \notin [0,1]. \end{cases}$$

Then A is a compact-supported subset of $\mathcal{F}_c(\mathbb{R})$. Furthermore,

$$L_\alpha f_n = \begin{cases} [0,1] & if \quad \alpha \leq 1 - \frac{1}{n} \\ [n(\alpha - 1) + 1, 1] & if \quad 1 - \frac{1}{n} < \alpha \leq 1. \end{cases}$$

Now, let $0 < \epsilon < 1$, $\alpha = 1$ and $\delta > 0$. We observe that if $\mid 1 - \beta \mid < \delta$, there exists $n \in \mathbb{N}$, such that $\beta \leq 1 - \frac{1}{n} < 1$. Thus,

$$H(L_1 f_n, L_\beta f_n) = H(\{1\}, [0,1]) = 1 > \epsilon.$$

Therefore, A is not level-equicontinuous and, consequently, A is not relatively compact in $\mathcal{F}_c(\mathbb{R})$.

EXAMPLE 4.8. For $t \in [0,1)$, define

$$u_t(x) = \begin{cases} 0 & if \;\; 0 \le x \le t \\ \frac{x-t}{1-t} & if \;\; t \le x < 1 \\ 0 & elsewhere. \end{cases}$$

Then it is clear that family $\{u_t\}$ is compact-supported. Moreover

$$L_\alpha u_t = [\alpha(1-t)+t, 1].$$

Consequently,

$$\begin{aligned} H(L_\alpha u_t, L_\beta u_t) &= H([\alpha(1-t)+t,1],[\beta(1-t)+t,1]) \\ &= \mid 1-t \mid\mid \alpha - \beta \mid \\ &\le \mid \alpha - \beta \mid . \end{aligned}$$

Thus, given $\epsilon > 0$ we take $\delta = \epsilon$ and we obtain

$$\mid \alpha - \beta \mid < \delta \Rightarrow H(L_\alpha u_t, L_\beta u_t) < \epsilon,$$

for every t.

This implies that $A = \{u_t/\ t \in [0,1)\}$ is a level-equicontinuous subset of $\mathcal{F}_c(\mathbb{R})$ and therefore, by Theor. 4.5, A is relatively compact.

EXAMPLE 4.9. Let K be a compact subset of $\mathcal{X}$ and consider

$$\widetilde{K} = \left\{\chi_{\{x\}}/\ x \in K\right\}.$$

Then $\widetilde{K}$ is relatively compact in $\mathcal{F}_c(\mathcal{X})$.

In fact, it is clear that $\widetilde{K}$ is compact-supported. Moreover,

$$H(L_\alpha \chi_{\{x\}}, L_\beta \chi_{\{x\}}) = H(\{x\}, \{x\}) = 0,\ \forall x \in K.$$

Therefore, $\widetilde{K}$ is a level-equicontinuous and, consequently, $\widetilde{K}$ is a relatively compact subset of $\mathcal{F}_c(\mathcal{X})$.

EXAMPLE 4.10. Let $m > 0$ and consider (see [7-11-12-16]):

$$E_m = \{u \in \mathcal{F}_c(\mathcal{X})/\ H(L_\alpha u, L_\beta u) \leq m \mid \alpha - \beta \mid\}.$$

Because E_m is not compact-supported then E_m is not relatively compact in

$\mathcal{F}_c(\mathcal{X})$.

Nevertheless, if K is a compact suset of $\mathcal{X}$ and

$$E_{K,m} = \{u \in E_m / \; L_0 u \subseteq K\},$$

then $E_{K,m}$ is a relatively compact subset of $\mathcal{F}_c(\mathcal{X})$. In fact, it is clear that $E_{K,m}$ is compact-supported and given $\epsilon > 0$ we can to choose $\delta = \epsilon/m$, obtaining

$$\mid \alpha - \beta \mid < \delta \Rightarrow H(L_\alpha u, L_\beta u) \leq m \mid \alpha - \beta \mid < \epsilon, \; \forall u \in E_{K,m}.$$

That is, A is level-equicontinuous and, consequently, relatively compact in $\mathcal{F}_c(\mathcal{X})$.

5. A NOTE ON THE CAUCHY PROBLEM IN $\mathcal{F}(\mathcal{X})$

For simplicity, let $T = [a, b] \; \subseteq \mathbb{R}$ be a compact interval and consider the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(a) = \hat{0} \; , \tag{15}$$

where $f : T \times \mathcal{F}(\mathcal{X}) \rightarrow \mathcal{F}(\mathcal{X})$ is a continuous mapping and $\hat{0} = \chi_{\{\mathbf{0}\}}$.

This problem have been studied by several authors, including Diamond&Kloeden

[3], Friedman et al. [7], Kaleva [12] and Kloeden [16].

In general, it is well known that the mere continuity of f is not sufficient for to assure the existence of solutions for (15). Hence, additional conditions must be satisfies by f (see [16]). In this direction, recently Nieto in [20, Th. 3.1], assuming that f is continuous and bounded and $\mathcal{X} = \mathbb{R}^n$, proves an existence theorem for (15) and, for this, he proves that, under this conditions, the operator $F : \mathcal{C}(T, \mathcal{F}(\mathbb{R}^n)) \to \mathcal{C}(T, \mathcal{F}(\mathbb{R}^n))$ defined by

$$F[x](t) = \int_a^t f(s, x(s))ds, \quad \forall t \in T, \; \forall x \in \mathcal{C}(T, \mathcal{F}(\mathbb{R}^n)), \tag{16}$$

is compact, i.e.,

$$A \subseteq \mathcal{C}(T, \mathcal{F}(\mathbb{R}^n)) \textit{ bounded} \Rightarrow F(A) \textit{ relatively compact in } \mathcal{C}(T, \mathcal{F}(\mathbb{R}^n)).$$

This result is essential for the existence of at least one fixed point of F, which is solution of (15) (see [20]).

We remark that :

a) $\mathcal{C}(T, \mathcal{F}(\mathcal{X}))$ denote the class of all continuous mapping from T to $\mathcal{F}(\mathcal{X})$ equiped with the metric $d(x, y) = \sup_{t \in T} D(x(t), y(t))$. It is well known that $(\mathcal{C}(T, \mathcal{F}(\mathcal{X})), d)$

is a complete metric space (see [11]).

b) The integral used in (16) is the integral defined by Puri&Ralescu [22] for fuzzy random variables (for details see [3-12-22-26]).

Our purpose in this section is, on the one hand, to show that this conditions on f are not sufficients for to ensure the compactness of F when $\mathcal{X}$ is an infinite dimensional Banach space (in other words, if $\mathcal{X}$ is an infinite dimensional Banach space, then it is possible to construct a continuous and bounded function f such that the induced operator F in (16)) is not compact), and on the other, to found sufficient conditions on f to assure the compactness of the operator F.

LEMMA 5.1. *Let $\mathcal{X}$ be an infinite dimensional Banach space and $B = \mathcal{B}(\mathcal{X}) = \{x \in \mathcal{X} / \ \|x\| \leq 1\}$, the unit ball in $\mathcal{X}$. Then $\tilde{B} = \{\chi_{\{x\}} / \ x \in B\}$ is not relatively compact in $\mathcal{F}(\mathcal{X})$.*

Proof. Suppose that $\tilde{B}$ is relatively compact in $\mathcal{F}(\mathcal{X})$ and let (x_n) be a sequence in $\mathcal{B}(\mathcal{X})$. Then $(\chi_{\{x_n\}})$ is a sequence in $\tilde{B}$, therefore there exists a subsequence $(\chi_{\{x_{n_p}\}})$ and $u \in \mathcal{F}(\mathcal{X})$ such that $\chi_{\{x_{n_p}\}} \xrightarrow{D} u$ as $p \to \infty$.

This implies that $(\chi_{\{x_{n_p}\}})$ is D-Cauchy and, because $D((\chi_{\{x_{n_p}\}}, \chi_{\{x_{n_q}\}}) =$

$H(\{x_{n_p}\},\{x_{n_q}\}) \to 0$ as $p,q \to \infty$, we have that $(\{x_{n_p}\})$ is a H-Cauchy sequence in $\mathcal{K}(\mathcal{X})$. Thus, due completeness of $\mathcal{K}(\mathcal{X})$, there exists $K \in \mathcal{K}(\mathcal{X})$ such that $\{x_{n_p}\} \to K$ as $p \to \infty$. But then, given $\epsilon > 0$ there exists p_0 such that $H(\{x_{n_p}\},K) < \epsilon$ for all $p \geq p_0$, and this implies that $K \subseteq N(\{x_{n_p}\},\epsilon)$ for all $p \geq p_0$, and this implies that $diam(K) = 0$, i.e., $K = \{x\}$ for some $x \in \mathcal{X}$.

Thus, $\{x_{n_p}\} \to \{x\}$ and, therefore, $x_{n_p} \to x$. Finally, we observe that $\|x\| \leq \|x_{n_p}\| + \|x - x_{n_p}\| \leq 1+\epsilon$ for all $p \geq p_0$, and this implies that $x \in \mathcal{B}(\mathcal{X})$. Hence, we can to conclude that $\mathcal{B}(\mathcal{X})$ is sequencially compact, which is impossible.

Counterexample 5.2. Let $\mathcal{X}$ be an infinite dimensional Hilbert space and consider $B = \mathcal{B}(\mathcal{X})$, the unit ball in $\mathcal{X}$. Then B is a closed and convex and, because $\mathcal{X}$ is an Hilbert space, then we know that for each $x \in \mathcal{X}$ there exists an unique $x_0 = P_B(x) \in B$ such that $\|x - x_0\| = \inf_{y \in B} \|x - y\|$ (P_B is called the projection operator on B (see [5]). It is well known that $P_B : \mathcal{X} \to \mathcal{X}$ is uniformly continuous. Moreover,

$$\|P_B(x) - P_B(y)\| \leq \|x - y\|, \ \forall x, y \in \mathcal{X}. \tag{17}$$

If $u \in \mathcal{F}(\mathcal{X})$ then, by using the level-family of u, we define $P : \mathcal{F}(\mathcal{X}) \to$

$\mathcal{F}(\mathcal{X})$ as $L_\alpha P(u) = \overline{co}P_B(L_\alpha u)$, for all $\alpha \in [0,1]$. It is clear that the family $(\overline{co}P_B(L_\alpha u))_{\alpha\in[0,1]}$ verifies the hypothesis of Negoita&Ralescu Representation Theorem (see [19]) and, consequently, there exists an unique $v = P(u) \in \mathcal{F}(\mathcal{X})$ such that $L_\alpha v = \overline{co}P_B(L_\alpha u),\ \forall \alpha \in [0,1]$.

Now, define $f : [0,1] \times \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ by

$$f(t,u) = P(u),\ \forall (t,u) \in [0,1] \times \mathcal{F}(\mathcal{X}).$$

We observe that

$$\begin{aligned} D(f(t,u), f(t',v) &= D(P(u), P(v)) \\ &= \sup_{\alpha\in[0,1]} H\left(\overline{co}P_B(L_\alpha u), \overline{co}P_B(L_\alpha v)\right) \\ &\leq \sup_{\alpha\in[0,1]} H(P_B(L_\alpha u), P_B(L_\alpha v)) \quad (see\ [22]) \\ &\leq \sup_{\alpha\in[0,1]} H(L_\alpha u, L_\alpha v) \quad (by\ (17)) \\ &= D(u,v) \end{aligned}$$

Therefore, f is a continuous mapping. Besides, because $\overline{co}P_B(L_\alpha u) \subseteq B$ for all $\alpha \in [0,1]$, then

$$\|f(t,u)\| = \|P(u)\| = \sup_{\alpha\in[0,1]} \|\overline{co}P_B(L_\alpha u)\| \leq \|B\|, \ \forall (t,u) \in [0,1] \times \mathcal{F}(\mathcal{X}),$$

and this implies that f is bounded.

Now, we will prove that the integral operator F induced by f and defined by (16) is not compact.

For this, for each $(t,x) \in [0,1] \times B$, define $\xi_{\{x\}}(t) = \chi_{\{x\}}$ (note that $\xi_{\{x\}} : [0,1] \to \mathcal{F}(\mathcal{X})$ is a constant function) and consider $A = \{\xi_{\{x\}}/ \ \|x\| \leq 1\}$. Since

$$d(\xi_{\{x\}}, \xi_{\{y\}}) = \sup_{t\in[0,1]} D(\xi_{\{x\}}(t), \xi_{\{y\}}(t)) = \|x-y\| \leq 2,$$

it is clear that A is a bounded subset of $\mathcal{C}(T, \mathcal{F}(\mathcal{X}))$.

Now, taking $t = 1$, we have that

$$\begin{aligned} F(A)(1) &= \left\{ \int_0^1 \xi_{\{x\}}(s)ds/ \ \xi_{\{x\}} \in A \right\} \\ &= \left\{ \int_0^1 \chi_{\{x\}}ds/ \ \|x\| \leq 1 \right\} \\ &= \{\chi_{\{x\}}/ \ \|x\| \leq 1\} = \tilde{B}, \end{aligned}$$

which, by Lemma 5.1, is not relatively compact in $\mathcal{F}(\mathcal{X})$. Consequently, by Th. 4.1, F is not compact.

THEOREM 5.3. *Let $T = [a, b] \subseteq \mathbb{R}$ be a compact interval, and assume that $f : T \times \mathcal{F}_c(\mathcal{X}) \to \mathcal{F}_c(\mathcal{X})$ is a compact and bounded mapping, i.e.,*

a) If $I \subseteq T$ and $E \subseteq \mathcal{F}_c(\mathcal{X})$ bounded $\Rightarrow f(I \times E)$ relatively compact in $\mathcal{F}_c(\mathcal{X})$ (i.e., due Th. 4.5, $f(I \times E)$ is a compact-supported and level-equicontinuous subset of $\mathcal{F}_c(\mathcal{X})$).

b) There exists a real positive constant r such that

$$\|f(s, x(s))\| \leq r \text{ on } T \times \mathcal{F}_c(\mathcal{X}) \tag{18}$$

Then $F : \mathcal{C}(T, \mathcal{F}_c(\mathcal{X})) \to \mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$ is compact.

Proof. We remark that, due (18) and $L_\alpha \int_a^t f(s, x(s))ds = \int_a^t L_\alpha f(s, x(s))ds$ for all $\alpha \in [0, 1]$, we obtain that $F[x](t) = \int_a^t f(s, x(s))ds \in \mathcal{F}(\mathcal{X})$ (see [22-23-26]). Moreover, since x is continuous and $[a, t] \times x([a, t])$ is compact in $T \times \mathcal{F}_c(\mathcal{X})$ then, by hypothesis, $f([a, t] \times x([a, t]))$ is relavively compact in $\mathcal{F}_c(\mathcal{X})$ and, consequently, by Th. 4.5, is level-equicontinuous. Thus, given $\epsilon > 0$, exists $\delta > 0$ such that

$$| \alpha - \beta |< \delta \Rightarrow H(L_\alpha f(s, x(s')), L_\beta f(s, x(s'))) < \epsilon \ , \ \forall s \in [a, t], \ \forall s' \in T. \quad (19)$$

Now, if $\alpha_p \to \alpha$ then there exist $p_0 \in \mathbb{N}$ such that $| \alpha_p - \alpha_0 |< \delta$ for every $p \geq p_0$. Consequently, due (19),

$$\begin{aligned} \mathrm{H}(\mathrm{L}_{\alpha_p} \int_a^t \mathrm{f(s,x(s))ds}, \mathrm{L}_\alpha \int_a^t \mathrm{f(s,x(s))ds}) &= \mathrm{H}(\int_a^t \mathrm{L}_{\alpha_p} \mathrm{f(s,x(s))ds}, \int_a^t \mathrm{L}_\alpha \mathrm{f(s,x(s))ds}) \\ &\leq \int_a^t \mathrm{H}(\mathrm{L}_{\alpha_p}\mathrm{f(s,x(s))}, \mathrm{L}_\alpha \mathrm{f(s,x(s))})\mathrm{ds} \\ &\leq | b - a | \epsilon \ , \ \forall \ p \geq p_0. \end{aligned}$$

This shows that, actually, $\int_a^t f(s, x(s))ds \in \mathcal{F}_c(\mathcal{X})$ and, consequently, F it is well-defined.

Now, we will prove that F is compact.

For this, we need to prove that if A is bounded in $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$, then $F(A)$ is relatively compact in $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$ and, due Th. 4.1 (Ascoli-Arzelá theorem), this is equivalent to prove:

i) $F(A)$ is an equicontinuous subset of $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$

ii) $F(A)(t)$ is relatively compact in $\mathcal{F}_c(\mathcal{X})$, for each $t \in T$.

In fact, if A is a bounded subset of $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$ and $\epsilon > 0$ then, taking $| t_1 - t_2 | < \frac{\epsilon}{r}$, we obtain

$$\begin{aligned} D\left(F[x](t_1), F[x](t_2)\right) &\leq \int_{t_1}^{t_2} \|f(s, x(s))\| ds \\ &= \int_{t_1}^{t_2} D(f(s, x(s)), \chi_{\{\mathbf{0}\}}) ds \\ &\leq | t_1 - t_2 | \, r < \epsilon , \end{aligned}$$

for all $x \in A$, and this proves that $F(A)$ is an equicontinuous subset of $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$.

Now, we will show that $F(A)(t)$ is relatively compact in $\mathcal{F}_c(\mathcal{X})$ and, due Th. 4.5, this is equivalent to prove that $F(A)(t)$ is a level-equicontinuos and compact-supported subset of $\mathcal{F}_c(\mathcal{X})$.

In fact, fixing $t \in T$ we see that $F(A)(t) \subseteq \mathcal{F}_c(\mathcal{X})$ and, if $u \in F(A)(t)$, then $u = \int_a^t f(s, y(s)) ds$ for some $y \in A$. Since $f([a,t] \times A)$ is relatively compact in $\mathcal{F}_c(\mathcal{X})$ then, due Th.4.5, $f([a,t] \times A)$ is level-equicontinuous. Thus, given $\epsilon > 0$ we have that there exists $\delta > 0$ such that

$$| \alpha - \beta | < \delta \Rightarrow H\left(L_\alpha f(s, x(s)), L_\beta f(s, x(s))\right) < \epsilon \, , \; \forall (s, x) \in [a,t] \times A.$$

In particular,

$$\begin{aligned} H(L_\alpha u, L_\beta u) &= H\left(L_\alpha \int_a^t f(s,y(s))ds, L_\beta \int_a^t f(s,y(s))ds\right) \\ &= H\left(\int_a^t L_\alpha f(s,y(s))ds, \int_a^t L_\beta f(s,y(s))ds\right) \\ &\leq \int_a^t H\left(L_\alpha f(s,y(s)), L_\beta f(s,y(s))\right) ds \\ &\leq \mid t-a \mid \epsilon \leq \mid b-a \mid \epsilon \,, \end{aligned}$$

for all $\mid \alpha - \beta \mid < \delta$.

Therefore, $F(A)(t)$ is level-equicontinuous in $\mathcal{F}_c(\mathcal{X})$.

Finally, due to relative compactness of $f([a,t] \times A)$, we have that there exists a compact $K \subseteq \mathcal{X}$ such that $L_0 f(s,x(s)) \subseteq K, \ \forall (s,x) \in [a,t] \times A$. Thus,

$$L_0 \int_a^t f(s,x(s))ds = \int_a^t L_0 f(s,x(s))ds \subseteq \int_a^t K ds = (t-a)K,$$

which proves that $F(A)(t)$ is compact-supported.

Thus, F is a compact mapping.

COROLLARY 5.4. *In the same conditions of Th. 5.3, the Cauchy problem (15)* *posseses at least one solution on T.*

Proof. It is well known that $x \in \mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$ is a solution of (15) if and only if x is a fixed point of the operator F. Now, following Nieto [20, Th. 3.1], consider $m = (b-a)r$ and define

$$A = \left\{ \xi \in \mathcal{C}(T, \mathcal{F}_c(\mathcal{X})) /\ d(\xi, \xi_{\{\mathbf{0}\}}) \leq m \right\}.$$

We recall that $\xi_{\{\mathbf{0}\}}(t) = \chi_{\{\mathbf{0}\}},\ \forall t \in T$.

Then, we have that A is a convex subset of $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$. In fact, if $\xi_1, \xi_2 \in A$ and $\lambda \in [0,1]$ then, by using properties of H (Prop. 2.1) and the addition in $\mathcal{F}_c(\mathcal{X})$,we obtain

$$\begin{aligned}
\mathrm{d}(\lambda\xi_1 + (1\text{-}\lambda)\xi_2, \xi_{\{\mathbf{0}\}}) &= \sup_{t\in T} \mathrm{D}(\lambda\xi_1(\mathrm{t}) + (1\text{-}\lambda)\xi_2(\mathrm{t}), \xi_{\{\mathbf{0}\}}(\mathrm{t})) \\
&= \sup_{t\in T} \mathrm{H}(\mathrm{L}_\alpha[\lambda\xi_1(\mathrm{t}) + (1\text{-}\lambda)\xi_2(\mathrm{t})], \mathrm{L}_\alpha\xi_{\{\mathbf{0}\}}(\mathrm{t})) \\
&= \sup_{t\in T}[\sup_{\alpha\in[0,1]} \mathrm{H}(\lambda\mathrm{L}_\alpha\xi_1(\mathrm{t}) + (1\text{-}\lambda)\mathrm{L}_\alpha\xi_2(\mathrm{t}), \{\mathbf{0}\})] \\
&\leq \sup_{t\in T}[\sup_{\alpha\in[0,1]} \mathrm{H}(\lambda\mathrm{L}_\alpha\xi_1(\mathrm{t}), \{\mathbf{0}\}) + \mathrm{H}((1\text{-}\lambda)\mathrm{L}_\alpha\xi_2(\mathrm{t}), \{\mathbf{0}\})] \\
&= \sup_{t\in T}[\ \lambda \sup_{\alpha\in[0,1]} \mathrm{H}(\mathrm{L}_\alpha\xi_1(\mathrm{t}), \{\mathbf{0}\}) + (1\text{-}\lambda) \sup_{\alpha\in[0,1]} \mathrm{H}(\mathrm{L}_\alpha\xi_2(\mathrm{t}), \{\mathbf{0}\})] \\
&= \sup_{t\in T}[\ \lambda\mathrm{D}(\xi_1(\mathrm{t}), \xi_{\{\mathbf{0}\}}(\mathrm{t})) + (1\text{-}\lambda) D(\xi_2(\mathrm{t}), \xi_{\{\mathbf{0}\}}(\mathrm{t}))]
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \sup_{t\in T} \mathrm{D}(\xi_1(\mathrm{t}), \xi_{\{\mathbf{0}\}}(\mathrm{t})) + (1\text{-}\lambda) \sup_{t\in T} D(\xi_2(\mathrm{t}), \xi_{\{\mathbf{0}\}}(\mathrm{t})) \\
&\leq \lambda m + (1-\lambda)m = m.
\end{aligned}
$$

Thus, A is closed and convex in $\mathcal{C}(T, \mathcal{F}_c(\mathcal{X}))$. Furthermore, we observe that $F(A) \subseteq A$. In fact, if $\xi \in A$ then

$$
\begin{aligned}
d(F(\xi), \xi_{\{\mathbf{0}\}}) &= \sup_{t\in T} D(F(\xi)(t), \xi_{\{\mathbf{0}\}}(t)) \\
&= \sup_{t\in T} D(\int_a^t f(s, \xi(s))ds, \chi_{\{\mathbf{0}\}}) \\
&= \sup_{t\in T} [\sup_{\alpha\in[0,1]} H(L_\alpha \int_a^t f(s, y(s))ds, L_\alpha \chi_{\{\mathbf{0}\}}) \\
&= \sup_{t\in T} [\sup_{\alpha\in[0,1]} H(L_\alpha \int_a^t f(s, y(s))ds, \{\mathbf{0}\}) \\
&= \sup_{t\in T} [\sup_{\alpha\in[0,1]} \|L_\alpha \int_a^t f(s, y(s))ds\| \\
&= \sup_{t\in T} \| \int_a^t f(s, y(s))ds\| \\
&\leq \sup_{t\in T} \int_a^t \|f(s, y(s))\|ds \\
&\leq (b-a)r = m.
\end{aligned}
$$

Thus, by using of Schauder fixed point theorem for compact mappings (see [6, Th. 3.2, p.415), we have that there exits $x \in A$ such that $F(x) = x$, which is a solution of (15).

6. CONCLUSION

If $\mathcal{X}$ is a Banach space and $\mathcal{F}_L(\mathcal{X})$ is the class of level-Lipschitz fuzzy sets on $\mathcal{X}$ then, by using of multivalued Bernstein polynomials and its properties, in Section 3 we have proved that $(\mathcal{F}_L(\mathcal{X}), D)$ is dense in $(\mathcal{F}_c(\mathcal{X}), D)$, the class of level-continuous fuzzy sets on $\mathcal{X}$ and, as a direct consequence of this result, we obtain the existence of an isometric embedding $j : \mathcal{F}_c(\mathcal{X}) \to \mathcal{C}([0,1] \times B(\mathcal{X}^*))$.

Now, by using of this isometry, in Section 4 we have proved an Ascoli-Arzelá type theorem for the characterization of relatively compact subsets in $(\mathcal{F}_c(\mathcal{X}), D)$, in the setting of infinite-dimensional Banach spaces. Furthermore, several examples are presented.

In connection with these results, particular note should be given to the possibilities of using this tools for the study of fuzzy abstract differential equations on $\mathcal{F}(\mathcal{X})$. For instance, in Section 5 we used the above characterization for the study of the Cauchy problem on $\mathcal{F}(\mathcal{X})$ and, by using the compactness of certain integral operator F,we obtain an extension of the Nieto results ([20]) on the existence of solutions for the Cauchy problem for fuzzy differential equations on $\mathcal{F}(\mathbb{R}^n)$ to $\mathcal{F}(\mathcal{X})$.

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