QUATERNIONIC MATRICES: INVERSION AND DETERMINANT*

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Abstract. We discuss the Schur complement formula for quaternionic matrices, M, and give an efficient method to calculate the matrix inverse. We also introduce the functional $\mathcal{D}[M]$ which extends to quaternionic matrices the non-negative number $|\det[M]|$.

Key words. Quaternions, Matrices, Determinants.

AMS subject classifications. 15A09, 15A33

1. Introduction. Much of the spectral theory of complex matrices does not extend to quaternion matrices without further modifications [1, 2, 3]. In particular, the determinant cannot be defined, and the eigenvalues [4] have several possible extensions which do not necessarily respect the fundamental theorem of algebra [5, 6, 7].

Quaternionic mathematical structures have recently appeared in the physical litterature mainly in the context of quantum mechanics [8, 9, 10] and group theory [11, 12]. The relevance of quatenionic calculus to several iussues in theoretical phycis has rekindled the interest in quaternionic linear algebra. A usefull thecnical procedure in approaching quaternionic applications in physics has been the replacement of quaternionic matrices by complex matrices of double size [13, 14], and the consequent spectral analysis of the latter matrix [4, 15, 16]. This indirect procedure, usually, suffices to solve the problem at hand, but it sheds little light on the structure of quaternionic matrices.

We aim to discuss inversion and determinant for quaternionic matrices in view of new classifications of quaternionic groups and possible applications in grand unification theory [17, 18]. The main results of this paper can be summarized by

Inversion. The formula $M^{-1} = \mathrm{Adj}[M]/\det[M]$ cannot be generalized for quaternionic matrices. Nevertheless, the Schur complement formula provides an efficient method to calculate the matrix inverse.

Determinant. The determinat for $n \times n$ quaternionic matrices, M, cannot be defined in a consistent way. Instead, the non-negative number $|\det[M]|$ can be extended to quaternionic matrices by the functional $\mathcal{D}[M]$ defined in terms of quaternionic M-entries.

2. Notation. Quaternions, introduced by Hamilton [19], can be represented by four real quantities

$$q = a + i\,b + j\,c + k\,d \;, \qquad a, b, c, d \in \mathbb{R} \;, \tag{1}$$

and three imaginary units i, j, k satisfying

$$i^2 = j^2 = k^2 = ijk = -1 \;.$$

*Submitted to *Electronic Journal of Linear Algebra* on 18 July 1999.

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We will denote by

$$\mathrm{Re}[q] := a \qquad \text{and} \qquad \mathrm{Im}[q] := q - a = i\,b + j\,c + k\,d \ ,$$

the real and imaginary parts of q.

The quaternion skew-field $\mathbb{H}$ is an associative but non-commutative algebra of rank 4 over $\mathbb{R}$, endowed with an involutory operation, called quaternionic conjugation,

$$\bar{q} = a - i\,b - j\,c - k\,d = \mathrm{Re}[q] - \mathrm{Im}[q] \ ,$$

satisfying

$$\overline{pq} = \overline{q}\,\overline{p} \ , \qquad q, p \in \mathbb{H} \ .$$

The quaternion norm $|q|$ is defined by

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2 \ .$$

Among the properties of the norm, to be used in subsequent sections, we mention here the following

$$|pq| = |qp| = |q|\,|p| \qquad \text{and} \qquad |1 - pq| = |1 - qp| \ .$$

Every nonzero quaternion q has a unique inverse

$$q^{-1} = \bar{q}/|q|^2 \ .$$

Two quaternions p and q are similar if

$$q = s^{-1} p\, s \ , \qquad s \in \mathbb{H} \ .$$

By replacing s by $u = s/|s|$, we may always assume s to be unitary. The usual complex conjugation in $\mathbb{C}$ may be obtained by choosing $s = j$ or $s = k$. A necessary and sufficient condition for the similarity of p and q is given by

$$\mathrm{Re}[q] = \mathrm{Re}[p] \qquad \text{and} \qquad |\mathrm{Im}[q]| = |\mathrm{Im}[p]| \ .$$

An equivalent condition is $Re[q] = Re[p]$ and $|q| = |p|$. Every similarity class contains a complex number, unique up to conjugation. Namely, every quaternion q is similar to

$$\mathrm{Re}[q] \pm i\,|\mathrm{Im}[q]| \ .$$

Since $\mathrm{Re}[q] = \mathrm{Re}[\bar{q}]$ and $\mathrm{Im}[q] = \mathrm{Im}[\bar{q}]$, the quaternions q and $\bar{q}$ are similar. In fact, it can be seen that for $s \in \mathbb{H}$ to satisfy

$$q = s^{-1}\bar{q}\, s \ ,$$

given $q \in \mathbb{H}$, it is necessary and sufficient that

$$\mathrm{Re}[qs] = \mathrm{Re}[s] = 0 \ .$$

However, there exists no single $s \in \mathbb{H}$ such that $q = s^{-1}\bar{q}\, s$ for all $q \in \mathbb{H}$.

3. Matrix inversion. Let $\mathcal{R}$ be an associative ring. Denote by $\mathcal{R}^{n\times n}$ the ring of n-dimensional matrices over R and by I_n the identity matrix in $\mathcal{R}^{n\times n}$. A matrix $M\in\mathcal{R}^{n\times n}$ is called invertible if $MN=NM=I_n$ for some $N\in\mathcal{R}^{n\times n}$, which is necessarily unique. It is possible to show [16] that for quaternionic matrices $MN=I_n$ implies $NM=I_n$.

The Schur complments procedure [20] is a powerful computational tool in calculating inverses of matrices over rings. Let us write a generic n-dimensional matrix M in block form

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Assuming that $A\in\mathcal{R}^{k\times k}$ is invertible, one has

$$M=\begin{pmatrix} I_k & 0 \\ CA^{-1} & I_{n-k} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A_S \end{pmatrix}\begin{pmatrix} I_k & A^{-1}B \\ 0 & I_{n-k} \end{pmatrix} ,$$

with

$$A_S := D - CA^{-1}B .$$

The invertibility of A ensures that the matrix M is invertible if and only if D_S is invertible, and the inverse is given by

$$M^{-1}=\begin{pmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & {A_S}^{-1} \end{pmatrix}\begin{pmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{pmatrix} . \tag{2}$$

The inversion of an n-dimensional matrix is thus reduced to inversion of two smaller matrices,

$$A\in\mathcal{R}^{k\times k} \qquad \text{and} \qquad A_S\in\mathcal{R}^{(n-k)\times(n-k)} .$$

We shall call A_S the *Schur complement* of A in M.

To invert n-dimensional quaternionic matrices we may apply the following procedure. Let us consider a quaternionic matrix $M\in\mathbb{H}^{n\times n}$

$$M=\begin{pmatrix} a & B \\ C & D \end{pmatrix} .$$

If $\mathbb{H}\ni a\neq 0$, we can apply directly Eq. (2). Otherwise, we perform a column/row permutation by constructing a new matrix

$$P_1\,M\,Q_1=\begin{pmatrix} a_1 & B_1 \\ C_1 & D_1 \end{pmatrix} , \qquad \mathbb{H}\ni a_1\neq 0 .$$

By using Eq. (2) and applying recursively the column/row permutations, P_m and Q_m, we find

$$M^{-1}=L\,\Omega\,U , \tag{3}$$

where

$$\Omega = a_1 \oplus \ldots \oplus a_n \ ,$$
$$L = \prod_{m=1}^{n-1} \left[I_{m-1} \oplus P_m \begin{pmatrix} 1 & -a_m^{-1} B_m \\ 0 & I_{n-m} \end{pmatrix} \right]$$
$$U = \prod_{m=1}^{n-1} \left[I_{m-1} \oplus \begin{pmatrix} 1 & 0 \\ -C_m a_m^{-1} & I_{n-m} \end{pmatrix} Q_m \right] .$$

4. Determinant. The identity $\det[MN] = \det[M]\det[N]$, and the fact that M is invertible if and only if $\det[M] \neq 0$, are central in the study of complex matrices. For two-dimensional quaternionic matrices, at first glance, one could introduce four different definitions

(4) $$ab - cd \ , \quad ab - dc \ , \quad ba - cd \ , \quad ba - dc \ .$$

However, (4) has little to do with invertibility. For example, for the invertible matrix

(5) $$\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix} \ ,$$

two expressions in (4) vanish and the other two are non-zero. Another example is the invertible matrix

(6) $$\tfrac{1}{\sqrt{2}} \begin{pmatrix} i & j \\ j & i \end{pmatrix} \ ,$$

whose four "determinants", calculated by (4), are all zero. In fact, as shown below, the complex functional

$$\det \ : \ \mathbb{C}^{\,n \times n} \to \mathbb{C} \ ,$$

cannot extend to quaternions. More precisely, there is no functional

$$\mathrm{d} \ : \ \mathbb{H}^{\,n \times n} \to \mathbb{H} \ ,$$

which is multiplicative, $\mathrm{d}[MN] = \mathrm{d}[M]\,\mathrm{d}[N]$, for all $M\,,\,N \in \mathbb{H}^{\,n \times n}$, and such that $\mathrm{d}\,[\oplus_{i=1}^{n} z_i] = \prod_{i=1}^{n} z_i$ for all $z_i \in \mathbb{C}$. As a counter-example, consider the 2×2 matrices

$$M = \begin{pmatrix} 1+i & 0 \\ 0 & i \end{pmatrix} \ , \quad N = \begin{pmatrix} 1+i & 0 \\ 0 & -i \end{pmatrix} \ , \quad S = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \ .$$

Since $SM = NS$, we conclude that $\mathrm{d}[S]\,\mathrm{d}[M] = \mathrm{d}[N]\,\mathrm{d}[S]$, hence $\mathrm{d}[M]$ and $\mathrm{d}[N]$ should be similar. This is a contradiction because obviously $\mathrm{Re}\,\{\mathrm{d}[M]\} \neq \mathrm{Re}\,\{\mathrm{d}[N]\}$.

Nevertheless, the *real* functional

$$|\det[M]| \ : \ \mathbb{C}^{\,n \times n} \to \mathbb{R}_+$$

can be extended to quaternionic matrices.

THEOREM 4.1. *There exists a unique non trivial functional*

$$\mathcal{D}_n \ : \ \mathbb{H}^{n \times n} \to \mathbb{R}_+ \ ,$$

which is multiplicative, i.e.

$$\mathcal{D}_n[MN] = \mathcal{D}_n[NM] = \mathcal{D}_n[M]\,\mathcal{D}_n[N] \ . \tag{7}$$

This functional has the following properties:

I. $\mathcal{D}_n\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \mathcal{D}_k[A]\,\mathcal{D}_{n-k}[D-CA^{-1}B]$ *as long as* $A \in \mathbb{H}^{k \times k}$ *and* $\mathcal{D}_k[A] \neq 0$*;*

II. $\mathcal{D}_n[I+MN] = \mathcal{D}_n[I+NM]$*;*

III $\mathcal{D}_n[M]\mathcal{D}_n[M^{-1}] = 1$ *if* N *invertible,* $\mathcal{D}_n[M] = 0$ *otherwise.*

Proof. For $n=1$ we define

$$\mathcal{D}_1[q] = |q| \ , \qquad q \in \mathbb{H} \ .$$

Obviously Eq. (7) and properties II-III are trivially satisfied. For $n \geq 2$ we use induction to define $\mathcal{D}_n$. Assume that $\mathcal{D}_k$ has been defined for all $k < n$. For all $0 < i \, , \, j < n$, set

$$\mathcal{M}_{[i,j]} = \{ M \in \mathbb{H}^{n \times n} \ \ : \ \ M_{ij} \neq 0 \} \ .$$

We define the functional $\mathcal{D}_n^{ij} \ : \ \mathcal{M}_{[i,j]} \to \mathbb{R}_+$ by

$$\mathcal{D}_n^{ij}[M] := |M_{ij}| \ \mathcal{D}_{n-1}\left[M_{ij,s}\right] \ ,$$

where $M_{ij,s} \in \mathbb{H}^{(n-1) \times (n-1)}$ is the Schur complement of M_{ij} in M. We claim that all the n^2 functionals $\mathcal{D}_n^{ij}$, $0 \leq i, j \leq n$, are compatible, namely

$$\mathcal{D}_n^{ij}[M] = \mathcal{D}_n^{kl}[M] \ , \quad \forall M \in \left\{ \mathcal{M}_{[i,j]} \cap \mathcal{M}_{[k,l]} \right\} \ . \tag{8}$$

There are four cases to check:

Case 1) $i=k$, $j=l$: Trivially true.

Case 2) $i=k$, $j \neq l$: We shall only consider the case $i=k=j=1$ and $l=2$. All other cases follow by row/column permutations. So, we have

$$M = \begin{pmatrix} a & b & E \\ C & D & F \\ G & H & J \end{pmatrix} \ , \qquad a, b \in \mathbb{H} \ , \quad ab \neq 0 \ .$$

The two relevant Schur complements are

$$M_{11,S} = \begin{pmatrix} D & F \\ H & J \end{pmatrix} - \begin{pmatrix} C \\ G \end{pmatrix} a^{-1} \begin{pmatrix} b & E \end{pmatrix} ,$$

and

$$M_{12,S} = \begin{pmatrix} C & F \\ G & J \end{pmatrix} - \begin{pmatrix} D \\ H \end{pmatrix} b^{-1} \begin{pmatrix} a & E \end{pmatrix} .$$

It is easy to see that

$$M_{11,S} = M_{12,S} \begin{pmatrix} -a^{-1}b & -a^{-1}E \\ 0 & I \end{pmatrix} , \quad \mathcal{D}_{n-1} \left[\begin{pmatrix} -a^{-1}b & -a^{-1}E \\ 0 & I \end{pmatrix} \right] = |a^{-1}b| .$$

Therefore, using eq. (7) for $\mathcal{D}_{n-1}$, *which holds by induction*, we have

$$\begin{aligned} \mathcal{D}_n^{11}[M] &= |a| \, \mathcal{D}_{n-1} [M_{11,S}] = |a| \, \mathcal{D}_{n-1} [M_{12,S}] \, |a^{-1}b| = |b| \, \mathcal{D}_{n-1} [M_{12,S}] \\ &= \mathcal{D}_n^{12}[M] , \end{aligned}$$

as required.

Case 3) $i \neq k$, $j = l$: This case is treated along similar lines.

Case 4) $i \neq k$, $j \neq l$: We shall only consider the case $i = j = 1$ and $k = l = 2$. All other cases follow by row/column permutations. So, we have

$$M = \begin{pmatrix} a & b & E \\ c & d & F \\ G & H & J \end{pmatrix} , \quad a, b, c, d \in \mathbb{H} , \quad ad \neq 0 .$$

We want to prove that $\mathcal{D}_n^{11}[M] = \mathcal{D}_n^{22}[M]$. For $b \neq 0$ or $c \neq 0$, by using the results in 2) and 3),

$$\mathcal{D}_n^{11}(M) = \mathcal{D}_n^{12}(M) = \mathcal{D}_n^{22}(M) \;\; \text{or} \;\; \mathcal{D}_n^{11}(M) = \mathcal{D}_n^{21}(M) = \mathcal{D}_n^{22}(M) ,$$

trivially holds. Let us consider the only remaining case, $b = c = 0$:

$$M = \begin{pmatrix} a & 0 & E \\ 0 & d & F \\ G & H & J \end{pmatrix} , \quad a, d \in \mathbb{H} , \quad ad \neq 0 .$$

The functional $\mathcal{D}_n^{11}$ is given by

$$\mathcal{D}_n^{11}[M] = |a| \, \mathcal{D}_{n-1}^{11} \left[\begin{pmatrix} d & F \\ H & J - Ga^{-1}E \end{pmatrix} \right] = |a| \, |d| \, \mathcal{D}_{n-2} [J - Ga^{-1}E - Hd^{-1}F] .$$

Similar calculations for the functional $\mathcal{D}_n^{22}$ lead to

$$\mathcal{D}_n^{22}[M] = |d| \, \mathcal{D}_{n-1}^{11} \left[\begin{pmatrix} a & E \\ G & J - Hd^{-1}F \end{pmatrix} \right] = |a| \, |d| \, \mathcal{D}_{n-2} [J - Hd^{-1}F - Ga^{-1}E] .$$

Consequently, $\mathcal{D}_n^{11}[M] = \mathcal{D}_n^{22}[M]$, the compatibility (8) has been estabilished, and we may thus speak unambiguously of $\mathcal{D}_n[M]$.

- Proof of Property I. We shall use induction on k. The case $k = 1$ is equivalent to (8) and has just been proved. For $k \geq 2$ we consider

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \mathbb{H}^{k \times k} ,$$

and assume without loss of generality that $a = A_{11} \neq 0$. Then,

$$\mathcal{D}_n \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] = \mathcal{D}_n \left[\begin{pmatrix} a & B_A & B_1 \\ C_A & D_A & B_2 \\ C_1 & C_2 & D \end{pmatrix} \right] .$$

By Schur complements, the invertibility of A implies that of $A_S = D_A - C_A a^{-1} B_A$. By the induction hypothesis, we get

$$\begin{aligned} \mathcal{D}_n \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] &= |a|\, \mathcal{D}_{n-1} \left[\begin{pmatrix} D_A - C_A a^{-1} B_A & B_2 - C_A a^{-1} B_1 \\ C_2 - C_1 a^{-1} B_A & D - C_1 a^{-1} B_1 \end{pmatrix} \right] \\ &= |a|\, \mathcal{D}_{k-1} \left[D_A - C_A a^{-1} B_A \right] \mathcal{D}_{n-k} \left[X \right] \\ &= \mathcal{D}_k \left[A \right] \mathcal{D}_{n-k} \left[X \right] \ , \end{aligned}$$

where X is a Schur complement defined by

$$\begin{aligned} X &= D - C_1 a^{-1} B_1 - (C_2 - C_1 a^{-1} B_A)\,(D_A - C_A a^{-1} B_A)^{-1}\,(B_2 - C_A a^{-1} B_1) \\ &= D - \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} B_A \\ 0 & I \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a_S^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -C_A a^{-1} & I \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= D - C A^{-1} B \ . \end{aligned}$$

With this, Property I is proved.

- Proof of Property II. Let us introduce the following matrix

$$\begin{pmatrix} I_1 & N \\ M & I_2 \end{pmatrix} \quad \in \mathbb{H}^{2n \times 2n} .$$

By using property I and Schur complements with respect to I_1 and I_2, we have

$$\mathcal{D}_{2n} \left[\begin{pmatrix} I_1 & N \\ M & I_2 \end{pmatrix} \right] = \mathcal{D}_n[I_1]\, \mathcal{D}_n[I_2 - M I_1^{-1} N] \ ,$$

and

$$\mathcal{D}_{2n} \left[\begin{pmatrix} I_1 & N \\ M & I_2 \end{pmatrix} \right] = \mathcal{D}_n[I_2]\, \mathcal{D}_n[I_2 - N I_1^{-1} M] \ ,$$

and the result follows immediately.

• Proof of Property III. We begin by considering M invertible and constructing the matrix

$$\begin{pmatrix} M & I \\ -I & 0 \end{pmatrix} \qquad \in \mathbb{H}^{2n \times 2n} \ .$$

We find,

$$\mathcal{D}_n[I]\,\mathcal{D}_n[-I] = \mathcal{D}_{2n}\left[\begin{pmatrix} M & I \\ -I & 0 \end{pmatrix}\right] = \mathcal{D}_n[M]\,\mathcal{D}_n[IM^{-1}I] \ ,$$

and noting that $\mathcal{D}_n[I] = \mathcal{D}_n[-I] = 1$ the first part of property III is demonstrated. To complete the proof we show that

$$\mathcal{D}_n(M) \neq 0 \quad \Leftrightarrow \quad M \text{ invertible} \ .$$

We move by induction on n. The case $n = 1$ is trivial, so assume $M \in \mathbb{H}^{n \times n}$, $n \geq 2$. If $M \equiv 0$, we know that M is not invertible and $\mathcal{D}_n[M] = 0$ by definition. Otherwise, we may assume without loss of generality that

$$M = \begin{pmatrix} a & B \\ C & D \end{pmatrix} \ , \qquad a \in \mathbb{H} \ , \quad a \neq 0 \ .$$

Now,

$$\mathcal{D}_n[M] = |a|\,\mathcal{D}_{n-1}[D - Ca^{-1}B] = |a|\,\mathcal{D}_{n-1}[a_S] \ ,$$

hence, using the induction hipothesis,

$$\mathcal{D}_n[M] \neq 0 \quad \Leftrightarrow \quad \mathcal{D}_{n-1}[a_S] \neq 0 \quad \Leftrightarrow \quad a_S \ \text{ invertible} \quad \Leftrightarrow \quad M \ \text{ invertible} \ .$$

• Proof of the Multiplicative Property. Let us prove eq. (7). If $\mathcal{D}_n(M) = 0$ then M is not invertible, MN is not invertible, hence $\mathcal{D}_n(MN) = 0$. Otherwise, consider

$$\begin{pmatrix} M^{-1} & -I \\ N & 0 \end{pmatrix} \qquad \in \mathbb{H}^{2n \times 2n} \ .$$

Schur complements with respect to M^{-1} gives

$$\mathcal{D}_{2n}\left[\begin{pmatrix} M^{-1} & -I \\ N & 0 \end{pmatrix}\right] = \mathcal{D}_n[M^{-1}]\,\mathcal{D}_n[NM] = \mathcal{D}_n[NM]\,/\,\mathcal{D}_n[M] \ .$$

Now Schur complements with respect to $-I$ gives

$$\mathcal{D}_{2n}\left[\begin{pmatrix} M^{-1} & -I \\ N & 0 \end{pmatrix}\right] = \mathcal{D}_n[-I]\,\mathcal{D}_n[N] = \mathcal{D}_n[N] \ .$$

In conclusion

$$\mathcal{D}_n[MN] = \mathcal{D}_n[NM] = \mathcal{D}_n[M]\,\mathcal{D}_n[N] \ .$$

□

4.1. Remarks. We discuss additional properties of the functional $\mathcal{D}$. By using the results of the previous section it is immediate to show that

`IV.` For triangular matrices $\mathcal{D}_n[M] = \prod_{i=1}^n |M_{ii}|$;

Next, we calculate $\mathcal{D}$ in terms of singular values. The singular values decomposition, SVD, for complex matrices extends to quaternionic matrices in a straightforward way. A matrix $U \in \mathbb{H}^{n \times n}$ is called unitary if $U^+U = I$, where $(U^+)_{ij} = \overline{U_{ij}}$. Every $n \times n$ quaternionic matrix, M, has the SVD $M = U\Sigma V$ where U and V are unitary, $\Sigma = \Sigma_1 \oplus 0$, $\Sigma_1 = \sigma_1 \oplus \ldots \oplus \sigma_k$, where $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_k \geq 0$ are the singular values of M [13, 21, 22, 23]. In these terms the following holds:

`V.` $\mathcal{D}_n[U] = 1$ if U is unitary;
`VI.` $\mathcal{D}_n[M] = \prod_{i=1}^n \sigma_i$;
`VII.` $\mathcal{D}_n[M^+] = \mathcal{D}_n[M]$.

It can be seen that the uniqueness claim in theorem (4.1) follows from property VI. Indeed, every non-trivial multiplicative functional $\mathcal{L} : \mathbb{H}^{n \times n} \to \mathbb{R}_+$ should satisfy $\mathcal{L}\left(\oplus_{i=1}^n \sigma_i\right) = \prod_{i=1}^n \sigma_i$ and $\mathcal{L}\left(U\right) = 1$ for U unitary. Using the SVD, it then follows that $\mathcal{L} = \mathcal{D}$.

A weak version of the Jordan canonical form extends to quaternionic matrices. Namley, every matrix $M \in \mathbb{H}^{n \times n}$ is similar, over the quaternions, to a *complex* Jordan matrix J, defining a set of n complex eigenvalues. However the eigenvalues $\lambda_i \in \mathbb{C}$ are determined only up to complex conjugation. For further discussion see [13]. The Jordan canonical form is associated with *right eigenvalues* $M\psi = \psi\, q$, $\psi \in \mathbb{H}^{n \times 1}$, $q \in \mathbb{H}$, which are determined only up to quaternionic similarity. This is further discussed in [4, 15]. Thus, $\mathcal{D}$ has the additional property

`VIII.` $\mathcal{D}_n[M] = \prod_{i=1}^n |\lambda_i|$ if $\lambda_i \in \mathbb{C}$ are the right eigenvalues associated with M.

Let us denote by $\mathcal{Z}[M]$ the complexification [13, 14] of the quaternionic matrix M, i.e.

$$\mathcal{Z}[M] := \begin{pmatrix} M_1 & -M_2^* \\ M_2 & M_1^* \end{pmatrix} , \qquad M = M_1 + j\, M_2 \;, \quad M_{1,2} \in \mathbb{C}^{n \times n} \;.$$

It has been shown in [13] that if J is the complex Jordan form of M then $J \oplus J^*$ is the Jordan form of $\mathcal{Z}[M]$. Consequently the spectrum of $\mathcal{Z}[M]$ is $\{\lambda_1, \lambda_1^*, \ldots, \lambda_n, \lambda_n^*\}$.

A quaternionic matrix H is called hermitian if $H^+ = H$. Obviously for hermitian quaternionic matrices, the eigenvalues are uniquely determined and real. This follows from the fact that $\mathcal{Z}[M]$ is also hermitian. In this case we can define the following *real determinant*

$$|H|_r = \prod_{i=1}^n \lambda_i \;.$$

It is easy to show that H is positive definite, i.e. $x^+Hx > 0$ for all non zero $x \in \mathbb{H}^{n\times 1}$, if and only if all the eigenvalues are positive, if and only if all the *real determinants* of the principal minors are positive.

We conclude this section by comparing the functionals $\mathcal{D}[M]$ and $|H|_r$, introduced in this paper, with the *q-determinant* [16] defined by

$$|M|_q = \det\{\mathcal{Z}[M]\} \; .$$

From previous considerations, we have

$$|M|_q = \prod_{i=1}^{n} |\lambda_i|^2 = \mathcal{D}[M^+]\,\mathcal{D}[M] = \mathcal{D}^2[M] = |M^+M|_r \; .$$

5. The case of 2×2 matrices. In this last section, we discuss inversion, adjoint and determinant for 2×2 quaternionic matrices.

- **Inversion.** Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an invertible 2×2 matrix with quaternionic entries. When a, b, c, d are all non-zero, four parallel applications of the Schur complement formula lead to a concrete description of the inverse:

$$M^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \; , \tag{9}$$

where

$$\begin{array}{ll} \tilde{a} = (a - bd^{-1}c)^{-1}, & \tilde{b} = (c - db^{-1}a)^{-1} \; , \\ \tilde{c} = (b - ac^{-1}d)^{-1} \; , & \tilde{d} = (d - ca^{-1}b)^{-1} \; . \end{array} \tag{10}$$

What happens if some of the entries is zero? Assume for example that $a = 0$. The invertibility of M implies that $b, c \neq 0$. Consequently, the element $d - ca^{-1}b$ has *infinite* modulus. In this case, we can define

$$\tilde{d} := \lim_{a\to 0} (d - ca^{-1}b)^{-1} \; .$$

A simple calculation,

$$|\tilde{d}| := \lim_{a\to 0} \frac{1}{|d - ca^{-1}b|} = \lim_{a\to 0} \frac{1}{|c|\,|c^{-1}d - a^{-1}b|} = \lim_{a\to 0} \frac{|a|}{|c|\,|ac^{-1}d - b|}$$
$$= \lim_{a\to 0} \frac{|a|}{|c|\,|b|} = 0 \; ,$$

shows that $\tilde{d} = 0$. Thus,

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & 0 \end{pmatrix} \; , \quad \tilde{a} = -c^{-1}db^{-1} \; , \; \tilde{b} = c^{-1} \; , \; \tilde{c} = b^{-1} \; .$$

We conclude that Eqs. (10) remain valid under appropriate conventions, when some entries in M are zero. We do not have a clear generalization of this phenomenon for $n > 2$.

• **Adjoint.** Eqs. (9, 10) are valid in every associative ring $\mathcal{R}$. In case $\mathcal{R}$ is also commutative, Eqs. (9,10) reduce to the well known formula

$$M^{-1} = \frac{\text{Adj}[M]}{\det[M]} \ . \tag{11}$$

In calculating the inverse of real and complex matrices, (11) is of great theoretical importance. So far, we have failed to generalize this formula to quaternion matrices. At first sight, it might make sense to conjecture a non-commuting generalization of the general form,

$$M^{-1} = P \, \text{Adj}[M] \, Q \ , \tag{12}$$

with

$$P = \text{diag}[p_1, p_2] \quad \text{and} \quad Q = \text{diag}[q_1, q_2] \ ,$$

quaternionic diagonal matrices. Nevertheless, the constraints

$$\begin{array}{rclcrcl} p_1 & = & \tilde{a} q_1 d^{-1} \ , & \quad & p_2 & = & -\tilde{c} q_1 c^{-1} \ , \\ p_1 & = & -\tilde{b} q_2 b^{-1} \ , & \quad & p_2 & = & \tilde{d} q_1 a^{-1} \ , \end{array}$$

which, for commuative fields, are satisfied if

$$P = \frac{I}{\det[M]} \quad \text{and} \quad Q = I \ ,$$

are not always satisfied in the quaternionic world. It is easily checked that the unitary matrix (5) cannot be written in the form (12). Whether a further weakening, beyond (12), of formula (11) is valid for quaternionic matrices remains an open problem. Even the definition of Adj$[M]$, $M \in \mathbb{H}^{n \times n}$, $n > 2$, is not clear.

• **Determinant.** . For $n = 2$, an application of theorem (4.1) yields

$$\mathcal{D}_2 \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = |a| \, |d - ca^{-1}b| = |b| \, |c - db^{-1}a| = |c| \, |b - ac^{-1}d| = |d| \, |a - bd^{-1}c| \ .$$

From this definition, the functional $\mathcal{D}_2$ for the matrix (5,6) is

$$\mathcal{D}_2 \left[\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix} \right] = \mathcal{D}_2 \left[\tfrac{1}{\sqrt{2}} \begin{pmatrix} i & j \\ j & i \end{pmatrix} \right] = 1 \ ,$$

as required by unitarity. For hermitian quaternionic matrices, the real determinant is given by

$$\left| \begin{pmatrix} \alpha & q \\ \bar{q} & \delta \end{pmatrix} \right|_r = \lambda_1 \, \lambda_2 = \alpha \, \beta - |q|^2 \ , \qquad \alpha \, , \, \beta \in \mathbb{R} \ , \quad q \in \mathbb{H} \ .$$

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