Existence of vortex sheets with reflection symmetry in two space dimensions

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Abstract

The main purpose of this work is to establish existence of a weak solution to the incompressible 2D Euler equations with initial vorticity consisting of a Radon measure with distinguished sign in H^{-1}, compactly supported in the closed right-half-plane, superimposed to its odd reflection in the left-half-plane. We make use of a new a priori estimate to control the interaction between positive and negative vorticity at the symmetry axis. We prove that a weak limit of a sequence of approximations obtained by either regularizing the initial data or by the vanishing viscosity method is a weak solution of the incompressible 2D Euler equations. We also establish the equivalence at the level of weak solutions between mirror symmetric flows in the full plane and flows in the half-plane. Finally, we extend our existence result to odd L^1 perturbations, without distinguished sign, of our original initial vorticity.

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Introduction

In 1991, J.-M. Delort proved existence of weak solutions to the incompressible 2D Euler equations with vortex sheet initial data, under the assumption that the initial vorticity be the sum of a Radon measure with distinguished sign and an arbitrary L^p function, $p > 1$, see [2]. This result was later proved in different ways, see [7, 17], extended to $p = 1$, see [3, 17, 19] and to the convergence to a weak solution of approximations generated by vanishing viscosity, see [14, 17]. Delort's result was also extended to convergence of approximations obtained by vortex methods, see [13, 18]. The problem of existence for vortex sheet initial data without the distinguished sign assumption has remained open. The distinguished sign hypothesis is needed because under it conservation of energy implies the avoidance of concentrations in vorticity (see [14, 17]). However, the distinguished sign assumption is a significant restriction on the scope of the available theory because important features of irregular flow seem to be connected with intricate intertwining of regions of positive and negative vorticity.

In the present work we will prove an existence result for flows with initial vorticity which is odd with respect to a straight line, and which is of a distinguished sign on each side of the line. Setting the initial data in this way creates a situation where vorticity of different signs is allowed to interact, but not to intertwine. This is the first instance where the global (in time) existence of vortex sheet evolution is rigorously established with vorticity density changing sign, albeit under very special circumstances. The difficulty is to show that vorticity concentration does not occur along the symmetry axis. The key new ingredient in our proof is an a priori estimate on the velocity at the symmetry axis, which allows us to control the interaction between positive and negative vorticity. Our result shows that the difficulty in the analytical treatment of vortex sheets without distinguished sign stems from the intertwining of the positive and negative vorticity and not merely from their interaction. One concrete situation where a vortex sheet initial data problem with reflection symmetry appears is the modelling of the wake due to an elliptically loaded airplane wing in the Trefftz plane, as done by

R. Krasny in [11].

The incompressible 2D Euler equations are covariant with respect to reflection symmetry. This means that a smooth solution of the equations on a half-plane may be extended by reflection to a smooth solution in the full plane. This observation is imbedded in the method of images, see [16]. Thus a natural approach to our existence problem would be to extend Delort's existence theory to flows in the half-plane and use the covariance of the 2D Euler equations with respect to the reflection symmetry in order to obtain a symmetric weak solution in the full plane. This approach does not work. The extension of Delort's theorem for bounded domains to half-plane flows is rather routine, without any new additional estimates compared with the case without boundaries. However, the weak solution thus obtained assumes the boundary condition (velocity tangent to the boundary) in a way that is not strong enough to guarantee that the flow constructed by reflection is a weak solution in the full plane.

For flows in a bounded domain, Delort's theorem guarantees the existence of a weak solution in an interior sense, i.e. the solution satisfies the weak formulation of the equations with respect to test functions that are compactly supported in the interior of the domain, and it satisfies the boundary condition in a trace sense. On one hand, the boundary condition is linear and hence it is well behaved with respect to weak convergence, which is necessary in Delort's treatment of boundaries. On the other hand, this separate treatment of the boundary condition leaves open the possibility of vorticity concentrating at the boundary. We introduce another notion of weak solution, which we call boundary-coupled weak solution, where we use test functions that vanish at the boundary, but not their derivatives. We prove that the existence of a reflection-symmetric weak solution to the full-plane problem is equivalent to the existence of a boundary-coupled weak solution to the half-plane problem. One corollary of this result, together with the existence of a weak solution to the full-plane problem with reflection symmetry, is that there exists a boundary-coupled weak solution to the incompressible 2D Euler equations in the half-plane with nonnegative measures as initial vorticity.

The problem of existence of vortex sheet evolution must be understood in the context of the pioneering work of R. DiPerna and A. Majda in [4, 5, 6]. The concern with concentrations in kinetic energy, which lies at the core of their analysis, has played no role in our work, in the same way that it was not present in Delort's work. Energy concentration is possible and, in fact, it is an outstanding open problem whether it does actually occur dynamically. Rather, our analysis revolves around the possibility of concentrations in vorticity. This was already the case with Delort's work as explained by S. Schochet in [17]; one key fact in Delort's result is that concentrations in vorticity are excluded by logarithmic decay of circulation in small circles. The main result in [17] is a concentration-cancellation theorem, in which approximate vorticities which concentrate at a single point $x = x(t)$, with $x(\cdot)$ a $C^{1/2}$ function of time, are shown to possess a weak limit which is a weak solution. In contrast, our theorem is not a concentration-cancellation result. Although vorticity concentrations are not ruled out, the possibility of their occurrence is bypassed. We show that, in a time-averaged sense, no concentrations occur. This is enough to pass to the limit in the nonlinear term of the weak vorticity formulation.

The remainder of this article is divided in six sections. In the first one, we construct symmetric smooth solutions. In the second one we obtain the new a priori estimate that makes our analysis possible. In the third section we construct inviscid approximate solution sequences and we apply the a priori estimates obtained in the second section to the weak vorticity formulation, proving our main existence result. In the fourth section we prove the validity of the method of images, obtaining existence of a boundary-coupled weak solution for the half-plane problem. In Section 5 we prove the convergence of (a subsequence of the) viscous approximations to a weak solution in the full plane and, in the last section, we extend our main result to odd L^1 perturbations, without sign restriction, of the initial vorticities previously considered and present our conclusions.

1 Symmetric smooth solutions

In order to construct symmetric weak solutions to the 2D Euler system we must first be able to construct symmetric approximate solution sequences. In this article, we consider two families of approximate solution sequences, obtained by mollifying the initial data and either exactly solving the Euler equations or exactly solving the Navier-Stokes equations. It is easy to mollify preserving the symmetry of the initial data, so that we need to show that the Navier-Stokes and the Euler systems preserve the symmetry. This can be accomplished by a standard energy estimate argument, which we will outline below.

The 2D incompressible Navier-Stokes equations in the full plane and in vorticity form are given by:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \varepsilon \Delta \omega, & \text{in } \mathbb{R}^2 \times (0, \infty) \\ \text{div } u = 0, \ \text{curl } u = \omega & \text{in } \mathbb{R}^2 \times [0, \infty) \\ \omega(x, 0) = \omega_0(x) & \text{in } \mathbb{R}^2 \\ |u|(x, t) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1}$$

The incompressible 2D Euler equations (in vorticity form) correspond to $\varepsilon = 0$. If ω_0 is smooth and compactly supported, then for any $\varepsilon \geq 0$ there exists a unique smooth solution to the problem above, see [15]. It is easy to see that the solution is also compactly supported in space for all time, if $\varepsilon = 0$. If $\varepsilon > 0$, $\omega(x, t)$ is exponentially decaying as $|x| \to \infty$. This is a consequence of the properties of the parametrix for the linearized parabolic problem, see [8].

The elliptic system div $u = 0$, curl $u = \omega$ in the full plane, together with the condition $|u| \to 0$ at ∞ can be inverted explicitly, so as to express velocity in terms of vorticity by:

$$u(x, t) = (K * \omega(\cdot, t))(x) \equiv \int_{\mathbb{R}^2} \frac{(x - y)^{\perp}}{2\pi |x - y|^2} \omega(y, t) dy, \tag{2}$$

with $(z_1, z_2)^{\perp} = (-z_2, z_1)$. This identity is called the Biot-Savart law. Using the Biot-Savart law, the system (1) becomes a scalar nonlocal equation with the vorticity ω as the single unknown.

Definition 1 *A Radon measure $\mu \in \mathcal{BM}(\mathbb{R}^2)$ is said to be NMS (nonnegative mirror-symmetric) if μ restricted to the right half-plane $\{x_1 > 0\}$ is nonnegative and μ is odd with respect to the $\{x_1 = 0\}$ axis, i.e. the duality pairing $\langle \mu, \varphi \rangle$ vanishes for any $\varphi \in C_0(\mathbb{R}^2)$ which is even with respect to the first variable.*

We will use the following notation for the mirror symmetry: $x = (x_1, x_2) \mapsto x^* \equiv (-x_1, x_2)$. We note that the restriction of a bounded Radon measure in the plane to any open subset Ω of the plane with smooth boundary gives rise to a bounded Radon measure on Ω. This is an immediate consequence of the characterization of $\mathcal{BM}(\Omega)$ as the dual of the space $C_0(\Omega)$, the closure of $C_c^{\infty}(\Omega)$ with respect to the sup-norm.

Proposition 1 *Let $\varepsilon \geq 0$ and let $\omega_0 \in C_c^\infty(\mathbb{R}^2)$ be NMS. If $\omega = \omega(x,t)$ is the unique solution of (1) with initial data ω_0, then $\omega(\cdot,t)$ is NMS for all $t \geq 0$.*

Proof: Define $\widetilde{\omega}(x,t) \equiv \omega(x,t) + \omega(x^*,t)$, so that $\widetilde{\omega}(x,0) \equiv 0$.

Then $\widetilde{\omega}$ satisfies the following equation:

$$\widetilde{\omega}_t(x,t) + u(x,t) \cdot \nabla\widetilde{\omega}(x,t) =$$

$$\varepsilon\Delta\widetilde{\omega}(x,t) + u(x,t) \cdot (\nabla\omega)(x^*,t) - u(x^*,t) \cdot (\nabla\omega)(x^*,t).$$

We rewrite the r.h.s. of this equation, using the explicit form of the Biot-Savart law (2) to obtain:

$$\widetilde{\omega}_t(x,t) + u(x,t) \cdot \nabla\widetilde{\omega}(x,t) = \varepsilon\Delta\widetilde{\omega}(x,t) + (K * \widetilde{\omega})(x,t) \cdot (\nabla\omega)(x^*,t).$$

Multiply this identity by $2\widetilde{\omega}$, integrate by parts over all $\mathbb{R}^2$ to obtain:

$$\frac{d}{dt}(\|\widetilde{\omega}\|_{L^2}^2) \leq \int_{\mathbb{R}^2} \widetilde{\omega}(K * \widetilde{\omega})(x,t) \cdot (\nabla\omega)(x^*,t)dx \leq$$

$$\|\widetilde{\omega}\|_{L^2}\|K * \widetilde{\omega}\|_{L^{2p/(p-2)}}\|\nabla\omega\|_{L^p},$$

where $2 < p < \infty$ is arbitrary. The exponent $2p/(p-2)$ is precisely the critical Sobolev exponent corresponding to $p' = p/(p-1)$. Therefore, since $1 < p' < 2$, one can use the Hardy-Littlewood-Sobolev inequality (see Proposition 1, [10]), to get:

$$\|K * \widetilde{\omega}\|_{L^{2p/(p-2)}} \leq C_p\|\widetilde{\omega}\|_{L^{p'}}.$$

In the case $\varepsilon = 0$, $\widetilde{\omega}(\cdot,t)$ has compact support, and hence the $L^{p'}$-norm above is dominated by the L^2-norm, so that, by Gronwall's inequality, $\widetilde{\omega}$ vanishes identically. This means that $\omega(\cdot,t)$ is odd with respect to the first variable, which implies, by the explicit form of the Biot-Savart law, that u is tangent to the $\{x_1 = 0\}$-axis, so that each half-plane is invariant under the flow. Hence $\omega(\cdot,t)$ is nonnegative in the right half plane, so that $\omega(\cdot,t)$ is NMS.

If $\varepsilon > 0$, one can use the exponential decay of $\widetilde{\omega}(\cdot,t)$ at infinity and Hölder's inequality to conclude that:

$$\|\widetilde{\omega}\|_{L^{p'}} \leq C(R^\beta\|\widetilde{\omega}\|_{L^2} + e^{-KR}),$$

where C, K are positive constants, $R > R_0$, for R_0 sufficiently big and $\beta = (2 - p')/p'$, so that $0 < \beta < 1$. One inputs this information into the differential inequality and uses Gronwall's inequality to conclude that $\|\widetilde{\omega}(\cdot, t)\|_{L^2}^2 \leq Ce^{-\widetilde{K}R}$, for arbitrary $R > R_0$, so that $\widetilde{\omega} \equiv 0$. Finally, we observe that $\omega = \omega(x,t)$ is now odd with respect to x_1, so that $\omega(0, x_2, t) \equiv 0$. Hence, ω satisfies a linear Fokker-Planck equation on the half plane, with a homogeneous Dirichlet boundary condition, and nonnegative initial condition. By the maximum principle, the nonnegativity is preserved for future times, so that ω is NMS in this case as well.

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2 A priori estimates in the inviscid case

We consider the problem of existence of a weak solution to the incompressible 2D Euler equations in the full plane with initial velocity $u_0 \in L^2(\mathbb{R}^2)$ such that the vorticity $\omega_0 \equiv \text{curl } u_0$ is NMS and compactly supported. We will look for solutions satisfying the weak vorticity formulation of the equations, as was introduced by S. Schochet in [17], which we make explicit in the definition below.

First we note that, if $\mu \in \mathcal{BM}(\mathbb{R}^2)$ then it is possible to make sense of $K * \mu$ as a distribution. To see this, observe that the map $\varphi \mapsto K * \varphi$ is a continuous linear operator from $C_c^\infty(\mathbb{R}^2)$ into $C_0(\mathbb{R}^2)$, where C_0 denotes the continuous functions vanishing at infinity. Hence, using that $K(z) = -K(-z)$ we may define $K * \mu$ as a distribution by the relation

$$\langle K * \mu, \varphi \rangle = -\langle \mu, K * \varphi \rangle.$$

Definition 2 *We say that $\omega \in L^\infty([0,\infty); \mathcal{BM}(\mathbb{R}^2))$ is a weak solution of the incompressible 2D Euler equations with initial data $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ if:*

(a) the velocity $u \equiv K * \omega \in L^\infty_{loc}([0,\infty); (L^2_{loc}(\mathbb{R}^2))^2)$, and

(b) for any test function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2)$ we have:

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi_t \omega(x,t) dx dt + \int_0^\infty \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\varphi(x,y,t) \omega(x,t) \omega(y,t) dy dx dt +$$

$$\int_{\mathbb{R}^2} \varphi(x,0) \omega_0(x) dx = 0,$$

where

$$H_\varphi(x,y,t) \equiv \frac{\nabla\varphi(x,t) - \nabla\varphi(y,t)}{4\pi|x-y|} \cdot \frac{(x-y)^\perp}{|x-y|}.$$

We will say that ω is a finite-energy weak solution if it is a weak solution with $u \in L^\infty_{loc}([0,\infty);(L^2(\mathbb{R}^2))^2)$.

Remark: The derivation of the relation above for smooth solutions is contained in [17]. We abuse notation several times in the definition, since ω is only a parametrized family of measures and not a function. It is the content of Lemma 2.1 in [17] that for a given weak solution in the sense above one can construct a classical weak solution for the velocity formulation of the Euler equations, since the velocity being in L^2_{loc} implies that the vorticity has no discrete part.

To prove our main existence result we will require a new a priori estimate for the smooth symmetric solutions of the inviscid problem, whose existence we examined in Proposition 1.

Lemma 1 *Let ω_0 be a smooth, compactly supported function which is NMS. Let $u = (u_1, u_2)$ and ω be the smooth solution of the inviscid vorticity equation (1, $\varepsilon = 0$) with initial data ω_0. If $\varphi = \varphi(x_1, x_2)$ is a smooth function with bounded derivatives up to second order in the closed right-half plane, then:*

$$\frac{d}{dt}\int_{x_1>0} \varphi(x)\omega(x,t)dx = -\frac{1}{2}\int_{-\infty}^{+\infty} (u_2)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2 +$$

$$\int_{x_1>0} (u_1^2 - u_2^2)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1} - \varphi_{x_2x_2})dx.$$

Proof: We recall that since ω has compact support and total mass zero it follows that $u(\cdot,t) \in L^2(\mathbb{R}^2)$, and that $|u|(\cdot,t) = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$. With this we have:

$$\frac{d}{dt}\int_0^\infty \int_{-\infty}^\infty \varphi(x)\omega(x,t)dx_2dx_1 = \tag{3}$$

$$\int_0^\infty \int_{-\infty}^\infty \varphi\omega_t dx_2dx_1 = -\int_0^\infty \int_{-\infty}^\infty \varphi \text{div } (u\omega)dx_2dx_1 =$$

$$\int_0^\infty \int_{-\infty}^\infty u\cdot\nabla\varphi\omega dx_2dx_1 - \lim_{R\to\infty}\int_{\{|x|=R, x_1\geq 0\}} \varphi\omega u\cdot\hat{n}dS =$$

$$\int_0^\infty \int_{-\infty}^\infty u \cdot \nabla\varphi \omega dx_2 dx_1,$$

since the boundary terms vanish due to $\omega(\cdot,t)$ having compact support together with the fact that $u \cdot \hat{n} = -u_1$ on the boundary $x_1 = 0$, which also vanishes. We re-write the term $u \cdot \nabla\varphi\omega$ as $-u \cdot \nabla\varphi \text{div } u^\perp$ and integrate by parts to obtain:

$$(3) = \int_0^\infty \int_{-\infty}^\infty \nabla(u\cdot\nabla\varphi)\cdot u^\perp dx_2 dx_1 - \lim_{R\to\infty}\int_{\{|x|=R, x_1\geq 0\}} (u\cdot\nabla\varphi)u^\perp \cdot \hat{n} dS =$$

$$\int_0^\infty \int_{-\infty}^\infty \nabla(u\cdot\nabla\varphi)\cdot u^\perp dx_2 dx_1 - \int_{\{x_1=0\}} (u\cdot\nabla\varphi)u^\perp \cdot (-1,0) dS =$$

$$\int_0^\infty \int_{-\infty}^\infty \nabla(u\cdot\nabla\varphi)\cdot u^\perp dx_2 dx_1 - \int_{-\infty}^\infty (u_2)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2 =$$

$$\int_0^\infty \int_{-\infty}^\infty \left(\nabla\left(\frac{|u|^2}{2}\right)\cdot\nabla^\perp\varphi + (u_1^2-u_2^2)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right)dx_2dx_1 +$$

$$-\int_{-\infty}^\infty (u_2)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2 = \lim_{R\to\infty}\int_{\{|x|=R, x_1\geq 0\}} \frac{|u|^2}{2}\nabla^\perp\varphi\cdot\hat{n}dS +$$

$$\int_0^\infty \int_{-\infty}^\infty (u_1^2-u_2^2)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})dx_2dx_1 +$$

$$-\int_{-\infty}^\infty (u_2)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2 =$$

$$\int_0^\infty \int_{-\infty}^\infty (u_1^2-u_2^2)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})dx_2dx_1 +$$

$$-\frac{1}{2}\int_{-\infty}^\infty (u_2)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2,$$

as we wished.

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Remark: This lemma is inspired on an a priori estimate derived by D. Chae and O. Y. Imanuvilov in [1] for 3D axisymmetric inviscid flow.

Lemma 1 used with $\varphi = \arctan(x_2)$, integrated in time yields, for any $0 < L < \infty$, the following a priori estimate on $u_2(0,x_2,t)$:

$$\int_0^T \int_{-L}^L |u_2(0,x_2,t)|^2 dx_2 dt \leq C, \tag{4}$$

with C depending on $\|\omega_0\|_{L^1}$, $\|u_0\|_{L^2}$, T and L. We will use this a priori estimate to show that the total mass of vorticity in a disk around a point on the interface $\{x_1 = 0\}$ decays as the disk shrinks to a point. More precisely, we have:

Lemma 2 *Let u, ω be the smooth solution of (1, $\varepsilon = 0$) in Lemma 1. Set $x^0 = (0, a) \in \mathbb{R}^2$. If $L > 0$ and $\delta > 0$ are such that $(a - \delta, a + \delta) \subset (-L, L)$ then*

$$\int_{B(x^0,\delta)} |\omega(y,t)|dy \leq C\sqrt{\delta}\left(\int_{-L}^{L} |u_2(0,x_2,t)|^2 dx_2\right)^{1/2},$$

where C is a universal constant.

Proof: Let us begin by noting that, by the Biot-Savart law (2), the tangential component of velocity on $\{x_1 = 0\}$, under the symmetry considered, is:

$$u_2(0,x_2,t) = -\int_0^\infty \int_{-\infty}^\infty \frac{y_1}{\pi((y_1)^2 + (x_2 - y_2)^2)}\omega(y,t)dy_2dy_1,$$

and hence is nonpositive. Therefore we have:

$$\int_{a-\delta}^{a+\delta} |u_2(0,x_2,t)|dx_2 = \int_{a-\delta}^{a+\delta}\int_0^\infty \int_{-\infty}^\infty \frac{y_1}{\pi((y_1)^2 + (x_2 - y_2)^2)}\omega(y,t)dy_2dy_1dx_2 =$$

$$\int_0^\infty \int_{-\infty}^\infty \omega(y,t)\frac{y_1}{\pi}\int_{a-\delta}^{a+\delta} \frac{1}{((y_1)^2 + (x_2 - y_2)^2)}dx_2dy_2dy_1 =$$

$$\int_0^\infty \int_{-\infty}^\infty \omega(y,t)\frac{g(y_1,y_2)}{\pi}dy,$$

where $g(y_1,y_2) \equiv \arctan\frac{a+\delta-y_2}{y_1} - \arctan\frac{a-\delta-y_2}{y_1} \geq 0$,

$$\geq \int_{\{|y-x^0|<\delta,\, y_1>0\}} \omega(y,t)\frac{g(y_1,y_2)}{\pi}dy,$$

since $\omega \geq 0$ in $\{y_1 > 0\}$.

Next consider, for any fixed $h > 0$, the function $f_h(z) \equiv \arctan(z + h) - \arctan(z - h)$, with $|z| < h$. It is easy to check that, in this range, $f_h(z) \geq \arctan(2h)$. Now if $y = (y_1, y_2) \in \{|y - x^0| < \delta,\ y_1 > 0\}$ then clearly $|(a - y_2)/y_1| < \frac{\delta}{y_1}$ and $\delta/y_1 > 1$. If we set $h = \delta/y_1$ and $z = (a - y_2)/y_1$ then we have $g(y_1,y_2) = f_h(z) \geq \arctan(2h) \geq \arctan(2)$. We have hence:

$$\int_{a-\delta}^{a+\delta} |u_2(0,x_2,t)|dx_2 \geq \frac{\arctan 2}{\pi}\int_{\{|y-x^0|<\delta,\, y_1>0\}} \omega(y,t)dy = C\int_{B(x^0,\delta)} |\omega(y,t)|dy. \tag{5}$$

On the other hand, using the Cauchy-Schwartz inequality we obtain

$$\int_{a-\delta}^{a+\delta} |u_2(0,x_2,t)|dx_2 \le C\sqrt{\delta}\left(\int_{a-\delta}^{a+\delta} |u_2(0,x_2,t)|^2 dx_2\right)^{1/2},$$

which together with (5) gives what we wish as long as $(a-\delta, a+\delta) \subset (-L, L)$. $\blacksquare$

Observe that the result in Lemma 2 concerns only the Biot-Savart law.

3 Convergence theorem

The objective of this section is to prove the existence of an NMS weak solution of the incompressible 2D Euler equations, in the sense of Definition 2, with NMS initial vorticity. We begin with the construction of an approximate solution sequence, by mollifying the initial vorticity and exactly solving the Euler equations.

Let $\rho = \rho(r) \in C_c^\infty([0,\infty))$ be nonnegative, monotonic decreasing inside its support, with total integral $1/2\pi$, and fix the Friedrichs mollifier $\phi(x) = \rho(|x|)$.

Let $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ be NMS. Consider the sequence of smooth, compactly supported functions $\{\omega_0^n\}$ obtained by convolving ω_0 with $\phi^n = \phi^n(x) = n^2\phi(nx)$. Let $u_0^n = K * \omega_0^n$ and let (u^n, ω^n) be the smooth solution of $(1, \varepsilon = 0)$ with initial data ω_0^n.

In [4] Section 1.C, DiPerna and Majda proved that the sequences $\{u^n\}$ and $\{\omega^n\}$ are an approximate solution sequence, in the sense of Definition 1.1 in [4]. Since ω_0 is NMS, the total mass of ω_0^n is automatically zero. As DiPerna and Majda observe, ω_0^n is uniformly bounded in $L^1(\mathbb{R}^2)$ and, since ω_0^n has total mass zero, u_0^n is uniformly bounded in $L^2(\mathbb{R}^2)$. Due to our choice of monotonic, circularly symmetric mollifiers, the ω_0^n are NMS. Indeed, the mirror symmetry is an obvious consequence of the circular symmetry of ϕ, whereas the sign condition follows from straightforward pointwise estimates on $\phi^n * \omega_0$, using both the symmetry and the monotonicity of ϕ. Therefore, by Proposition 1, the $\omega^n(\cdot,t)$ are NMS for all time.

It was shown in [4] that the following estimates hold for any $T > 0$:

(E1) $\sup_{0\le t\le T} \|\omega^n(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le C$.

(E2) $\sup_{0\le t\le T} \|u^n(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C$.

(E3) There exists $1 < M < \infty$ such that, $\{u^n\}$ is uniformly bounded in $Lip([0,T]; H^{-M}_{\mathrm{loc}}(\mathbb{R}^2))$.

We fix the approximate solution sequence $\{u^n\}$, $\{\omega^n\}$ throughout this section.

The key issue in the proof of existence, as formulated by Schochet in [17] is the possibility of concentrations in the sequence of vorticities. In order to control the occurrence of concentrations in the sequence $\{\omega^n\}$ we will put together the a priori estimate derived in the previous section with a version of the $\log^{-1/2}$ decay of circulation in small circles first observed by A. Majda in [14]. The a priori logarithmic decay in circulation turns out to be a *local* feature of flows with distinguished sign vorticity, which was pointed out by Schochet in [17]; this locality is crucial to our analysis.

Lemma 3 *For every $T > 0$ and $\mathcal{K} \subseteq \mathbb{R}^2$ compact there exists a constant $C > 0$ such that for every $0 < \delta < 1$:*

$$\int_0^T \left(\sup_{x \in \mathcal{K}} \int_{B(x,\delta)} |\omega^n(y,t)| dy \right) dt \leq C |\log \delta|^{-1/2}.$$

Proof: Let $R > 0$ be such that $\mathcal{K} \subseteq B(0,R)$. We recall an estimate due to S. Schochet (see Theorem 3.6, estimate (3.12) of [17]):

$$\left| \int_{\mathbb{R}^2} \eta_\delta(x-y) \omega^n(y,t) dy \right| \leq C \|u^n(\cdot,t)\|_{L^2} |\log \delta|^{-1/2},$$

where η_δ was defined as:

$$\eta_\delta(z) = \begin{cases} 1 \text{ if } |z| \leq \delta \\ \\ \dfrac{\log(\sqrt{\delta}/|z|)}{\log(1/\sqrt{\delta})} \text{ if } \delta \leq |z| \leq \sqrt{\delta} \\ \\ 0 \text{ if } |z| \geq \sqrt{\delta}. \end{cases}$$

It can be easily seen that, if $x = (b,c)$, with $|b| > \sqrt{\delta}$, then $B(x,\sqrt{\delta}) \subset \{x_1 > 0\} \cup \{x_1 < 0\}$ and therefore, since $\omega^n(\cdot,t)$ is of a distinguished sign in this disk, we get:

$$\int_{B(x,\delta)} |\omega^n(y,t)| dy \leq C |\log \delta|^{-1/2}, \tag{6}$$

by estimate (E2).

Let $a \in \mathbb{R}$, $\eta > 0$ and $L > 0$ be such that $(a-\eta, a+\eta) \subset (-L, L)$. It follows from Lemma 1, (4) and Lemma 2 that:

$$\int_0^T \int_{-L}^{L} |u_2^n(0,x_2,t)|^2 dx_2 dt \leq C(\|\omega_0^n\|_{L^1}, \|u_0^n\|_{L^2}, T, L) \equiv C, \tag{7}$$

$$\int_{B((0,a),\eta)} |\omega^n(y,t)|dy \leq C\sqrt{\eta}\left(\int_{-L}^{L} |u_2^n(0,x_2,t)|^2 dx_2\right)^{1/2}, \tag{8}$$

for any $T > 0$, (using (E1) and (E2) in (7)).

Now let $x = (b,c)$, with $|b| \leq \sqrt{\delta}$. Then, $B(x,\delta) \subset B((0,c), |b|+\delta)$, and $|c| \leq R$. Therefore, using (8) with $a = c$ and $\eta = |b| + \delta < 2\sqrt{\delta}$ (since $\delta < 1$) we have

$$\int_{B(x,\delta)} |\omega^n(y,t)|dy \leq \int_{B((0,c),|b|+\delta)} |\omega^n(y,t)|dy \leq$$

$$C\sqrt[4]{\delta}\left(\int_{-R-2}^{R+2} |u_2^n(0,x_2,t)|^2 dx_2\right)^{1/2},$$

since $(c - |b| - \delta, c + |b| + \delta) \subset (-R-2, R+2)$.

Therefore,

$$\sup_{x \in \mathcal{K}} \int_{B(x,\delta)} |\omega^n(y,t)|dy \leq \max\left\{C|\log\delta|^{-1/2}, C\sqrt[4]{\delta}\|u_2^n(0,\cdot,t)\|_{L^2(-R-2,R+2)}\right\},$$

$$\leq C(|\log\delta|^{-1/2})\left(\|u_2^n(0,\cdot,t)\|_{L^2(-R-2,R+2)} + 1\right).$$

We integrate in time over $[0,T]$ and use (7) to conclude the proof.

∎

We recognize the result above as describing the absence of concentrations in a time-averaged sense. If, instead of the integral in time we had the same estimate pointwise almost everywhere in time then our main existence result would follow from Lemma 3.7 in [17]. However, the integral estimate in Lemma 3 does not imply the a.e. in time pointwise boundedness of $\sup_{x\in\mathcal{K}} \int_{B(x,\delta)} |\omega^n(y,t)|dy$ or, in simpler terms, one cannot, from a sequence bounded in $L^1([0,T])$, extract a subsequence which is a.e. pointwise bounded in $[0,T]$, see [9] for a counterexample.

Let $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2)$ be a test function. Let

$$v \in L^\infty([0,\infty), \mathcal{BM}(\mathbb{R}^2)) \cap Lip([0,\infty); H^{-M-1}(\mathbb{R}^2))$$

be such that $K * v \in L^\infty_{\text{loc}}([0,\infty);(L^2_{\text{loc}}(\mathbb{R}^2))^2)$. We introduce the following notation for the terms which appear in the weak vorticity formulation in Definition 2:

$$\mathcal{W}(\varphi, v) \equiv \mathcal{W}_L(\varphi, v) + \mathcal{W}_{NL}(\varphi, v),$$

$$\mathcal{W}_L(\varphi, v) \equiv \int_0^\infty \int_{\mathbb{R}^2} \varphi_t v(x,t)\,dxdt + \int_{\mathbb{R}^2} \varphi(x,0)v(x,0)\,dx,$$

$$\mathcal{W}_{NL}(\varphi, v) \equiv \int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi(x,y,t)v(x,t)v(y,t)\,dydxdt,$$

where we are once again abusing notation since v is merely a parametrized family of measures.

Theorem 1 *There exists a finite-energy weak solution ω of the 2D incompressible Euler equations with initial vorticity ω_0.*

Proof: Let $M > 1$ be the exponent such that estimate (E3) holds for u^n. Then, for any $T > 0$, ω^n is uniformly bounded in $Lip([0,T]; H^{-M-1}_{\text{loc}}(\mathbb{R}^2))$, so that a simple application of the Aubin-Lions compactness lemma yields a subsequence, which we do not relabel, converging strongly in $C([0,T]; H^{-L}_{\text{loc}}(\mathbb{R}^2))$, for any $L < M+1$. Furthermore, there exists a subsequence of $\{\omega^n\}$ which converges weak-$*$ in $L^\infty_{\text{loc}}([0,\infty);\mathcal{BM}(\mathbb{R}^2))$ such that the corresponding u^n converge weak-$*$ in $L^\infty_{\text{loc}}([0,\infty);(L^2(\mathbb{R}^2))^2)$. Fix such a subsequence and let ω be the weak-$*$ limit of $\{\omega^n\}$. We will show that ω is a weak solution.

Of course, for each ω^n the weak vorticity formulation is an identity so that, for any test function $\varphi \in C^\infty_c((0,\infty)\times\mathbb{R}^2)$ we have $\mathcal{W}(\varphi,\omega^n) \equiv 0$. We will show that

$$\lim_{n\to\infty} \mathcal{W}(\varphi,\omega^n) = \mathcal{W}(\varphi,\omega),$$

and hence the conclusion will follow.

The linear term, $\mathcal{W}_L(\varphi,\omega^n)$, is weakly continuous under these (simultaneous) weak-$*$ limits and thus converges to the corresponding term $\mathcal{W}_L(\varphi,\omega)$.

Recall that the function H_φ appearing in the nonlinear term $\mathcal{W}_{NL}(\varphi,\omega^n)$ is globally bounded in $[0,\infty)\times\mathbb{R}^2\times\mathbb{R}^2$. We assume that the support of φ is contained in the cylinder $[0,T]\times B(0,R_0)$. Then we have that H_φ vanishes identically whenever $|x| > R_0$ and $|y| > R_0$. The function H_φ is not continuous (it is discontinuous on the diagonal $x = y$), so that the weak continuity of the nonlinear term is more delicate.

Fix $0 < \delta < 1$. We choose a test function $\zeta_\delta \in C_c^\infty(\mathbb{R}^2)$, such that $0 \leq \zeta_\delta(z) \leq 1$, $\zeta_\delta(z) \equiv 1$ if $|z| < \delta/2$ and $\zeta_\delta(z) \equiv 0$ if $|z| > \delta$. Then the nonlinear term can be rewritten as:

$$\int_0^\infty \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 - \zeta_\delta(x-y)) H_\varphi(x,y,t) \omega^n(x,t) \omega^n(y,t) dy dx dt +$$

$$\int_0^\infty \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \zeta_\delta(x-y) H_\varphi(x,y,t) \omega^n(x,t) \omega^n(y,t) dy dx dt$$

$$\equiv I_\delta(\omega^n) + J_\delta(\omega^n).$$

Since, for each $\delta > 0$, $(1 - \zeta_\delta(x-y)) H_\varphi(x,y,t)$ is a continuous function vanishing at infinity, we have that $I_\delta(\omega^n)$ converges to $I_\delta(\omega)$; see Lemma 3.2 in [17].

Let us now estimate $J_\delta(\omega^n)$. First we observe that

$$J_\delta(\omega^n) = \int_0^T \int_{\{|x| \leq R_0+1\}} \int_{B(x,\delta)} \zeta_\delta(x-y) H_\varphi(x,y,t) \omega^n(x,t) \omega^n(y,t) dy dx dt.$$

We use Lemma 3 with $\mathcal{K} = \overline{B(0, R_0+1)}$ to estimate $J_\delta(\omega^n)$:

$$|J_\delta(\omega^n)| \leq \|H_\varphi\|_{L^\infty} \int_0^T \left(\sup_{|x| \leq R_0+1} \int_{B(x,\delta)} |\omega^n(y,t)| dy \right) \left(\int_{|x| \leq R_0+1} |\omega^n(x,t)| dx \right) dt$$

$$\leq C |\log \delta|^{-1/2}.$$

Thus $\sup_n |J_\delta(\omega^n)| \to 0$ as $\delta \to 0$.

It remains to prove that $I_\delta(\omega)$ converges to $\mathcal{W}_{NL}(\varphi, \omega)$ as $\delta \to 0$. This is a repeat of the analysis done for $J_\delta(\omega^n)$, this time performed for $J_\delta(\omega)$, since we have:

$$\int_{y \in B(x,\delta)} d|\omega|(y,t) \leq \liminf_{n \to \infty} \int_{B(x,\delta)} |\omega^n(y,t)| dy \leq C |\log \delta|^{-1/2},$$

if $x = (b,c)$ with $|b| > \sqrt{\delta}$, and:

$$\int_{y \in B(x,\delta)} d|\omega|(y,t) \leq C \sqrt[4]{\delta} \liminf_{n \to \infty} \left(\int_{-R_0-3}^{R_0+3} |u_2^n(0, x_2, t)|^2 dx_2 \right)^{1/2} \equiv F(t) \sqrt[4]{\delta},$$

if $x = (b,c)$, $|x| \leq R_0 + 1$ and $|b| \leq \sqrt{\delta}$. Note that $F \geq 0$ and, by Fatou's Lemma, $F \in L^1([0,T])$. Hence F is finite a.e.-$[0,T]$. As before these two estimates yield that $J_\delta(\omega) \to 0$ as $\delta \to 0$. The pointwise evaluations in time above are valid since it can be shown that ω belongs to $C_{\rm loc}([0,\infty); w^* - \mathcal{BM}(\mathbb{R}^2))$ by a straightforward adaptation of the result in Appendix C of [12] to the weak-$*$ topology of $\mathcal{BM}(\mathbb{R}^2)$.

Finally, since $\{u^n\}$ converges weak-$*$ in $L^\infty_{\rm loc}([0,\infty); (L^2(\mathbb{R}^2))^2)$ to a limit u, since $u^n = K * \omega^n$, and since $\{\omega^n\}$ converges weak-$*$ in $L^\infty_{\rm loc}([0,\infty); \mathcal{BM}(\mathbb{R}^2))$ it follows that $u = K * \omega$ and that it belongs to $L^\infty_{\rm loc}([0,\infty); (L^2(\mathbb{R}^2))^2)$. Hence ω is a finite-energy weak solution, which completes the proof.

∎

Remark: This proof is an adaptation of the proof of Theorem 3.3 in [17]. Those portions of the argument above which repeat the reasoning presented in [17] have been merely outlined. The new feature in our proof is the use of time-averaged control of vorticity concentration to pass to the limit in the weak formulation.

4 The method of images

The purpose of this section is to formulate a version of the method of images that is valid at the level of weak solutions. To do that, we introduce a notion of weak solution on domains with boundary, stronger than the one used by Delort in [2], which we call boundary-coupled weak solution.

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth, simply connected domain with boundary $\partial\Omega$. We introduce the set of admissible test functions as:

$$\mathcal{A} \equiv \left\{ \varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}) \mid \varphi \equiv 0 \text{ on } \partial\Omega \right\}.$$

Let $G = G(x,y)$ be the Green's function for the Laplacian on Ω and $K_\Omega \equiv \nabla_x^\perp G$. We will use the notation $K_\Omega[f] = K_\Omega[f](x) \equiv \int_\Omega K_\Omega(x,y) f(y) dy$.

As before we first note that, if $\mu \in \mathcal{BM}(\Omega)$ then it is possible to make sense of $K_\Omega[\mu]$ as a distribution. This is true because the map $\varphi \mapsto K_\Omega^*[\varphi](\cdot) \equiv \int_\Omega K_\Omega(x,\cdot)\varphi(x)dx$ is a continuous linear operator from $C_c^\infty(\Omega)$ into $C_0(\Omega)$. To see this, observe that the vector field $\psi = K_\Omega^*[\varphi]$ is the unique solution of the problem:

$$\begin{cases} -\Delta\psi = \nabla^\perp\varphi \text{ in } \Omega, \\ \psi = 0 \text{ in } \partial\Omega, \end{cases}$$

which can be seen by an integration by parts and the symmetry of the Green's function.

Hence we may define $K_\Omega[\mu]$ as a distribution by the relation

$$\langle K_\Omega[\mu], \varphi\rangle = \langle \mu, K_\Omega * [\varphi]\rangle.$$

Let $\omega_0 \in \mathcal{BM}(\Omega)$ be such that $K_\Omega[\omega_0] \in (L^2(\Omega))^2$.

Definition 3 *The function $\omega \in L^\infty([0,\infty); \mathcal{BM}(\Omega))$ is called a boundary-coupled weak solution of the incompressible 2D Euler equations with initial data ω_0 if:*

(a) the velocity $u \equiv K_\Omega[\omega]$ belongs to $L^\infty_{loc}([0,\infty); (L^2(\Omega))^2)$, and

(b) for any test function $\varphi \in \mathcal{A}$ we have:

$$\int_0^\infty \int_\Omega \varphi_t \omega(x,t)dxdt + \int_0^\infty \int_\Omega \int_\Omega H^\Omega_\varphi(x,y,t)\omega(x,t)\omega(y,t)dydxdt +$$

$$\int_\Omega \varphi(x,0)\omega_0(x)dx = 0,$$

where

$$H^\Omega_\varphi(x,y,t) \equiv \frac{1}{2}(\nabla\varphi(x,t)\cdot K_\Omega(x,y) + \nabla\varphi(y,t)\cdot K_\Omega(y,x)).$$

Remark: The restriction to simply connected domains is important, because classical solutions of the Euler equations on a domain with nontrivial topology do not satisfy the definition above.

In the specific case of the half-plane $\mathbb{H} = \{x_1 > 0\}$,

$$K_\mathbb{H}(x,y) = \frac{(x-y)^\perp}{2\pi|x-y|^2} - \frac{(x-y^*)^\perp}{2\pi|x-y^*|^2},$$

where $x^* = (-x_1, x_2)$. Let $\omega_0 = \omega_0(x)$ be a Radon measure in $\mathcal{BM}(\mathbb{H})$ with bounded support such that $K_\mathbb{H}[\omega_0] \in (L^2(\mathbb{H}))^2$. Given a measure ω on $\mathbb{H}$ we denote by $\widetilde{\omega}$ its odd extension to the full plane with respect to the variable x_1.

Theorem 2 *The parametrized family of measures $\omega = \omega(x,t)$ is a boundary-coupled weak solution of the 2D Euler equations in the half-plane $\mathbb{H}$ with initial data ω_0 if and only if $\widetilde{\omega}$ is a finite-energy weak solution of the 2D Euler equations in the full plane with initial data $\widetilde{\omega_0}$.*

Proof: Let $\omega \in \mathcal{BM}(\mathbb{H})$. We begin with the following claim.

Claim: We have that $K * \tilde{\omega} = K_{\mathbb{H}}[\omega]$ in $\mathcal{D}'(\mathbb{H})$.

Proof of Claim: Let $\varphi \in C_c^\infty(\mathbb{H})$. Then:

$$\langle K * \tilde{\omega}, \varphi \rangle = -\langle \tilde{\omega}, K * \varphi \rangle = -2\langle \omega, (K * \varphi)_o \rangle_{\mathbb{H}}, \tag{9}$$

where the subscript o means the odd part of the function.

Let $x \in \mathbb{H}$. Then

$$(K * \varphi)_o(x) = \frac{K * \varphi(x) - K * \varphi(x^*)}{2} =$$

$$\frac{1}{4\pi} \int_{\mathbb{H}} \left(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x^*-y)^\perp}{|x^*-y|^2} \right) \varphi(y) dy =$$

$$-\frac{1}{2} \int_{\mathbb{H}} K_{\mathbb{H}}(y,x) \varphi(y) dy = -\frac{1}{2} K_{\mathbb{H}}^*[\varphi](x).$$

Hence, by virtue of (9) we have

$$\langle K * \tilde{\omega}, \varphi \rangle = -2\langle \omega, -\frac{1}{2} K_{\mathbb{H}}^*[\varphi] \rangle_{\mathbb{H}} = \langle K_{\mathbb{H}}[\omega], \varphi \rangle_{\mathbb{H}},$$

which proves the claim. ∎

Next note that $K * \tilde{\omega}$ is a mirror-symmetric vector field with respect to $x_1 = 0$ i.e. its first component is odd and its second component is even. This observation together with the Claim imply that if $\omega \in L^\infty_{\text{loc}}([0,\infty); \mathcal{BM}(\mathbb{H}))$ then we have:

$K_{\mathbb{H}}[\omega] \in L^\infty_{\text{loc}}([0,\infty); (L^2(\Omega))^2)$ if and only if $K * \tilde{\omega} \in L^\infty_{\text{loc}}([0,\infty); (L^2(\mathbb{R}^2))^2)$.

Next we prove the equivalence of the weak formulations in part (b) of Definitions 2 and 3.

Let $\omega = \omega(x) \in \mathcal{BM}(\mathbb{R}^2)$ be odd with respect to x_1, with no discrete part. Consider also $\varphi = \varphi(x) \in C_c^\infty(\mathbb{R}^2)$ be a test function, also odd with respect to x_1. Then,

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\varphi(x,y) \omega(x) \omega(y) dx dy =$$

$$\int_{\mathbb{H}}\int_{\mathbb{H}} \left(H_\varphi(x,y) - H_\varphi(x^*,y) - H_\varphi(x,y^*) + H_\varphi(x^*,y^*)\right)\omega(x)\omega(y)dxdy.$$

It is an intricate but straightforward algebraic manipulation to check that, for all $x, y \in \mathbb{H}$:

$$H_\varphi(x,y) - H_\varphi(x^*,y) - H_\varphi(x,y^*) + H_\varphi(x^*,y^*) = 2H_\varphi^{\mathbb{H}}(x,y). \tag{10}$$

Let us now assume that $\omega = \omega(x,t)$ is a boundary-coupled weak solution in the half plane. Let $\varphi = \varphi(x,t)$ be a test function in $C_c^\infty([0,\infty)\times\mathbb{R}^2)$. Write $\varphi(x,t) = \varphi_o(x,t) + \varphi_e(x,t)$, where φ_o is odd and φ_e is even with respect to x_1. Let $\tilde{\omega}$ be the odd extension of ω. We will show that $\tilde{\omega}$ is a weak solution. Due to the symmetries, one has:

$$\mathcal{W}_L(\varphi,\tilde{\omega}) = \mathcal{W}_L(\varphi_o,\tilde{\omega}) = 2\int_0^\infty \int_{\mathbb{H}} (\varphi_o)_t \omega(x,t)dxdt + 2\int_{\mathbb{H}} (\varphi_o)(x,0)\omega_0(x)dx,$$

For the nonlinear part,

$$\mathcal{W}_{NL}(\varphi,\tilde{\omega}) = \mathcal{W}_{NL}(\varphi_o,\tilde{\omega}) + \mathcal{W}_{NL}(\varphi_e,\tilde{\omega}).$$

Direct calculation shows that:

$$\mathcal{W}_{NL}(\varphi_e,\tilde{\omega}) = \int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_{\varphi_e}(x,y,t)\tilde{\omega}(x,t)\tilde{\omega}(y,t)dxdydt =$$

$$\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_{\varphi_e}(x^*,y^*,t)\tilde{\omega}(x,t)\tilde{\omega}(y,t)dxdydt = 0,$$

since an easy calculation verifies that:

$$H_{\varphi_e}(x^*,y^*,t) = -H_{\varphi_e}(x,y,t).$$

From (10), we know that:

$$\mathcal{W}_{NL}(\varphi_o,\tilde{\omega}) = 2\int_0^\infty \int_{\mathbb{H}}\int_{\mathbb{H}} H_{\varphi_o}^{\mathbb{H}}(x,y,t)\omega(x,t)\omega(y,t)dydxdt.$$

Since ω is a boundary-coupled weak solution in the half plane, and the restriction of φ_o to $\mathbb{H}$ is an admissible test function in $\mathcal{A}$, we have shown that $\mathcal{W}_L(\varphi,\tilde{\omega}) + \mathcal{W}_{NL}(\varphi,\tilde{\omega}) = 0$, that is, $\tilde{\omega}$ is a weak solution in the full plane.

Conversely, we assume that $\tilde{\omega}$ is an odd weak solution in the full plane. We wish to show that the restriction ω of $\tilde{\omega}$ to $\mathbb{H}$ is a boundary-coupled weak solution in $\mathbb{H}$. Let $\varphi \in \mathcal{A}$ be an admissible test function. Let $\tilde{\varphi}$ be the odd extension of φ. It is easy to see that $\widetilde{\varphi} \in C_c^\infty([0,\infty);W_c^{2,\infty}(\mathbb{R}^2))$. We mollify $\widetilde{\varphi}$, obtaining a sequence of odd test functions $\widetilde{\varphi}^n$ which, for each fixed time, converges in $W^{1,\infty}(\mathbb{R}^2)$ to $\widetilde{\varphi}$ and is bounded in $W^{2,\infty}(\mathbb{R}^2)$.

For each n, we have that $\mathcal{W}(\widetilde{\varphi}^n,\widetilde{\omega}) = 0$. Since:

$$\mathcal{W}(\widetilde{\varphi},\widetilde{\omega}) = 2\int_0^\infty \int_{\mathbb{H}} \varphi_t \omega(x,t)dxdt + 2\int_{\mathbb{H}} \varphi(x,0)\omega_0(x)dx +$$

$$2\int_0^\infty \int_{\mathbb{H}} \int_{\mathbb{H}} H_\varphi^{\mathbb{H}}(x,y,t)\omega(x,t)\omega(y,t)dydxdt,$$

it is enough to show that:

$$\lim_{n\to\infty} \mathcal{W}(\widetilde{\varphi}^n,\widetilde{\omega}) = \mathcal{W}(\widetilde{\varphi},\widetilde{\omega}).$$

For the linear part, the uniform convergence of $\widetilde{\varphi}_t^n$ and $\widetilde{\varphi}_0^n$ to $\widetilde{\varphi}_t$ and $\widetilde{\varphi}_0$ respectively is enough to conclude that $\mathcal{W}_L(\widetilde{\varphi}^n,\widetilde{\omega}) \to \mathcal{W}_L(\widetilde{\varphi},\widetilde{\omega})$ as $n \to \infty$. For the nonlinear part, we separate a neighborhood of the diagonal as in Theorem 1, and use the uniform convergence of the $\nabla\widetilde{\varphi}^n$ to pass to the limit far from the diagonal. Near the diagonal we use the boundedness of $\widetilde{\varphi}^n$ in $W^{2,\infty}$ together with the fact that $\widetilde{\omega}$ has no discrete part. This concludes the proof.

∎

A direct consequence of Theorem 1 and Theorem 2 is the existence of a boundary-coupled weak solution for the vortex sheet initial data problem in the half plane with nonnegative initial vorticity. This is a stronger existence result than the natural extension of Delort's result to the half plane since our weak solution does not allow vorticity concentration at the boundary.

The result in Theorem 2 can be used independently of the existence theory developed in the first three sections of this work. We illustrate this with the corollary below, which applies to L^1 vorticities without sign restriction.

Corollary 1 *Let $\omega_0 \in L^1(\mathbb{H})$ have bounded support and be such that $K_{\mathbb{H}}[\omega_0] \in (L^2(\mathbb{H}))^2$. Then there exists a boundary-coupled weak solution of the incompressible 2D Euler equations in the half-plane with ω_0 as initial vorticity.*

Proof: We consider $\widetilde{\omega_0}$ the odd extension of ω_0. From the hypothesis and the Claim in the proof of Theorem 2 we have that $K * \widetilde{\omega_0} \in (L^2(\mathbb{R}^2))^2$. We consider the Friedrichs mollifier ϕ introduced in Section 3 and let $\widetilde{\omega_0}^n \equiv \phi^n * \widetilde{\omega_0}$. Consider the exact smooth solutions $\widetilde{u}^n$, $\widetilde{\omega}^n$ of (1, $\varepsilon = 0$). It follows from the argument in Proposition 1 that $\widetilde{\omega}^n$ is odd for all time. From the available existence theory, see [3, 17, 19], it follows that there exists a subsequence to the $\widetilde{\omega}^n$ converging weakly to a finite energy weak solution $\widetilde{\omega}$ of the incompressible 2D Euler equations. Furthermore, it is easy to see that $\widetilde{\omega}$ is odd a.e. in time. Hence, by Theorem 2, the restriction of $\widetilde{\omega}$ to $\mathbb{H}$ is a boundary-coupled weak solution in $\mathbb{H}$. ∎

5 Viscous approximations

The objective of this section is to derive a version of Theorem 1 for approximations obtained by the vanishing viscosity method. As in Section 3, we begin with vortex sheet initial data $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ which is NMS and we consider the same sequence of smooth, compactly supported mollifications $\{\omega_0^n\}$, and $u_0^n \equiv K * \omega_0^n$. As before, ω_0^n is still NMS. We fix a sequence ε_n with $\varepsilon_n \to 0$ as $n \to \infty$ and we let (u^n, ω^n) be the solution of $(1, \varepsilon = \varepsilon_n)$. By Proposition 1 we have that $\omega^n(\cdot, t)$ is NMS for all time. In [4], Section 2.A, DiPerna and Majda proved estimates (E1), (E2) and (E3) for this sequence of approximations.

Theorem 3 *There exists a subsequence of $\{\omega^n\}$ converging weakly to an NMS weak solution of the 2D incompressible Euler equations with ω_0 as the initial vorticity.*

Proof: We begin by adapting the proof of Lemma 1. Fix a function φ, smooth and bounded on the closed half plane $\overline{\mathbb{H}}$ up to its second order derivatives. In addition, we will assume that $\varphi \geq 0$. We multiply equation $(1, \varepsilon = \varepsilon_n)$ by φ and perform the same integration by parts as in Lemma 1, tracking the viscous term to obtain:

$$\frac{d}{dt}\int_{x_1>0} \varphi(x)\omega^n(x,t)dx = -\frac{1}{2}\int_{-\infty}^{+\infty} (u_2^n)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2 +$$

$$\int_{x_1>0} ((u_1^n)^2 - (u_2^n)^2)\varphi_{x_1x_2} - u_1^n u_2^n(\varphi_{x_1x_1} - \varphi_{x_2x_2})dx +$$

$$\varepsilon_n \int_{x_1>0} \omega^n \Delta\varphi dx - \varepsilon_n \int_{-\infty}^{+\infty} \varphi(0,x_2)\omega_{x_1}^n(0,x_2,t)dx_2,$$

where one has used that ω^n vanishes exponentially fast at infinity, together with its derivatives and that it vanishes on the symmetry axis $\{x_1 = 0\}$ as well. Next, since ω^n is odd with respect to x_1, nonnegative for $x_1 > 0$, we have that $\omega^n_{x_1}(0, x_2, t) \geq 0$. It follows from this and the assumption that $\varphi \geq 0$ that the following inequality holds:

$$\frac{d}{dt}\int_{x_1>0} \varphi(x)\omega^n(x,t)dx \leq -\frac{1}{2}\int_{-\infty}^{+\infty}(u_2^n)^2(0,x_2,t)\varphi_{x_2}(0,x_2)dx_2+$$

$$\int_{x_1>0}((u_1^n)^2-(u_2^n)^2)\varphi_{x_1x_2}-u_1^n u_2^n(\varphi_{x_1x_1}-\varphi_{x_2x_2})dx+\varepsilon_n\int_{x_1>0}\omega^n\Delta\varphi dx.$$

This inequality, used with $\varphi(x) = \arctan(x_2)+\pi/2$, yields, for any $0 < L, T < \infty$, the uniform estimate (7) for the sequence $\{u^n(0, x_2, t)\}$ after integration in time.

Note that the proof of Lemma 2 depends only on the Biot-Savart law, which is exactly valid for the Navier-Stokes approximations. Hence, one can prove the uniform estimate (8) for ω^n as well. Furthermore, as before, it follows from these uniform estimates (7) and (8), plus (E1), (E2) and (E3) that the statement of Lemma 3 applies to the sequence $\{\omega^n\}$.

Finally, one can check easily that the only difference in the argument of Theorem 1 and the present situation is that, instead of $W(\varphi, \omega^n)$ being equal to zero, one has:

$$W(\varphi,\omega^n)+\varepsilon_n\int_0^{\infty}\int_{\mathbb{R}^2}\omega^n\Delta\varphi dxdt = 0,$$

which means the addition of a linear term, presenting no new difficulties in the passage to the weak limit. Thus, the proof of the Theorem 3 is considered completed.

∎

Since the Navier-Stokes equations are also covariant with respect to the reflection symmetry, it is natural to ask whether the result above can be reduced by symmetry in the spirit of Section 4, showing that a boundary-coupled weak solution to the Euler equations in the half plane can be obtained as a vanishing viscosity limit of solutions of the Navier-Stokes equations on the half plane. The answer is decidedly negative, at least with the techniques we have explored in this work, and the reason is that the method of images does not work, even for smooth solutions of the Navier-Stokes equations. It is easy to see that a smooth solution of the Navier-Stokes equations on the half plane with no-slip boundary condition gives rise to a solution of the Navier-Stokes equation on the full plane by odd extension, but the converse is not true. This is the case because the tangential component of velocity does not vanish identically on the symmetry axis for NMS solutions of the Navier-Stokes equation, which would be necessary in order for the symmetry-reduced solution to satisfy the no-slip boundary condition. Furthermore, a smooth divergence-free velocity field satisfying the no-slip boundary condition always has vanishing total circulation, so that no NMS solutions in the full plane can be obtained by reflection.

6 Extensions and conclusions

Let us consider an odd L^1 perturbation of the initial data of Theorem 1, without the sign restriction. Our existence proof can be adapted to this situation. We outline the adaptation below.

Theorem 4 *Let $\omega_0 \equiv \omega_0' + \omega_0''$, with $\omega_0' \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ NMS and $\omega_0'' \in L^1_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$, odd with respect to x_1. Then there exists a finite-energy weak solution of the 2D Euler equations with ω_0 as initial data.*

Proof: We begin by considering the approximation ω^n, constructed in the same way as in Theorem 1, with initial data:

$$\omega_0^n = \phi^n * \omega_0 = \phi^n * \omega_0' + \phi^n * \omega_0'' \equiv \omega_0'^n + \omega_0''^n.$$

Corresponding to this decomposition of the initial vorticity, we introduce ω'^n and ω''^n as the solutions to the transport equation $v_t + u^n \cdot \nabla v = 0$ with initial data $\omega_0'^n$ and $\omega_0''^n$ respectively.

First note that Lemma 1 remains valid, as the proof did not use the distinguished sign assumption on the half-plane. Hence, $u_2^n(0,\cdot,\cdot)$ is still bounded in $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$. We will use the following notation: $u'^n = K * \omega'^n$ and $u''^n = K * \omega''^n$.

Next we note that the precise decay rate of the time-averaged maximal vorticity function in Lemma 3 is not necessary for the convergence proof. It suffices to show that it decays to zero uniformly in n as δ tends to zero. Thus, to prove our theorem, it suffices to show the following claim:

Claim: Let L be a fixed positive number, and $a \in (-L, L)$. For any given positive ε, there exists $\delta_0 > 0$, independent of n, such that if $\delta \leq \delta_0$, then

$$\sup_{0 \leq t \leq T} \int_{a-\delta}^{a+\delta} |u_2''^n(0, x_2, t)| dx_2 \leq \varepsilon.$$

Here the constant δ_0 might depend on L and T.

In Lemma 2 we saw that one could estimate the mass of vorticity near the symmetry axis by the mass of tangential velocity at the symmetry axis. The proof of this claim is based on a converse of this statement, i.e. that velocity at the symmetry axis may be estimated by vorticity nearby.

Assuming this claim, one can conclude that the same statement in the claim holds true for

$$\int_0^T \int_{a-\delta}^{a+\delta} |u_2'^n(0, x_2, t)| dx_2,$$

since $u_2^n(0, \cdot, \cdot)$ is uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$. On the other hand, the estimate (5) shows that

$$\int_{B(x^0,\delta)} |\omega'^n(y,t)| dy \leq C \int_{a-\delta}^{a+\delta} |u_2'^n(0, x_2, t)| dx_2,$$

where $x_0 = (0, a)$. It follows from this and the similar argument for Lemma 3 that for any fixed $T > 0$ and $\mathcal{K} \subseteq \mathbb{R}^2$,

$$\int_0^T \left(\sup_{n, x \in \mathcal{K}} \int_{B(x,\delta)} |\omega'^n(y,t)| dy \right) dt$$

goes to zero as δ approaches zero. Since ω''^n is transported by an area-preserving flow and since the initial data $\{\omega_0''^n\}$ is uniformly integrable, then the same conclusion on time-averaged decay of the maximal vorticity function of ω'^n holds true for ω''^n. Consequently, the same conclusion applies to ω^n, and the proof of theorem can be completed as before.

Thus, it remains to prove the claim. To this end, we first note that it follows from the transport equation for ω''^n and and the uniform integrability of the initial data that

$$\sup_{0 \leq t \leq T} \|\omega''^n(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq C_1 \tag{11}$$

and

$$\sup_{0 \leq t \leq T} \int_\Omega |\omega''^n(y,t)| dy \leq C_2(\Omega), \tag{12}$$

where C_1 and $C_2(\Omega)$ are positive constants independent of n, and $C_2(\Omega) \to 0$ as $|\Omega|$, the Lebesgue measure of Ω, goes to zero.

Next, as in the proof of Lemma 2, one can obtain from the Biot-Savart law that

$$\int_{a-\delta}^{a+\delta} |u_2''^n(0, x_2, t)| dx_2 \le \int_\Omega |\omega''^n(y, t)| \frac{g(y_1, y_2)}{\pi} dy, \tag{13}$$

where $\Omega = \{(y_1, y_2), y_1 \ge 0\}$, and $g(y_1, y_2)$ is the same function as given in the proof of Lemma 2. We now decompose the integral on the right hand side of the above inequality as

$$\int_\Omega |\omega''^n(y, t)| g(y_1, y_2) dy = \sum_{i=1}^{4} \int_{\Omega_i} |\omega''^n(y, t)| g(y_1, y_2) dy,$$

where

$$\Omega_1 = \{(y_1, y_2),\ y_1 \ge 1\},$$

$$\begin{aligned} \Omega_2 &= \{(y_1, y_2),\ 0 \le y_1 \le 1, |y_2| \ge M\}, \\ \Omega_3 &= \{(y_1, y_2),\ 0 \le y_1 \le h, |y_2| \le M\}, \\ \Omega_4 &= \{(y_1, y_2),\ h \le y_1 \le 1, |y_2| \le M\}, \end{aligned}$$

with positive constants $h(\le 1)$ and $M(\ge L)$ to be chosen later.

Now we estimate each integral above separately. First, it follows from the mean value theorem that

$$g(y_1, y_2) \le \frac{2\delta}{y_1} \le 2\delta,$$

for $y \in \Omega_1$. Thus, for $\delta \le \delta_1 = \frac{\varepsilon}{8TC_1}$, one has that

$$\sup_{0\le t\le T} \int_{\Omega_1} |\omega''^n(y, t)| g(y_1, y_2) dy \le 2\delta C_1 \le \frac{\varepsilon}{4T}. \tag{14}$$

Second, since

$$\sup_{0\le t\le T} \int_{\Omega_2} |\omega''^n(y, t)| g(y_1, y_2) dy \le C_1 \sup_{y\in\Omega_2} g(y_1, y_2),$$

and $\sup_{\Omega_2} g(y_1, y_2) \to 0$ as $M \to +\infty$, we can fix a positive M so that for $\delta \le 1$, it holds that

$$\sup_{0\le t\le T} \int_{\Omega_2} |\omega''^n(y, t)| g(y_1, y_2) dy \le \frac{\varepsilon}{4T}. \tag{15}$$

Next, for this fixed M, we have that $|\Omega_3| = hM \to 0$ as $h \to 0$, thus it follows from (12) that

$$\sup_{0\le t\le T} \int_{\Omega_3} |\omega''^n(y,t)| g(y_1,y_2) dy \le \pi C_2(\Omega_3),$$

which goes to zero as $h \to 0^+$. Consequently, for $\delta \le 1$, one can choose a fixed h so that

$$\sup_{0\le t\le T} \int_{\Omega_3} |\omega''^n(y,t)| g(y_1,y_2) dy \le \frac{\varepsilon}{4T}. \tag{16}$$

Finally, with M and h so chosen and fixed, we set $\delta_2 = \frac{h\varepsilon}{8TC_1}$. Then for $\delta \le \delta_2$, one has that

$$\sup_{0\le t\le T} \int_{\Omega_4} |\omega''^n(y,t)| g(y_1,y_2) dy \le \frac{2\delta}{h} \int_{\Omega_4} |\omega''^n(y,t)| dy \le \frac{2\delta}{h} C_1 \le \frac{\varepsilon}{4T}. \tag{17}$$

Collecting all the estimates (14)-(17), we conclude that for any given ε positive, taking $\delta_0 = min\{\delta_1, \delta_2, 1\}$, one has that for $\varepsilon \le \varepsilon_0$,

$$\sup_{0\le t\le T} \int_{\Omega} |\omega''^n(y,t)| g(y_1,y_2) dy \le \frac{\varepsilon}{T}.$$

This, together with (13), yields the claim as can be checked trivially. Consequently, the proof of the theorem is completed.

∎

It follows from the result above together with Theorem 2 that there exists a boundary-coupled weak solution of the 2D incompressible Euler equations with an initial vorticity consisting of an L^1 perturbation of a measure with distinguished sign.

Finally, let us add a pair of concluding remarks concerning this work. We note that it is possible to extend Theorem 1 from NMS initial vorticities to initial vorticities with more complicated symmetry. One class of examples is the set of initial vorticities which are single-signed on a wedge with tip at the origin, with angle π/n, extended to an initial vorticity on the full plane which is simultaneously odd with respect to n straight lines intersecting at the origin, arranged in an n-fold symmetric pattern.

The results presented raise two natural questions which are currently under investigation by the authors. The first is whether one can prove existence of boundary-coupled weak solutions for the problem of inviscid flow on a general bounded domain. This would be a stronger version of Delort's existence result for the bounded domain. The other question is the problem of convergence of other approximation schemes, such as the vortex blob method, for vortex sheet initial data flows with reflection symmetry. Such a result would give rigorous justification for the computations of R. Krasny in [11] for the elliptically loaded wing.

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