COMMENTS UPON THE MASS OSCILLATION FORMULAS

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Standard formulas for mass oscillations are based upon the approximation, $t \approx L$, and the hypotheses that neutrinos have been produced with a definite momentum, p, or, alternatively, with definite energy, E. This represents an inconsistent scenario and gives an unjustified factor of two in mass oscillation formulas. Such an ambiguity has been a matter of speculations and mistakes in discussing flavour oscillations. We present a series of results and show how the problem of the factor two in the oscillation length is not a consequence of gedanken experiments, i.e. oscillations in time. The common velocity scenario yields the maximum simplicity and probably the right answer.

I. INTRODUCTION

One of the most popular fields of research in particle physics phenomenology of the last decades has been, and still is, that of neutrino oscillations [1]. Publications in this field have accompanied an ever increasing and stimulating series of experiments involving either solar, atmospheric or laboratory neutrinos [2].

The vast majority of the theoretical studies consider the possibility of massive neutrinos distinct from the flavour eigenstates created in the various production processes [3–5]. Neutrinos are not the only example of such a phenomenon. The first examples of flavour oscillations observed were in the the kaon system where the strong interaction is involved in the particle creation [6]. We shall continue to refer in this work to neutrinos, but the considerations are quite general.

What we shall call the "factor two problem" in the neutrino oscillation formulas has been already observed and discussed in previous papers [8,9], but the situation is still surprisingly confused and probably still subject to argument. We would like to close the question with this paper, but more realistically, we shall simply contribute to the general debate.

II. NEUTRINOS MIXING

To focalize the contents of this paper we begin by claiming that standard oscillation formulas, for mixing between mass eigenstates, are based upon the approximation, $t \approx L$, and the assumptions of definite momentum or definite energy for the neutrinos created.

To explain and understand the common mistakes in oscillation calculations, let us briefly recall the standard approach. The most important aspect of neutrino oscillations can be understood by studying the explicit solution for a system with only two types of neutrinos. For this two flavour problem, the flavour eigenstates, $|\nu\rangle$ e $|\tilde{\nu}\rangle$, are represented by a coherent linear superposition of mass eigenstates, $|\nu_1\rangle$ e $|\nu_2\rangle$,

$$\begin{aligned} |\nu\rangle &= \quad \cos\theta\, |\nu_1\rangle + \sin\theta\, |\nu_2\rangle \;, \\ |\tilde{\nu}\rangle &= -\sin\theta\, |\nu_1\rangle + \cos\theta\, |\nu_2\rangle \;. \end{aligned} \tag{1}$$

The time evolution of $|\nu\rangle$ is determined by solving the Schrödinger equation for the $|\nu_{1,2}\rangle$ component of $|\nu\rangle$ in the rest frame of that component

$$|\nu_n(\tau_n)\rangle = e^{-iM_n\tau_n}\, |\nu_n\rangle \qquad n=1,2 \;, \tag{2}$$

where M_n is the mass of $|\nu_n\rangle$ and τ_n is the time in the mass eigenstate frame. In terms of the time, t, and the position, L, in the laboratory, or any other, frame, the Lorentz-invariant phase factor in Eq. (2) must be rewritten as:

$$e^{-iM_n\tau_n} = e^{-i(E_n t - p_n L)} \;, \tag{3}$$

with E_n and p_n energy and momentum of mass eigenstates in the laboratory frame. In the standard approach the above equation is followed by this statement: In practice, our neutrino will be extremely relativistic, so we will be interested in evaluating the phase factor of Eq. (3) where $t \approx L$, where it becomes

$$e^{-i(E_n - p_n)L} \ . \tag{4}$$

This result is incorrect as we shall show below.

For example, in the latest presentation contained in the Review of Particle Physics [7], the probability that the wrong flavour will appear, is given by:

$$P(\nu \to \nu') \approx \sin^2 2\theta \ \sin^2\left(\frac{L}{4E}\,\Delta M^2\right) \approx \sin^2 2\theta \ \sin^2\left(1.27\,\frac{L\,[\mathrm{Km}]}{E\,[\mathrm{Gev}]}\,\Delta M^2\,[\mathrm{eV}^2]\right) \ , \tag{5}$$

where

$$\Delta M^2 \equiv M_1^2 - M_2^2 \ ,$$

is obtained calculating the phase factor for each mass eigenstate traveling in the x direction, $e^{-i(E_n t - p_n x)}$, with the approximation $t \approx L$ and the assumptions that $|\nu\rangle$ has been produced with a definite momentum, p,

$$E_n = \sqrt{p^2 + M_n^2} \approx p + \frac{M_n^2}{2p} \ ,$$

or, alternatively, with a definite energy, E,

$$p_n = \sqrt{E^2 - M_n^2} \approx E - \frac{M_n^2}{2E} \ .$$

The phase factor of Eq. (3) reads

$$e^{-i\frac{M_n^2}{2p}L} \quad \text{or} \quad e^{-i\frac{M_n^2}{2E}L} \ . \tag{6}$$

"Since highly relativistic neutrinos have $E \approx p$, the phase factors in Eq. (6) are approximately equal. Thus, it doesn't matter whether $|\nu\rangle$ is created with definite momentum or definite energy." [7].

Now, our point is that $t = L$ implies for consistency

$$\frac{p_1}{E_1} = \frac{p_2}{E_2} = 1 \ ,$$

which, if simultaneously applied, eliminates the phase-factor completely. Null phase factors, as in all cases of equal phase factors for each mass eigenstate, precludes any oscillation phenomena. Nor is such an approximation justified within a more realistic wave-packet presentation. Returning to the simplified plane wave discussion, one should simply write,

$$e^{-i(Et - pL)} = e^{-i(\frac{E}{v} - p)L} = e^{-i(E^2 - p^2)\frac{L}{P}} = e^{-iM^2\frac{L}{p}} \ ,$$

which differs from Eq. (6) by a factor of two in the argument. This simply doubles the coefficient of ΔM^2 in the standard oscillation formulas.

The above result has already been noted by Lipkin [8], who however observes this ambiguity for the case of equal 3-momentum of the neutrino mass eigenstates, the "non-experiments" as he calls them, but not for his chosen equal energy scenario. Lipkin's preference for equal energy has not convinced the majority of authors on this subject. The same fate has thus been reserved for his observations upon the factor two ambiguity. We believe that only experiment can determine if in a given situation the neutrinos are produced with the same momentum or energy or neither. For this reason we wish to present below the differences in the various assumptions which are particularly significant for non-relativistic velocities, admittedly not very practical for the neutrino. In any case we emphasize that the fore mentioned factor two appears not only in the scenario of common momentum but also for the equal energy assumption or "real experiments" as Lipkin [9] calls them.

In realistic situations the flavour neutrino is created in a wave packet at time $t = 0$ and thus over an extended region. We simplify our discussion by ignoring, where possible, this localization but we must note that it is essential to give an approximate significance to L, the distance from source to measuring apparatus, or t, the time of travel. Indeed, if for each microscopic region we assign a common 4-momentum plane wave factor then it is mathematically impossible that at time $t = 0$ the neutrino is created as a flavour state everywhere, since the plane-wave factor varies with x.

III. TIME OR SPACE OSCILLATIONS?

Assume that the state $|\nu\rangle$ is created at $t = 0$ in $x \approx 0$. Introduce the Lorentz invariant plane wave factor and apply them to the mass eigenstates at a later time t and for position x. Since the neutrino is created over an extended volume and the apparatus cannot be considered without dimension, the interference effects will involve in general amplitudes of states with different time and distance intervals. Different time intervals $t_1 \neq t_2$ may seem an unnecessary, unphysical, abstraction. However, it is needed for self-consistency. Even if in a given frame the creation is considered instantaneous it will not generally appear so for another observer, given the extended dimension of the wave function. For this latter observer there will exist times when the probability of measuring the created particle is between 0 and 1. This implies the introduction, in general, of a time dependence for the growth of a wave function at each x in all frames. Furthermore, if we fix $L_1 = L_2 \equiv L$, and have different velocities, $v_1 \neq v_2$, we must necessarily allow for $t_1 \neq t_2$. All this does not mean that the cases listed below are equally realistic.

We consider three broad classes, always within the approximation of an effective one dimensional treatment:

- Common momentum:
  $p_1 = p_2 = p \ , \quad E_1 \neq E_2 \qquad [\, v_1 \neq v_2 \,] \ ;$
- Common energy:
  $E_1 = E_2 = E \ , \quad p_1 \neq p_2 \qquad [\, v_1 \neq v_2 \,] \ ;$
- Different momentum and energies:
  $E_1 \neq E_2 \ , \quad p_1 \neq p_2 \qquad [\, v_1 \neq v_2 \quad \text{or} \quad v_1 = v_2 \ ] \ .$

The above cases by no means exhaust all possibilities but they are sufficient to cover almost all the assumptions made in the literature and lead to the subtle differences of the resulting formulas for $P(\nu \to \tilde{\nu})$ which we are interested in.

The space-time evolution of $|\nu\rangle$ and $|\tilde{\nu}\rangle$ is determined by the space-time development of the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. In the laboratory frame, we have

$$|\nu_n(t_n, L_n)\rangle = e^{-i(E_n t_n - p_n L_n)} |\nu_n\rangle \ , \qquad n = 1, 2 \ .$$

Consequently,

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2 \left(\tfrac{E_2 t_2 - E_1 t_1 - p_2 L_2 + p_1 L_1}{2} \right) \ .$$

We can eliminate the space or time dependence in the previous formula by using the relations

$$L_1 = \frac{p_1}{E_1} t_1 \qquad \text{and} \qquad L_2 = \frac{p_2}{E_2} t_2 \ .$$

For time oscillations, we have

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2 \left[\frac{M_2^2}{2E_2} t_2 - \frac{M_1^2}{2E_1} t_1 \right] \ , \tag{7}$$

whereas, for space oscillations, we obtain

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2 \left[\frac{M_2^2}{2p_2} L_2 - \frac{M_1^2}{2p_1} L_1 \right] \ . \tag{8}$$

For common momentum neutrino productions, we cosidere two different situations, common arrival time and fixed laboratory distance.

- Common momentum:
  - $* \ t_1 = t_2 = t \ ,$
    $P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2 \left[\tfrac{t}{2} \left(\tfrac{M_2^2}{E_2} - \tfrac{M_1^2}{E_1} \right) \right] \ ;$
  - $* \ L_1 = L_2 = L \ ,$
    $P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2 \left[\tfrac{L}{2p} \left(M_2^2 - M_1^2 \right) \right] \ .$

As already mentioned, for Lipkin [8] time oscillations represent non experiments or gedanken experiments because they measure time oscillations. For "real" experiments, in the scenario of common momentum neutrino production, we should use the formula:

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \;\sin^2 \left[\frac{L}{2p}\,\Delta M^2\right] \;. \tag{9}$$

In terms of the average neutrino energy

$$\bar{E} = \frac{E_1 + E_2}{2} = p\left[1 + \frac{M_1^2 + M_2^2}{4p^2} + \mathcal{O}\left(\frac{M^4}{p^4}\right)\right] \;,$$

we can rewrite the previous equation, in the ultra–relativistic limit, as:

$$P(\nu \to \tilde{\nu}) \approx \sin^2 2\theta \;\sin^2 \left[\frac{L}{2\bar{E}}\,\Delta M^2\right] \;. \tag{10}$$

Thus, we find a factor two difference between the oscillation coefficient in this formula and the standard mass oscillation formula of Eq. (5).

Let us now consider common energy neutrinos productions.

- Common energy:
  - $*\ t_1 = t_2 = t$,
    $P(\nu \to \tilde{\nu}) = \sin^2 2\theta \;\sin^2\left[\frac{t}{2E}\,\left(M_2^2 - M_1^2\right)\right]$;
  - $*\ L_1 = L_2 = L$,
    $P(\nu \to \tilde{\nu}) = \sin^2 2\theta \;\sin^2\left[\frac{L}{2}\,\left(\frac{M_2^2}{p_2} - \frac{M_1^2}{p_1}\right)\right]$.

Space oscillations are described by

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \;\sin^2 \left[\frac{L}{2}\,\Delta\left(\frac{M^2}{p}\right)\right] \;, \tag{11}$$

where

$$\Delta\left(\frac{M^2}{p}\right) = \frac{M_2^2}{p_2} - \frac{M_1^2}{p_1} \;.$$

This can be written as:

$$\Delta\left(\frac{M^2}{p}\right) = \frac{\Delta M^2}{E}\left[1 + \frac{M_1^2 + M_2^2}{2E^2} + \mathcal{O}\left(\frac{M^4}{E^4}\right)\right] \;.$$

Consequently, in the ultra-relativistic limit, Eq. (11) becomes:

$$P(\nu \to \tilde{\nu}) \approx \sin^2 2\theta \;\sin^2 \left[\frac{L}{2E}\,\Delta M^2\right] \;. \tag{12}$$

The factor two difference is thus also present in common energy scenarios.

The formulas in Eqs. (9,11) tend to the same result in the ultra-relativistic limit, Eqs. (10,12), and are in disagreement with the standard formula, Eq. (5). In theory, at least, the differences between them may be experimentally determined, especially for non-relativistic processes.

IV. OUR PREFERRED CHOICE

The scenario of different momentum and energies neutrinos productions, with common velocities, merits special attention, because only if $v_1 = v_2$ the formula $P(\nu \to \nu')$ is valid for all times. Otherwise, $P(\nu \to \nu')$ is valid only until the wave packet for the two mass eigenstates overlap substantially. This complication does not exists for $v_1 = v_2$. Indeed, with this condition, the wave packets travel together, for all observers, and we may even employ a common L

and common t. Furthermore, there exists in this case a rest frame, $v = 0$, for our flavour eigenstate common to that of the mass eigenstates. This situation is implicit in all calculations that use a common proper time τ [10].

By assuming a common velocity scenario, we must, necessarily, require different momentum and energies for the neutrinos produced. Due to the common velocity, the time evolution for the mass eigenstates in the common rest frame is

$$|\nu_n(\tau)\rangle = e^{-iM_n\tau}\,|\nu_n\rangle \ , \tag{13}$$

and only in this case is Eq. (3) really justified with its non indexed time and distance. In fact, for common velocities, the Lorentz-invariant phase factor can be rewritten in terms of the common time, t, and the common position, L, in the laboratory frame. We can eliminate the time dependence in the previous formula by using the relations

$$L = \frac{p_1}{E_1}\,t = \frac{p_2}{E_2}\,t \ .$$

Space oscillations, are, thus, described by

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2\left[\frac{L}{2}\left(\frac{M_2^2}{p_2} - \frac{M_1^2}{p_1}\right)\right] \ . \tag{14}$$

This equation is formally equivalent to Eq. (12) and, thus, at first glance, it seems to reproduce the factor two difference. This is a wrong conclusion! Indeed, in the scenario of common velocities

$$p_1 = M_1\gamma_v v \quad \text{and} \quad p_2 = M_2\gamma_v v \ ,$$

and consequently,

$$\Delta\left(\frac{M^2}{p}\right) = \frac{M_2 - M_1}{\gamma_v v} = \frac{\Delta M^2}{2\bar{p}} \ .$$

Space oscillations, in the common velocity scenario, are, thus, described by

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \ \sin^2\left[\frac{L}{4\bar{p}}\,\Delta M^2\right] \ , \tag{15}$$

and this recalls the standard result with the factor four in the denominator. However, it must be noticed that for this case

$$\frac{E_1}{E_2} = \frac{p_1}{p_2} = \frac{M_1}{M_2} \ ,$$

and this may be very far from unity. Thus, the use of $\bar{p}$ in Eq. (15) is not exactly the E intended in the standard formula, which was identical, or almost, for both neutrinos.

V. CONCLUSIONS

The common velocity scenario is aesthetically the most pleasing. We believe in fact that the creation of a particle may differ, for example in the extension of a wave function, from process to process, therefore only experiment can decide which, if any, of the above situations are involved. However, we wish to point out that the assumptions of same momentum, p, or same energy, E, can only be valid in, at most, one reference frame. It seems to us highly unlikely that this frame happens to coincide with our laboratory frame. This means that if our preferred common velocity scenario, which is frame independent, is not satisfied, we may legitimately doubt that any of the popular hypothesis coincide with any given experimental situation.

We conclude our discussion by giving the mass oscillation formula in terms of the "standard" phase factor

$$\frac{L}{4\bar{E}}\,\Delta M^2 \ ,$$

and the parameter

$$\alpha = \left(1 + \frac{\gamma_1 M_1}{\gamma_2 M_2}\right)\left(\frac{1}{v_2} - \frac{1}{v_1}\frac{\gamma_2 M_1}{\gamma_1 M_2}\right)\left(1 - \frac{M_1^2}{M_2^2}\right)^{-1} .$$

The new formula reads

$$P(\nu \to \tilde{\nu}) = \sin^2 2\theta \;\sin^2\left[\alpha\,\frac{L}{4\bar{E}}\,\Delta M^2\right] . \tag{16}$$

For common velocity, momentum and energy, the parameter α becomes

$$\begin{aligned} \alpha_v &\equiv \alpha\,[v_1 = v_2] = 1/v \,, \\ \alpha_p &\equiv \alpha\,[p_1 = p_2] = (v_1 + v_2)/v_1 v_2 \,, \\ \alpha_E &\equiv \alpha\,[E_1 = E_2] = 2\left(1 + \tfrac{1}{v_1 v_2}\right)/(v_1 + v_2) \,. \end{aligned}$$

Finally, in the ultra-relativistic limit, by killing the $\mathcal{O}\left(\frac{M^4}{p^4}\,,\,\frac{M^4}{E^4}\right)$ terms, we obtain

$$\begin{aligned} \alpha_v &\approx 1 + \left(M_1^2/p_1 + M_2^2/p_2\right)/4\bar{p} \,, \\ \alpha_p &\approx 2 + \left(M_1^2 + M_2^2\right)/2p^2 \,, \\ \alpha_E &\approx 2 + \left(M_1^2 + M_2^2\right)/E^2 \,. \end{aligned}$$

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