Numerical evidence for the nonunique evolution of vortex sheets in the plane

M.C. Lopes Filho*
J. Lowengrub†
H.J. Nussenzveig Lopes‡
Yuxi Zheng§

Abstract

We consider a special configuration of vorticity that consists of a pair of externally tangent circular vortex sheets. Each sheet has a circularly symmetric core of bounded vorticity that is concentric to the sheet. The cores precisely balance the vorticity mass of each sheet. This configuration is a stationary weak solution of the 2D incompressible Euler equations. Using the vortex blob method, we perform a series of numerical experiments which suggest that the approximations converge to a non-stationary solution as the numerical discretization and blob size parameters tend to zero. This kind of initial data is not covered by the currently available convergence results for the vortex blob method. However, we establish here a convergence theorem that applies to our data. This theorem and our numerical results strongly suggest that there exist two distinct weak solutions for this kind of initial data.

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1 Introduction

The existence of weak solutions of the incompressible two-dimensional Euler equations has been established for initial vorticities $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ provided that the negative (or the positive) part of ω_0 belongs to $L^1(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ (see [5, 13, 20, 24]). However, uniqueness of weak solutions is only known for initial vorticities in $L^\infty(\mathbb{R}^2)$ (see [25]) or very close to L^∞ (see [26]). The uniqueness of weak solutions with more general initial velocity/vorticity distributions remains an open question. In 1993, V. Scheffer ([18]) constructed a weak solution in which the velocity belongs to $L^2((-T,T) \times \mathbb{R}^2)$ with compact support in space-time. Subsequently, a similar example was produced by A. Shnirelman in [19], with a more explicit account of the analytical mechanism involved. Scheffer's and Shnirelman's examples rest on an inverse energy cascade where the energy "bubbles up" from the infinitesmal scales. These examples suggest non-uniqueness of weak solutions (namely, the weak solution consisting of velocity identically equal to zero) if one allows for sufficiently irregular flows. The problem of non-uniqueness in flows with more regularity and physical relevance such as velocities in $L^\infty((0,\infty), L^2(\mathbb{R}^2))$ remains open.

Promising candidates for such scenarios of non-uniqueness are flows involving vortex sheets. Vortex sheets are idealized models of narrow shear layers and are surfaces of discontinuity in which the tangential component of the velocity has a jump across the surface while the normal velocity is continuous. Thus, the fluid slips along the sheet while the sheet moves with the flow. Note that the vorticity due to a vortex sheet is in $\mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ so that single-signed vortex sheets are weak solutions of the incompressible 2D Euler equations. The reason that vortex sheets are such good candidates for providing potential examples of non-uniqueness is that they are highly unstable due to the Kelvin-Helmholtz instability. This instability reveals itself through the roll-up of the sheet (and analogous narrow shear layer) into large scale vortices. In fact, initial value problems with vortex sheet initial data are ill-posed in the sense of Hadamard in a wide class of Sobolev spaces [4, 8], although in certain analytic function spaces the problem is well-posed for short times [3, 7, 22].

Additional motivation for considering nonunique vortex sheet evolution comes from considering the corresponding electron sheet problem for the Vlasov-Poisson equations. Electron sheet evolution for the 1D Vlasov-Poisson equations bears much resemblance to vortex sheet evolution for the incompressible 2D Euler equations. For this electron sheet problem, A. Majda, G. Majda and one of the authors explicitly contructed two distinct electron sheet (weak) solutions with the same initial data [14].

Finally, even further motivation for considering nonunique vortex sheet evolution comes from the intriguing results of D. Pullin [16] and Pullin and W. Phillips [17]. In [16, 17], these authors exploited the difference in behavior of the vortex sheet problem in analytic and non-analytic function spaces to obtain numerical evidence of multiple self-similar vortex sheet solutions. For example, Pullin in [16] considered the self-similar evolution of a single, initially flat single-signed vortex sheet with x-coordinate given by $x(\Gamma,0) = \mathrm{sgn}\,(\Gamma)\,(|\Gamma|/2a)^{1/p}$ where Γ is the circulation between a point $(x(\Gamma,0),y(\Gamma,0))$ and $(x(0,0),y(0,0))$. Further, a is a constant and $0<p<2$. When $p=1$, the initial interface $x(\Gamma,0) = \Gamma/2a$ is an analytic function of Γ and the resulting vortex sheet is a stationary solution of the Euler equations. For all other values of p, there are singularities at the origin and the solutions are non-stationary. Pullin simulated the self-similar evolution of these vortex sheets by recasting the equations in self-similar form and solving the resulting integro-differential equations using several different numerical approximation techniques. Letting $p \to 1^+$, Pullin [16] and Pullin & Phillips [17] obtained a nontrivial self-similar single-spiral solution (in addition to the stationary solution). For $0.9666 < p < 1$, Pullin [16] obtained three different self-similar solutions – two single spiral and one double spiral solutions. While Pullin's evidence is suggestive of non-uniqueness for the initial value problem, it remains to obtain such an example directly.

In this paper, we present both theoretical and numerical evidence, including a numerical convergence study, for the existence of multiple weak solutions of the initial value problem branching out from initial data containing smooth, single-signed vortex sheets.

The specific initial vorticity configuration we consider is a case of two externally tangent *confined eddies*. A confined eddy is a compactly supported distribution of vorticity, with zero total vorticity, which is circularly symmetric with respect to some point in the plane. We note that the limit of a sequence of confined eddies with vanishing supports and constant total mass is a phantom vortex of DiPerna and Majda [6]. As was observed in [6], confined eddies are stationary weak solutions of the vorticity formulation of the incompressible 2D Euler equations. The velocity field associated with a confined eddy vanishes in the exterior of the support of the eddy. Hence, confined eddies may be externally superimposed, giving rise to an interesting class of stationary weak solutions of the 2D Euler equations, see Greengard and Thomann [9] for example. It was proved in [12] that for initial vorticities consisting of two or more confined eddies with *disjoint* supports, the solution of the 2D incompressible Navier-Stokes equations converges back to the superposition of confined eddies as the viscosity vanishes. We comment in passing that it is easy to extend the convergence proof to handle the case in which the approximate solutions are generated by the vortex blob approximation and to handle the case in which the approximate solutions are obtained by evolving a smoothed version of the initial data.

The two confined eddies we consider in this paper, ω_0^1 and ω_0^2, are centered at x^1 and x^2 respectively. Each eddy consists of a uniform measure supported on the circle $|x-x^i| = |x^1-x^2|/2$ (vortex sheet), with unit total measure, plus a core of bounded vorticity of total measure -1, internal to each respective circle (see the $T = 0$ plot in figure 1). The main purpose of the current work is to show numerical evidence, using the vortex blob approximation, that an unsteady weak solution evolves from this initial configuration. Thus, there is a second weak solution for this data– the first is the steady solution.

To give credence to our numerical results, we must examine the issue of convergence of the vortex blob approximation to a weak solution of the incompressible 2D Euler equations. For our specific set of initial data this is an open problem, not covered by the convergence result in [10], since the overall vorticity field does not have a distinguished sign (note that the vortex sheets are single-signed). The difficulty is somewhat surprising, since the vorticity is an L^∞ perturbation of a nonnegative measure and falls well within the class of vorticities for which other approximation schemes give rise to a weak solution, namely, regularizing the initial data and exactly solving the 2D Euler equations or using the vanishing viscosity approximation, see [5, 13, 20]. The crux of this difficulty is that oppositely signed point vortices may attract one another and collide, leading to a singularity in the blob equations as the blob size converges to zero. Since like-signed vortices do not collide, collisions may be ruled out if the positive and negative vortices remain well-separated. Under this separation assumption, we show the convergence of the vortex blob approximation to a weak solution of the Euler equations. Our numerical simulations verify this separation of positive and negative vortices. Our result is an extension of the theory developed by S. Schochet in [21].

Consequently, our argument for non-uniqueness rests on **(i)** the theoretical convergence result described above, **(ii)** the highly unstable steady initial configuration of vorticity and **(iii)** a series of numerical simulations that seem decidely convergent to some limit which is not the initial data. While this is logically sound, one must take care to demonstrate the convergence of the numerical simulations in the presence of such strong instability. We indeed show that certain crucial quantities, such as the separation between positive and negative vortices and various measures of unsteadiness, do converge as the numerical parameters vanish. Somewhat surprisingly, we observe exponential convergence of these quantities.

This paper is divided into six sections. In section 2, the theoretical basis of our work is presented. In particular, we consider the existence of a subsequence of the vortex blob approximation which converges to a weak solution of the 2D Euler equations. In section 3, we present the discretization of the confined eddies and show they fall within the class covered by our convergence theorem. In section 4, we briefly discuss our numerical algorithms and in section 5, we present our numerical results. Finally, some conclusions are presented in section 6.

2 Convergence of the vortex blob method

In this section we state and prove a convergence result for vortex blob approximations that applies to the initial data under consideration. Let $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ be such that $\omega_0 = \omega_0^+ - \omega_0^-$, where ω_0^+ and ω_0^- are nonnegative bounded measures with disjoint compact supports.

We introduce the vortex blob approximation. We restrict our discussion to the Krasny blob function:

$$\phi(x) = \frac{1}{\pi(1+|x|^2)^2},\ x \in \mathbb{R}^2,$$

although what follows applies to any divisible blob function, see [6]. Define $\phi_\eta(x) = (1/\eta)^2\phi(x/\eta)$ and denote

$$K_\eta(x) = \frac{x^\perp}{2\pi(|x|^2+\eta^2)} = K_0 * \phi_\eta,$$

where $x^\perp = (-x_2, x_1)$ and K_0 is K_η evaluated at $\eta = 0$.

We discretize the initial vorticity ω_0 by considering approximate vorticities $\widetilde{\omega}_{0,h}^+$ and $\widetilde{\omega}_{0,h}^-$ such that:

(H1) $\widetilde{\omega}_{0,h}^{\pm} \rightharpoonup \omega_0^{\pm}$ weak-$*$ in $\mathcal{BM}$, and there exists a single compact set containing the supports of $\widetilde{\omega}_{0,h}^{\pm}$.

(H2) $\widetilde{\omega}_{0,h}^{\pm} = \sum_{i=1}^{N^{\pm}} \xi_{i,h}^{\pm} \delta(x - \alpha_{i,h}^{\pm})$, where $N^{\pm} = N^{\pm}(h)$ and $\xi_{i,h}^{\pm} > 0$.

(H3) $\min_{i=1,\dots,N^+, j=1,\dots,N^-} |\alpha_{i,h}^+ - \alpha_{j,h}^-|$ is bounded away from zero, uniformly in h.

(H4) $\sum_{j,k=1, j\neq k}^{N^{\pm}} \xi_{j,h}^{\pm}\xi_{k,h}^{\pm} \log|\alpha_{j,h}^{\pm} - \alpha_{k,h}^{\pm}|$ is bounded, uniformly in h.

If the supports of the negative and positive parts of vorticity are initially disjoint, we can set up the initial discretization by choosing a rectangle containing the support of vorticity where we introduce a square mesh with cells Q^{ij} and centers α^{ij} in such a way that in each Q^{ij} the vorticity has a distinguished sign. We can construct discretizations:

$$\widetilde{\omega}_{0,h}^{\pm} = \sum \left(\int_{Q^{ij}} \omega^{\pm} dx \right) \delta(x - \alpha^{ij}),$$

where h is the diameter of Q^{ij} and $\omega^{\pm}$ are the positive and negative parts of ω. It is an easy consequence of Lemma 3.2 in [21] that the sequences $\widetilde{\omega}_{0,h}^{+}$ and $\widetilde{\omega}_{0,h}^{-}$ obtained satisfy the hypotheses (H1)-(H4).

The vortex blob system is given as follows. Let $X_{i,h,\eta}^{\pm}(t)$ be the positions of the vortex blobs at time t. Then,

$$\begin{cases} \dfrac{d}{dt} X_{i,h,\eta}^{\pm} = \displaystyle\sum_{j=1}^{N^+} \xi_{j,h}^{+} K_\eta(X_{i,h,\eta}^{\pm} - X_{j,h,\eta}^{+}) - \sum_{j=1}^{N^-} \xi_{j,h}^{-} K_\eta(X_{i,h,\eta}^{\pm} - X_{j,h,\eta}^{-}) \\ X_{i,h,\eta}^{\pm}(0) = \alpha_{i,h}^{\pm}. \end{cases} \tag{1}$$

For each $\eta > 0$ and $h > 0$ fixed, the flux for this system of ODE's is smooth and globally bounded, together with its derivatives. This implies that the solution $X_{i,h,\eta}^{\pm}(t)$ is defined for all time $t \geq 0$ and that there are no vortex collisions in finite time. We introduce our approximate vorticities by:

$$\omega_{h,\eta}(x,t) = \omega_{h,\eta}^{+}(x,t) - \omega_{h,\eta}^{-}(x,t) = \sum_{j=1}^{N^+} \xi_{j,h}^{+} \phi_\eta(x - X_{j,h,\eta}^{+}) - \sum_{j=1}^{N^-} \xi_{j,h}^{-} \phi_\eta(x - X_{j,h,\eta}^{-}).$$

The approximate velocities are then given by $u_h = u_h(x,t) = K_0 * \omega_{h,\eta}$. We fix a function $\eta = \eta(h)$ such that $\eta \to 0$ as $h \to 0$ and eliminate the explicit subscript η from what follows.

We consider two notions of weak solutions to the incompressible 2D Euler equations. The first refers to the *weak vorticity formulation*:

$$\int_t \int_x \varphi_t(x,t)\omega(x,t)dxdt +$$

$$\frac{1}{2}\int_t \int_x \int_y (\nabla\varphi(x,t) - \nabla\varphi(y,t)) \cdot \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(x,t)\omega(y,t)dxdydt = 0,$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^2 \times (0,T))$. The second is the *weak velocity formulation* which is perhaps the more standard weak formulation:

$$\int_t \int_x \varphi_t(x,t) \cdot u(x,t) + (u(x,t))^t D\varphi(x,t)u(x,t)dxdt = 0,$$

for any divergence-free test vector field $\varphi \in (C_0^\infty(\mathbb{R}^2 \times (0,T)))^2$, where the nonlinear term $u^t D\varphi u$ is the Jacobian matrix of φ, interpreted as a quadratic form and evaluated at u.

With the notation introduced above and assuming (H1)–(H4), we are now ready to state and prove our main result.

Theorem 1 *Let $T > 0$. Assume that there exists a constant $d > 0$, independent of h or $t \in [0,T]$, such that*

$$\min_{i=1,\ldots,N^+, j=1,\ldots,N^-, t\in[0,T]} |X^+_{i,h}(t) - X^-_{j,h}(t)| \geq d > 0. \tag{2}$$

Then, any weak-$*$ limit in $\mathcal{BM}$ of the sequence ω_h as $h \to 0$ is a weak solution of the weak vorticity formulation of the Euler equations. Furthermore, if the approximate velocities u_h are uniformly bounded in $L^\infty([0,T]; L^2_{loc}(\mathbb{R}^2))$ then any weak limit of the sequence of approximate velocities is a weak solution of the weak velocity formulation of the Euler equations.

Proof: Our proof relies heavily on the work done in [21]. Using Lemma 2.1 in [21], it is enough to show the nonconcentration condition:

$$\lim_{r\to 0}\lim_{h\to 0}\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^2}\left(\sum_{\{j \mid X^\pm_{j,h}(t)\in B(x,r)\}} \xi^\pm_{j,h}\right) = 0.$$

Define the function:

$$\psi_\eta = \psi_\eta(r) = \frac{1}{2}\log\frac{1}{r^2+\eta^2},$$

and consider:

$$H_h^{\pm} = \sum_{j,k=1}^{N^{\pm}} \xi_{j,h}^{\pm}\xi_{k,h}^{\pm}\psi_\eta(|X_{j,h}^{\pm} - X_{k,h}^{\pm}|),$$

$$W_h^{\pm} = \sum_{j,k=1}^{N^{\pm}} \xi_{j,h}^{\pm}\xi_{k,h}^{\pm}|X_{j,h}^{\pm} - X_{k,h}^{\pm}|^2,$$

$$H_h^I = \sum_{j=1}^{N^+}\sum_{k=1}^{N^-} \xi_{j,h}^{+}\xi_{k,h}^{-}\psi_\eta(|X_{j,h}^{+} - X_{k,h}^{-}|),$$

$$W_h^I = \sum_{j=1}^{N^+}\sum_{k=1}^{N^-} \xi_{j,h}^{+}\xi_{k,h}^{-}|X_{j,h}^{+} - X_{k,h}^{-}|^2.$$

We also introduce $J_h = J_h^+ + J_h^- - 2J_h^I$, with $J_h^{\pm} = H_h^{\pm} + W_h^{\pm}$ and $J_h^I = H_h^I + W_h^I$. It is an immediate extension of an observation by S. Schochet in [21] that J_h is a first integral of the vortex blob system. This first integral will provide us with the estimate needed to show the nonconcentration condition provided that we can control the evolution of the terms corresponding to the interaction of positive and negative vortices, i.e. J_h^I.

Assume without loss of generality that $d < 1$ and that $\eta(h) < \sqrt{1-d^2}$, for every $0 < h \leq 1$. From the separation assumption (2), we have:

$$H_h^I \leq \frac{1}{2}\log\frac{1}{d^2+\eta^2}\sum_{j=1}^{N^+}\sum_{k=1}^{N^-}\xi_{j,h}^{+}\xi_{k,h}^{-} \leq \log\frac{1}{d}\|\tilde{\omega}_{0,h}^{+}\|_{\mathcal{BM}}\|\tilde{\omega}_{0,h}^{-}\|_{\mathcal{BM}}.$$

Next we estimate W_h^I. First we rewrite W_h^I as:

$$W_h^I = \sum_{j=1}^{N^+}\xi_{j,h}^{+}|X_{j,h}^{+}|^2\sum_{k=1}^{N^-}\xi_{k,h}^{-} + \sum_{k=1}^{N^-}\xi_{k,h}^{-}|X_{k,h}^{-}|^2\sum_{j=1}^{N^+}\xi_{j,h}^{+} -$$

$$2\left(\sum_{k=1}^{N^-}\xi_{k,h}^{-}X_{k,h}^{-}\right)\cdot\left(\sum_{j=1}^{N^+}\xi_{j,h}^{+}X_{j,h}^{+}\right).$$

Let:

$$y^{\pm} = y^{\pm}(t) \equiv \left(\sum_{j=1}^{N^+}\xi_{j,h}^{+}|X_{j,h}^{+}|^2\right)^{1/2}.$$

We observe that these are the only quantities which need to be estimated in order to estimate W_h^I. Indeed, using the Cauchy-Schwarz inequality, we obtain:

$$\left|\sum_{j=1}^{N^\pm} \xi_{j,h}^\pm X_{j,h}^\pm\right| \leq \|\tilde{\omega}_{0,h}^\pm\|_{\mathcal{BM}}^{1/2}\, y^\pm.$$

We have, for y^+:

$$\frac{d}{dt}(y^+)^2 = \sum_{j=1}^{N^+} \left(\xi_{j,h}^+ 2X_{j,h}^+\right)\cdot\left(\sum_{\ell=1}^{N^+} \xi_{\ell,h}^+ K_\eta(X_{j,h}^+ - X_{\ell,h}^+) - \sum_{\ell=1}^{N^-} \xi_{\ell,h}^- K_\eta(X_{j,h}^+ - X_{\ell,h}^-)\right)$$

$$= -2\sum_{j=1}^{N^+}\sum_{\ell=1}^{N^-} \xi_{j,h}^+ \xi_{\ell,h}^- X_{j,h}^+ \cdot K_\eta(X_{j,h}^+ - X_{\ell,h}^-),$$

due to the antisymmetry of the kernel K and the fact that $K_\eta(z)\cdot z = 0$,

$$\leq \frac{1}{\pi d}\|\tilde{\omega}_{0,h}^-\|_{\mathcal{BM}} \sum_{j=1}^{N^+} \xi_{j,h}^+ |X_{j,h}^+|,$$

due to the separation assumption 2,

$$\leq \frac{1}{\pi d}\|\tilde{\omega}_{0,h}^-\|_{\mathcal{BM}} \|\tilde{\omega}_{0,h}^+\|_{\mathcal{BM}}^{1/2}\, y^+.$$

Therefore, since $y^+(t) > 0$,

$$y^+(t) \leq \frac{1}{2\pi d}\|\tilde{\omega}_{0,h}^-\|_{\mathcal{BM}} \|\tilde{\omega}_{0,h}^+\|_{\mathcal{BM}}^{1/2}\, T + y^+(0),$$

for all $0 \leq t \leq T$.

Analogously,

$$y^-(t) \leq \frac{1}{2\pi d}\|\tilde{\omega}_{0,h}^+\|_{\mathcal{BM}} \|\tilde{\omega}_{0,h}^-\|_{\mathcal{BM}}^{1/2}\, T + y^-(0),$$

for all $0 \leq t \leq T$.

Hence, $W_h^I \leq C$, where C is independent of t. Furthermore, since $\tilde{\omega}_{0,h}^\pm \rightharpoonup \omega_0^\pm$ weak-$*$ in $\mathcal{BM}$, and since the supports of $\tilde{\omega}_{0,h}^\pm$ are contained in a single compact set (hypothesis (H1)), it follows that the bounds obtained for H_h^I and W_h^I are independent of h as well. In addition, using hypotheses (H1), (H4) and the separation assumption (2) together with the fact that J_h is a conserved quantity, it follows that J_h is bounded independently of time and h. Therefore we see that $J_h^+ + J_h^- = J_h + 2J_h^I$ is a non-negative quantity which is uniformly bounded for $t \in [0,T]$ and $h \in (0,1]$.

Let $0 < r < d/2$ and $x \in \mathbb{R}^2$. Observe that there are either positive or negative vortices in $B(x,r)$, never both, due to the separation assumption (2). Hence we have:

$$\left(\sum_{\{j \mid X_{j,h}^{\pm} \in B(x,r)\}} \xi_{j,h}^{\pm} \right)^2 \leq \frac{-2}{\log(4r^2 + \eta^2)} (J_h^+ + J_h^-) \leq \frac{-C}{\log(4r^2 + \eta^2)}. \tag{3}$$

Note that, since we assumed that $\eta < \sqrt{1 - d^2}$ it follows that $4r^2 + \eta^2 < 1$.

The constant C above is positive and independent of t, x and h. Since the $\lim_{r \to 0} \lim_{h \to 0}$ of the right-hand-side above is zero, we thus have verified the nonconcentration condition.

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Remark: Theorem 1 is, strictly speaking, a new existence result. As an existence result, it may not seem very interesting because the hypothesis of separation is very restrictive and artificial. However, Theorem 1 as a convergence result for the vortex blob method is interesting because the separation assumption can be easily verified numerically. The new feature in the proof is the a priori estimate of the terms corresponding to interaction of vorticities with opposite sign in the first integral J^h.

3 Discretization of the Initial Data

Consider an initial measure ω_0 consisting of one-dimensional, homogenous Diracs supported on a pair of externally tangent unit circles (vortex sheets). More precisely, let ω_0^+ be given by:

$$\langle \omega_0^+, \phi \rangle = \frac{1}{2\pi} \int_{|x-(-1,0)|=1} \phi ds + \frac{1}{2\pi} \int_{|x-(1,0)|=1} \phi ds,$$

for any $\phi \in C_0(\mathbb{R}^2)$. Consider also

$$\omega_0^- = \frac{2}{\pi |x - (-1,0)|} \chi_{\{1/4 < |x-(-1,0)| < 1/2\}} + \frac{2}{\pi |x - (1,0)|} \chi_{\{1/4 < |x-(1,0)| < 1/2\}}.$$

Our first observation is that $\omega_0 = \omega_0^+ - \omega_0^-$ is a stationary weak solution of the incompressible 2D Euler equations, by virtue of being a superposition of confined eddies, see [12].

We consider an initial discretization of this measure as follows. Fix K an integer. We choose the blob size $\eta = 1/(2\sqrt{10}K)$. Let $N = 10K^2$, so that blob size is $1/(2\sqrt{N})$, and introduce:

$$\tilde{\omega}_0^{+,N}(x) = \sum_{i=0}^{N-1}(1/N)\delta(x-(\cos(2\pi i/N)-1-\eta/4,\ \sin(2\pi i/N)))+$$

$$\sum_{i=0}^{N-1}(1/N)\delta(x-(\cos(2\pi i/N)+1+\eta/4,\ \sin(2\pi i/N))).$$

This is the discretization of the two vortex sheets. Note that they are no longer tangent but are separated by the distance $\eta/2$.

We discretize the core of bounded vorticity by arranging the point vortices in a roughly uniform grid. Since the domain is the annulus $1/4 < |x| < 1/2$, the cores have an aspect ratio of about 10 in polar coordinates. We begin by choosing K points in $[1/4, 1/2]$: $r_i = 1/4+(i-1)/(4(K-1))$ for $i = 1, \ldots, K$. Next, we choose $10K$ points on $[0, 2\pi]$, by taking $\theta_j = 2\pi(j-1)/(10K)$, with $j = 1, \ldots, 10K$. We define $\alpha^{ij} = (r_i\cos(\theta_j), r_i\sin(\theta_j))$ and

$$\tilde{\omega}_0^{-,K}(x) = \sum_{i=1}^{K}\sum_{j=1}^{10K}\frac{1}{10K^2}[\delta(x-\alpha^{ij}-(-1-\eta/4,0))+$$

$$\delta(x-\alpha^{ij}-(1+\eta/4,0))].$$

Naturally, $\tilde{\omega}_0^K = \tilde{\omega}_0^{+,10K^2} - \tilde{\omega}_0^{-,K}$. In total, there are $M = 40K^2$ vortices. See the $T = 0$ graph in figure 1 for a plot of the vortices.

Proposition 1 *The sequence of approximations $\tilde{\omega}_0$ described above satisfies (H1)–(H4).*

Proof:

The hypothesis (H2) and (H3) are obviously satisfied by construction. Let us examine (H1). Let ϕ be a test function in $C_0(\mathbb{R}^2)$ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathcal{BM}$ and C_0. Then:

$$\langle\tilde{\omega}_0^{-,K},\phi\rangle = \frac{2}{\pi}\sum_{i=1}^{K}\sum_{j=1}^{10K}\frac{\pi}{20K^2}\left(\phi(\alpha^{ij}+(1+\eta/4,0))+\phi(\alpha^{ij}-(1-\eta/4,0))\right),$$

which is a pair of Riemann sums in polar coordinates, with small translations, and thus, since ϕ is continuous, converges as $K \to \infty$ to:

$$\frac{2}{\pi}\int_{1/4}^{1/2}\int_{0}^{2\pi} \phi(r\cos\theta+1, r\sin\theta)+\phi(r\cos\theta-1, r\sin\theta)d\theta dr \equiv \int \omega_0^- \phi dx.$$

The case of $\tilde{\omega}_0^{+,10K^2}$ can also be identified with a pair of translated Riemann sums, converging to the appropriate limit.

Hypothesis (H4) is more delicate. For the negative vortices, checking hypothesis (H4) involves a simple adaptation of the proof of Lemma 3.2 in [21] since the grid in the bounded cores is comparable to a square grid and the weights $\xi_{j,h}^-$ are precisely the total mass of ω_0^- in each grid cell. The fact that the $\alpha_{j,h}^-$ are not the centers of the grid cells, but one of their corners, is easy to deal with.

We now verify (H4) for the positive vortices. Let $\mathcal{H}_N$ be the logarithmic double sum involving the positive vortices that we wish to estimate. Let the complex position $z_{jN} \equiv \cos 2\pi j/N + i\sin 2\pi j/N$ for $j = 0, \ldots, N-1$. We decompose this discrete pseudoenergy into self-induction and interaction terms for the two vortex sheets and center the calculation at the origin to obtain:

$$\mathcal{H}_N = 2\sum_{j,k=1,j\neq k}^{N} \frac{1}{N^2}\log|z_{jN}-z_{kN}| + 2\sum_{j,k=1}^{N}\frac{1}{N^2}\log|z_{kN}-(z_{jN}+2+\eta/2)|.$$

We first compute the self-induction term:

$$\sum_{j,k=1,j\neq k}^{N}\frac{1}{N^2}\log|z_{jN}-z_{kN}| = \sum_{j=1}^{N}\log\left|\prod_{k=1,k\neq j}^{N}(z_{jN}-z_{kN})\right| =$$

$$\sum_{j=1}^{N}\log\left|\lim_{z\to z_{jN}}\frac{z^N-1}{z-z_{jN}}\right| = \frac{1}{N^2}\sum_{j=1}^{N}\log\left|N z_{jN}^{N-1}\right| = \frac{\log N}{N},$$

which converges to 0, and is therefore bounded, as $N \to \infty$.

Finally we estimate the interaction term. First observe that:

$$\sum_{j,k=1}^{N}\frac{1}{N^2}\log|z_{kN}-(z_{jN}+2+\eta/2)| \leq \log|4+\eta/2|.$$

On the other hand,

$$\sum_{j,k=1}^{N}\frac{1}{N^2}\log|z_{kN}-(z_{jN}+2+\eta/2)| = \frac{1}{N^2}\sum_{j=1}^{N}\log\left|(z_{jN}+2+\eta/2)^N-1\right| \geq$$

$$\frac{1}{N^2}\sum_{j=1}^{N}\log(|z_{jN}+2+\eta/2|^N-1)\geq\frac{1}{N^2}\sum_{j=1}^{N}\log N\eta/2=\log(N\eta/2)^{1/N}.$$

Hence, if $\eta=\eta(N)$ does not vanish too fast, for example, if it is bounded below by a negative exponential, the pseudoenergy is uniformly bounded as $N\to\infty$. Of course, this is true in our case $\eta=1/(2\sqrt{N})$.

∎

Therefore, Proposition 1 shows that if the separation assumption is valid, then Theorem 1 holds for the approximate solution sequences obtained by solving the vortex blob system (1) with the special initial conditions given above.

4 Temporal Discretization

In this section, we briefly describe our numerical algorithm. In order to solve the vortex blob system (1) numerically, we discretize the equations in time using a standard fourth-order Runge-Kutta time discretization. In all simulations presented in the next section, the time step is $\Delta t=0.01$. We performed two tests to verify temporal accuracy using this time step. In the first, we computed the pseudo-energy

$$\sum_{j,k=1,j\neq k}^{N^{\pm}}\xi_{j,h}^{\pm}\xi_{k,h}^{\pm}\log\left[|X_{j,h,\eta}^{\pm}-X_{k,h,\eta}^{\pm}|^2+\eta^2\right]$$

which is time invariant for system (1). Rather surprisingly, this quantity was preserved to at least 10 digits thoughout the simulations in all the cases we considered. In the second test, we computed $\max_k|X_{k,h,\Delta t}^{\pm}-X_{k,h,5\Delta t}^{\pm}|$ and, again surprisingly, found this to be less than 10^{-8} for all cases considered. We therefore conclude that the evolution is well-resolved using this time step. We suspect, though, that our particular choice of initial condition leads to the significant error cancellation in the scheme which results in our unexpectedly high temporal accuracy.

The most time consuming part of the simulation is performing summations of the form

$$S_i=\sum_{j=1}^{M}\xi_{j,h}K_\eta(X_{i,h,\eta}-X_{j,h,\eta}),\ \ i=1,\dots M$$

which requires $O(M^2)$ operations. To reduce computational cost, we implement the summations S_i in parallel as follows. Let P be the number of processors. Then, we compute M/P summations at each processor simultaneously and the result is broadcasted to all other processors. This has a perfect workload balance of M/P when M/P is an integer and has communication load of order $O(M)$. This approach is efficient because the communication costs scale only like the square root of the overall operation cost. All $O(M)$ operations, such as the time discretization, are performed sequentially. In the simulations we present in the next section, we used 32 processors on a 256 processor IBM-SP2 with 332 MHz 604e PowerPC processors at the Minnesota Supercomputer Institute. We use the Message Passing Interface (MPI) for communication among the processors. This strategy has been used successfully in several recent boundary integral computations [15, 11]. We regularly achieve efficiencies of over 90%; efficiency=$T_1/(PT_P)$, where T_P is the time required to use P processors.

5 Numerical Results

We now consider the evolution of our confined eddies. In figure 1, a time sequence of the dynamics using $K = 32$ is shown. This corresponds to $40,960 = 40{*}32^2$ point vortices which is the maximum number we used. Only the point vortices are graphed; there is no interpolation. One immediately observes that nontrivial dynamics occur. At early times, $t < 5$, the main deformations in the eddies are confined to the regions of near tangency of the outer two vortex sheets. There, the vortex sheets begin to wrap around one another leading to roll-up. At later times, the vortex cores also deform and the roll-up continues eventually drawing in nearly all the point vortices that were initially near the origin. Indeed, by time $t = 10$, there are only a few remaining positive point vortices separating the two negative vortex cores. In addition, the positive and negative vortices remain well-separated throughout the simulation suggesting that the separation assumption (2) is valid.

Since we are concerned with the convergence in the limit $K \to \infty$, we next compare our results using $K = 32$ to those using other lower resolutions. We will show that increasing K leads to larger deformations of the eddies and hence non-stationarity. We will quantify this by considering:

- The exterior normal velocity of the positive vortex sheets.

- The total vorticity near the origin.

Further, we will show the positive and negative vortices always remain well-separated.

In figure 2, evolution of the two confined eddies with $K = 16$ is shown for comparison. Again, only the $10,240$ point vortices are graphed which is why the figure is lighter than figure 1. While the dynamics follows that of the $K = 32$ simulation, there is certainly less roll-up. In figure 3, we directly compare the $K = 16$ (light) and $K = 32$ (dark) results at times $t = 2$, 5 and 10 in the regions of greatest deviation. It is clear that more structure and larger deformations are seen for $K = 32$.

As remarked above, we can observe this trend in another, more quantifiable way by considering the normal velocity of the outer vortex sheets. In figure 4, we present the normal velocity of the right vortex sheet at time $t = 2$ for three different resolutions $K = 32$, $K = 24$ and $K = 16$. Note that the normal velocity of the stationary solution is zero. The velocity is graphed as a function of the initial parametrization of the sheet and the point of near tangency is π. We observe that while the non-zero velocity is localized around the region of near tangency, the normal velocity increases in magnitude as K increases. In figure 5, the maximum of the normal velocity is shown as a function of $M \equiv 40 * K^2$. It clearly diverges like $\sqrt{M} = O(K)$ as $M \to \infty$. This is not surprising since the blob size $\eta = 1/\sqrt{M}$. We comment in passing that the support of the non-zero normal velocity shrinks as $K \to \infty$. Although we do not present it here, the integral of the positive part of the normal velocity at $t = 2$ remains constant as K increases suggesting convergence to a delta function in the limit. The integral of the negative part of the normal velocity, on the other hand, is a decreasing function of K.

As a measure of the non-stationarity of the vorticity distribution, consider the integral of the vorticity $I_{\omega,K}(t)$ in a ball centered at the origin with radius 0.1 given by

$$I_{\omega,K}(t) = \int_{|x|\leq .1} \tilde{\omega}^K(x,t)\ dx.$$

This quantity is weakly lower semicontinuous with respect to weak-$*$ convergence in Radon measures. Therefore, if $\liminf_K I_{\omega,K}(t)$ is smaller than its initial value, then any weak limit of the vorticity distribution is non-stationary. The quantity $I_{\omega,K}(t)$ is graphed in figure 6 for several different resolutions. Observe that $I_{\omega,K}(t)$ is a decreasing function of both time *and* resolution K. The graphs strongly suggests convergence to a non-stationary vorticity distribution.

We can estimate a rate of convergence of $I_{\omega,K}$ by comparing its values with $K < 32$ to those with $K = 32$. Let a measure of the error in $I_{\omega,K}$ be

$$E_{\omega,K}(t) = |I_{\omega,K}(t) - I_{\omega,32}(t)|.$$

In figure 7, $\log(E_{\omega,K}(t))$ is graphed as a function of M at different times. This plot suggests exponential convergence of $I_{\omega,K}(t)$ in M at every time t for K large enough, i.e.

$$E_{\omega,K}(t) \sim Ce^{-aM}$$

where C and a depend on time. At $t = 10$, for example, exponential convergence is seen for $K \geq 20$ while at the earlier times shown, exponential convergence is seen for $K \geq 16$. It is somewhat surprising to observe exponential convergence for the vortex blob method since proofs indicate $O(1/\sqrt{M})$ convergence [2, 10]. However, $I_{\omega,K}(t)$ is an integral quantity and thus averages numerical errors. Moreover, as we suggested in the previous section, our initial condition itself seems to lead to significant error cancellation.

It now remains to consider the separation distance between the supports of the positive and negative vorticity distributions. Accordingly in figure 8, we plot the minimum distance between positive and negative point vortices as a function of time for a number of different resolutions. The minimum distance is seen to be a decreasing function of time and resolution. At early times ($t \leq 2$), the decrease in time appears to be linear. However, at later times, there is a transition and the decrease seems to become only exponential in time. This is suggested by figure 9 where the logarithm of the minimum distance (with $K = 32$) is plotted versus time. We note that on this scale, the results from other resolutions are indistinguishable from the $K = 32$ plot. We comment in passing that it appears that there is a transition between two types of exponential decrease as evidenced by the two straight lines shown on the graph. This would suggest that for any finite time, the supports of the positive and negative vorticities remain separated.

In order to conclude that the separation assumption (2) is valid, we lastly must quantify the effect of varying resolution. Analogously to the case of $I_{\omega,K}(t)$ we considered above, we estimate a rate of convergence of the minimum distance by comparing the result for $K < 32$ with that from $K = 32$. The logarithm of this "error" in minimum distance is plotted in figure 10 as a function of M at several times. The convergence is seen to be exponential in M at all times and for all resolutions $K \geq 16$. Although this error does not measure the error between common individual point vorticies at different resolutions (which in view of figures 1-3 likely does not converge pointwise), it is nevertheless a local measure of computational error. Thus, it is somewhat surprising to observe exponential convergence. This provides strong evidence that the separation assumption (2) is valid for any time $T < \infty$.

Lastly, for $t \leq T = 10$, we can estimate the separation constant d from (2) by supposing that the error plotted in figure 10 is the true error. Then, letting ρ be the average of the slopes of each line segment in the $T = 10$ graph from figure 6, we consider the "best" fit line to the error

$$e_M = -e_{40*16^2} \cdot exp(-\rho[M - 40 * 16^2])$$

where e_{40*16^2} is the error between the $K = 16$ and the $K = 32$ result. Computationally, we find $\rho = 9.7661 \times 10^{-5}$ and the fit to be quite good. Then, we may estimate d by

$$d = \lim_{M\to\infty} \min \text{dist}_M \approx \min \text{dist}_{40*28^2} + e_{40*28^2} = 0.189762.$$

This number is slightly smaller than the $K = 32$ result which is $\min \text{dist}_{40*32^2} = 0.189794$.

6 Conclusions

Our numerical evidence strongly suggests that in the limit as $K \to \infty$, and hence $\eta \to 0$, the limiting vorticity distribution obtained from the vortex blob system is non-trivial and non-stationary. Moreover, the numerical evidence also suggests that the separation assumption (2) is valid for $T < \infty$. For $t \leq T = 10$, we provide a computational estimate of the separation distance d. In view of Theorem 1, this provides strong evidence for non-uniqueness since both the stationary solution and the $M \to \infty$ limit of our computational (non-stationary) solution are weak solutions of the Euler equations.

Finally, one may ask whether our computational solution may be obtained from the vanishing viscosity approximation. This is far from clear. While there have been studies comparing the inviscid roll-up of vortex sheets using the vortex blob approximation with simulations using small viscosity (see e.g. [23]), the connection between the blob and viscous regularizations is still not understood. The viscous regularization in the vortex sheet case does yield roll-up however. Thus, we conjecture that the vanishing viscosity approximation of our confined eddies does yield a non-stationary weak solution of the 2D Euler equations although this solution may not be the one we found. It is likely that other regularizations, such as the vortex layer approximation [1], yield other non-stationary weak solutions.

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Milton C. Lopes Filho
Departamento de Matematica, IMECC-UNICAMP.
Caixa Postal 6065, Campinas, SP 13081-970, Brasil
E-mail address: mlopes@ime.unicamp.br

John Lowengrub
School of Mathematics, Univ. of Minnesota
Minneapolis, MN and
Department of Mathematics, Univ. of North Carolina
Chapel Hill, NC
E-mail address: lowengrb@math.unc.edu

Helena J. Nussenzveig Lopes
Departamento de Matematica, IMECC-UNICAMP.
Caixa Postal 6065, Campinas, SP 13081-970, Brasil
E-mail address: hlopes@ime.unicamp.br

Yuxi Zheng
Departament of Mathematics, Indiana Univ.
Bloomington, IN 47405, USA
E-mail address: yzheng@indiana.edu